We may now assume that $A' = [p(A) \cap Y] \neq p(A)$. Let $H^n(X)$ denote the nth Čech cohomology group of a space X with integers as the coefficient group. Since A' is a proper closed subset of the suspension over the Polish circle $p(A)$, it follows that A' cannot separate E^3, and hence, by Alexander duality [10, p. 150] the cohomology group $H^2(A')$ is zero. Let Y/A' denote the space obtained from Y by identifying A' to a point. It follows from $\dim(Y-A')\leqslant 1$ that $\dim(Y/A')\leqslant 1$ [10, p. 32]. By Theorem VIII 4 of [10, p. 152], it follows that $H^2(Y/A')$ is zero. It can be easily shown by the continuity of the Čech cohomology theory that $H^2(Y, A')$ is isomorphic to $H^2(Y/A')$. By the long exact sequence of the pair (Y, A'), the sequence $0 = H^2(Y, A')\to H^2(Y)\to H^2(A') = 0$ is exact. This is a contradiction.

This finishes our proof that the decomposition space X/G is R-stable.

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An expansion of an $\aleph_0$-categorical model

by

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Abstract. We show the existence of an $\aleph_0$-categorical model M having an expansion M^* with the "same" elementary submodels, but which is not $\aleph_0$-categorical. In addition, (1) M contains $\aleph_0$ disjoint sets intersected by every elementary submodel; (2) for every countable $N^* \equiv M^*$, $\prec(N^*) \cong \prec(M^*)$, where $\prec(N^*)$ is the set of elementary submodels of N^* partially ordered by $\prec$.

Introduction. An expansion M^* of M is said to be *elementary* if the universe of every elementary submodel of M is the universe of an elementary submodel of M^*. This concept was introduced in [2] where it was shown that if M, N are countable, not isomorphic, N is not saturated, and M is prime, then there is an elementary expansion M^* of M such that there is no expansion N^* of N with $N^* \equiv M^*$. Of course, the interesting case is when $M \equiv N$. If in addition M is $\aleph_0$-categorical then $N \cong M$ and the above theorem does not apply. Nevertheless the properties of elementary expansions of $\aleph_0$-categorical models are worthy of investigation. Here we show (Theorem 1) the existence of an $\aleph_0$-categorical model M having a non-$\aleph_0$-categorical elementary expansion $M^* = (M, P_i)_{i<\omega}$, where the P_i are unary relation symbols interpreted as disjoint sets. Thus M contains $\aleph_0$ disjoint sets which are intersected by every elementary submodel. In a sense, this is as close as an $\aleph_0$-categorical model can get to being a minimal model (a model with no proper elementary submodels).

In addition there is a theory $T_1^* \subseteq \mathrm{Th}(M^*)$ such that every model of T_1^* can be realized as an elementary expansion of a model of $\mathrm{Th}(M)$. In particular, then, for all countable models elementary equivalent to M^*, the partially ordered sets of their elementary submodels are isomorphic.

Notation and definitions. We deal here with models M, N, etc. in languages $L(M)$, $L(N)$, etc. Most of the terminology and notation is standard. Anything not defined below can be found in Chang–Keisler [1]. We use the term language to mean a set of relation symbols. If L is a language, $\bar{L}$ is the set of first order formulas built up from L and the finitary connectives and quantifiers. An L-m-diagram is any subset of $\{R(x_{i_0}, \dots, x_{i_{k-1}})\colon R \in L, i_j < m\}$. If $\bar{a} = \langle a_0, \dots, a_{m-1}\rangle$ is a sequence of elements in an L-structure, M, written $\bar{a} \in M$ or $\bar{a} \subseteq M$, then the L-diagram of $\bar{a}$ is

$\{R(x_{i_0}, \ldots, x_{i_{k-1}})\colon M \models R(a_{i_0}, \ldots, a_{i_{k-1}}), R \in L\}$. We also write $\bar{a}(i) = a_i$. The length of a sequence $\bar{a}$ is denoted $l(\bar{a})$ and the number of free variables in a formula φ is denoted $l(\varphi)$. Two sequences $\bar{a}, \bar{b}$ of length m are L-isomorphic if they have the same L-diagram. If $\bar{a}, \bar{b} \in M$ then they are M-automorphic if there is an automorphism f of M such that $f(\bar{a}(i)) = \bar{b}(i)$ for all $i<m$.

We allow a certain confusion in distinguishing between the sequence $\bar{a}$ and the set $\{a_0, \ldots, a_{m-1}\}$. For example, depending on the context $f(\bar{a})$ might mean either

$$\langle f(a_0), \ldots, f(a_{m-1})\rangle \quad \text{or} \quad \{f(a_0), \ldots, f(a_{m-1})\}\,.$$

Also by $\bar{x} \supseteq \bar{y}$ we mean $\bar{y} = \langle y_0, \ldots, y_{m-1}\rangle$, $\bar{x} = \langle x_0, \ldots, x_{n-1}\rangle$ and every y_i is some x_j.

If $L \subseteq L^*$, T^* is a theory in L^*, then $T^* \upharpoonright L = T^* \cap \bar{L}$. If M is a model and $A \subseteq M$ then $M \upharpoonright A$ is the submodel of M with universe set A. If $L \subseteq L(M)$, $M \upharpoonright L$ is the restriction of M to L. $D_n(M)$ is the set of n-types realized in M.

The theorem.

THEOREM 1. *There exists a countable $\aleph_0$-categorical model M with a non-$\aleph_0$-categorical elementary expansion M^*.*

Remarks. The construction of the above M is an example of the following general situation: Let $L \subseteq L^*$ be two languages, T^* a theory in L^* (not necessarily complete.) Find $M^* \models T^*$ such that $M = M^* \upharpoonright L$ is $\aleph_0$-categorical. By Ryll-Nardzewski's Theorem and the compactness theorem there is such an M^* iff there are $t(n)<\omega$ for $n<\omega$ and formulas $\varphi_i^n(x_0, \ldots, x_{n-1})$ of L, $n<\omega$, $i<t(n)$, such that there are models $M_n^* \models T^*$ in which $\{\varphi_i^m\colon i<t(m)\}$ are atoms for all the m-types of $M_n^* \upharpoonright L$, $m<n$. This is also the best result, in the sense that a (non-complete) theory T_1 can be contrived so that for all $n<\omega$ there is a model $M_n \models T_1$ with finitely many n-types, even so that $|D_n(M_n)| = |D_n(M_k)|$ for $k \geqslant n$, but T_1 has no $\aleph_0$-categorical model.

Proof. Let $L_0 = \emptyset$, $L_0^* = \{P_i(x)\colon i<\omega\}$. Assuming L_j and L_j^* defined, for every formula $\varphi(y_0, \ldots, y_{l(\varphi)-1}) \in \overline{L_j^*}$ let $R_\varphi(x_0, \ldots, x_{l(R_\varphi)-1})$ be a new relation symbol and take $L_{j+1} = L_j \cup \{R_\varphi(x_0, \ldots, x_{l(R_\varphi)-1})\colon \varphi \in \overline{L_j^*}\}$, $L_{j+1}^* = L_j \cup L_0^*$. Let $L = \bigcup_{j<\omega} L_j$, $L^* = \bigcup_{j<\omega} L_j^*$ $(= L \cup L_0^*)$. Choose $l(R_\varphi)$ so that $l(R_\varphi) > \sum l(R_\psi)$ where the summation is over all R_ψ occurring in φ, and in addition $l(R_\varphi) \neq l(R_\psi)$ for $\varphi \neq \psi$.

Now let

$$\begin{aligned} T^* = {} & \{\neg\exists x(P_i(x) \wedge P_j(x))\colon i \neq j<\omega\} \cup \\ & \cup \{\forall \bar{y}[\varphi(\bar{y}) \to (\exists \bar{x} \supseteq \bar{y}) R_\varphi(\bar{x})\colon \varphi \in \overline{L^*}\} \cup \\ & \cup \{\forall \bar{x}(\forall \bar{u} \subseteq \bar{x})((R_\varphi(\bar{x}) \wedge \exists y \varphi(\bar{u}, y)) \to (\exists y \in \bar{x}) \varphi(\bar{u}, y))\colon \varphi \in \overline{L^*}\}. \end{aligned}$$

CLAIM. *Let $M^* \models T^*$, $M = M^* \upharpoonright L$.*

1. *If every two L-isomorphic sequences from M are M-automorphic, then M is $\aleph_0$-categorical.*

2. *M^* is an elementary expansion of M.*

Proof. 1. Since $l(R_\varphi) \neq l(R_\psi)$ for $\varphi \neq \psi$, for each n there is a finite number of L-isomorphism classes of n-tuples, and thus M-automorphism classes, and thus types.

2. Let $N \prec M$, $N^* = M^* \upharpoonright N$. In order to show $N^* \prec M^*$ it is sufficient, by the Tarski–Vaught test, to show that if $\bar{a} \in N$, $M^* \models \exists y \varphi(\bar{a}, y)$, then there is $b \in N$ such that $M^* \models \varphi(\bar{a}, b)$. Let $M^* \models \varphi(\bar{a}, c)$. Then $M^* \models (\exists \bar{x} \supseteq \bar{a}) R_\varphi(\bar{x})$. Thus $M \models$ (the same) and thus $N \models$ (the same). Let $\bar{d} \supseteq \bar{a}$ satisfy $N \models R_\varphi(\bar{d})$; of course $\bar{d} \subseteq N$. So $M \models R_\varphi(\bar{d})$ and $M^* \models R_\varphi(\bar{d})$. Since $M^* \models \exists y \varphi(\bar{a}, y)$ and $\bar{a} \subseteq \bar{d}$, we get $M^* \models (\exists y \in \bar{d}) \varphi(\bar{a}, y)$. Let b be that y. Certainly $b \in N$ and $M^* \models \varphi(\bar{a}, b)$.

So in order to prove Theorem 1 it is sufficient to prove:

LEMMA 1. *Given $j, n<\omega$ and finite subsets*

$$\begin{aligned} & \Psi_0 = \emptyset,\ \Psi_0^* \subseteq L_0^*,\ \Phi_0 \subseteq \overline{\Psi}_0^*, \\ & \Psi_1 = \{R_\varphi\colon \varphi \in \Phi_0\},\ \Psi_1^* = \Psi_1 \cup \Psi_0^*,\ \Phi_1 \subseteq \overline{\Psi}_1^*, \\ & \Psi_2 = \Psi_1 \cup \{R_\varphi\colon \varphi \in \Phi_1\},\ \Psi_2^* = \Psi_2 \cup \Psi_0^*, \ldots \\ & \ldots,\ \Psi_j = \Psi_{j-1} \cup \{R_\varphi\colon \varphi \in \Phi_{j-1}\},\ \Psi_j^* = \Psi_j \cup \Psi_0^*, \end{aligned}$$

there is a Ψ_j^*-model M^* such that

(1) $M^* \models (\neg\exists x)(P_{i_1}(x) \wedge P_{i_2}(x))$ *for all* $P_{i_1} \neq P_{i_2} \in \Psi_0^*$;

(2) $M^* \models (\forall \bar{y})(\varphi(\bar{y}) \to (\exists \bar{x} \supseteq \bar{y}) R_\varphi(\bar{x}))$, $\varphi \in \Phi_{j-1}$;

(3) $M^* \models (\forall \bar{x})(\forall \bar{u} \subseteq \bar{x})((R_\varphi(\bar{x}) \wedge \exists y \varphi(\bar{u}, y)) \to (\exists y \in \bar{x}) \varphi(\bar{u}, y))$ *for* $\varphi \in \Phi_{j-1}$;

(4) *If $\bar{a}, \bar{b}$ are n-tuples in M^* which are Ψ_j-isomorphic then they are $M^* \upharpoonright \Psi_j$-automorphic.*

If in addition $\mathrm{Th}(M^*)$ admits elimination of quantifiers, then in (2) and (3) above φ may be taken to be a diagram.

Thus it is sufficient to prove:

LEMMA 2. *Given $j, n<\omega$ and a finite subset $\Psi_0^* \subseteq L_0^*$, let Δ_0^* = the set of Ψ_0^*-m-diagrams, $m \leqslant n$,*

$$\Psi_1 = \{R_\delta\colon \delta \in \Delta_0^*\},\ \Psi_1^* = \Psi_1 \cup \Psi_0^*, \ldots,$$

Δ_{j-1}^* = the set of Ψ_{j-1}^*-m-diagrams, $m \leqslant n$,

$$\Psi_j = \{R_\delta\colon \delta \in \Delta_{j-1}^*\}, \quad \Psi_j^* = \Psi_j \cup \Psi_0^*.$$

Let

$$\Delta^* = \Delta_{j-1}^*, \quad \Psi = \Psi_j, \quad \Psi^* = \Psi_j^*.$$

Then there is a Ψ^*-model M^* such that

(0) *$\mathrm{Th}(M^*)$ admits elimination of quantifiers;*

(1) $M^* \models (\neg\exists x)(P_{i_1}(x) \wedge P_{i_2}(x))$ *for all* $P_{i_1} \neq P_{i_2} \in \Psi_0^*$;

(2) $M^* \models (\forall \bar{y})(\delta(\bar{y}) \to (\exists \bar{x} \supseteq \bar{y}) R_\delta(\bar{x})), \quad \delta \in \Delta^*$;

(3) $M^* \models (\forall \bar{x})(\forall \bar{u} \subseteq \bar{x})\big((R_\delta(\bar{x}) \wedge \exists y \delta(\bar{u}, y)) \to (\exists y \in \bar{x}) \delta(\bar{u}, y)\big), \quad \delta \in \Delta^*$;

(4) *If $\bar{a}, \bar{b}$ are n-tuples in M^* which are Ψ-isomorphic then they are $M^* \restriction \Psi$-automorphic;*

(5) *If $\bar{a}, \bar{b}$ are n-tuples in M^* which are Ψ^*-isomorphic then they are M^*-automorphic.*

Remark. Note that (5) implies (0).

Proof of Lemma 2. We define sets X_i, Y_i such that $X_i \subseteq Y_i \subseteq X_{i+1}$ and sets of functions

$$\mathscr{F}_i = \{f^\varepsilon_{i,\langle \bar{a}, \bar{b}\rangle}\colon \varepsilon = 1, -1,\ l(\bar{a}) = l(\bar{b}) = n,\ \bar{a}, \bar{b} \in Y_i \text{ are } \Psi\text{-isomorphic}\},$$

$$\mathscr{F}^*_i = \{f^{*\varepsilon}_{i,\langle \bar{a}, \bar{b}\rangle}\colon \varepsilon = 1, -1,\ l(\bar{a}) = l(\bar{b}) = n,\ \bar{a}, \bar{b} \in Y_i \text{ are } \Psi^*\text{-isomorphic}\},$$

$$\operatorname{Dom} g = Y_i,\ \operatorname{Rang} g \subseteq X_{i+1} \quad \text{for every } g \in \mathscr{F}_i \cup \mathscr{F}^*_i,\ i < \omega.$$

Say that an m-diagram δ is t-consistent if

(i) $\delta(\bar{x}) \to \neg\big(P_{i_1}(\bar{x}(k)) \wedge P_{i_2}(\bar{x}(k))\big)$ for all $P_{i_1} \neq P_{i_2} \in \Psi^*_0$, $k < m$;

(ii) $(\forall \bar{u} \subseteq \bar{x})[(\delta(\bar{x}) \wedge \delta_1(\bar{u})) \to (\exists \bar{y})(\bar{x} \supseteq \bar{y} \supseteq \bar{u} \wedge R_{\delta_1}(\bar{y}))]$ for all $\delta_1 \in \Delta^*$;

(iii) $(\forall \bar{z} \subseteq \bar{u} \subseteq \bar{x})[(\delta(\bar{x}) \wedge R_{\delta_1}(\bar{u}) \wedge \exists y \delta_1(\bar{z}, y)) \to (\exists y \in \bar{u}) \delta_1(\bar{z}, y)]$ for all $\delta_1 \in \Delta^*_t$.

Let X_0 be a finite set with Ψ^* defined on it so that every $(j-1)$-consistent n-diagram from Ψ^* is realized by some n-tuple from X_0, and X_0 satisfies (1) and (3). We leave it to the reader to see that this is possible. Let $Y_0 \supseteq X_0$ also be finite and satisfying (1) and (3) such that if $\bar{y} \in X_0$, $\delta \in \Delta^*$ then $Y_0 \models \delta(\bar{y}) \to (\exists \bar{x} \supseteq \bar{y}) R_\delta(\bar{x})$. Again we leave the details to the reader. Now assume Y_i is finite, satisfies (1) and (3), and if $\bar{y} \in X_i$, $\delta \in \Delta$, then $Y_i \models \delta(\bar{y}) \to (\exists \bar{x} \supseteq \bar{y}) R_\delta(\bar{x})$. Let $\mathscr{F}_i$ and $\mathscr{F}^*_i$ be as above. We want to define g on Y_i for every $g \in \mathscr{F}_i \cup \mathscr{F}^*_i$. If $\bar{a}, \bar{b} \in Y_{i-1}$ then $f^\varepsilon_{i-1,\langle \bar{a}, \bar{b}\rangle} \in \mathscr{F}_{i-1}$; define $f^\varepsilon_{i,\langle \bar{a}, \bar{b}\rangle} \restriction Y_{i-1} = f^\varepsilon_{i-1,\langle \bar{a}, \bar{b}\rangle}$. If $\bar{a} \cup \bar{b} \not\subseteq Y_{i-1}$, define

$$f^\varepsilon_{i,\langle \bar{a}, \bar{b}\rangle}(\bar{a}(k)) = \bar{b}(k) \quad \text{and} \quad f^{-1}_{i,\langle \bar{a}, \bar{b}\rangle}(\bar{b}(k)) = \bar{a}(k).$$

If $x = f^{-1}(y)$ define $f(x) = y$, and if $x = f(y)$ define $f^{-1}(x) = y$. The definitions are similar for $f^{*\varepsilon}_{i,\langle \bar{a}, \bar{b}\rangle}$. Let $X_{i+1} = Y_i \cup \{g(x)\colon g \in \mathscr{F}_i \cup \mathscr{F}^*_i$, x is not covered by any of the above cases in relation to $g\}$.

For $g \in \mathscr{F}_i$ define Ψ on $g(Y_i)$ so g will be a Ψ-isomorphism, and for $g \in \mathscr{F}^*_i$ define Ψ^* on $g(Y_i)$ so g will be a Ψ^*-isomorphism. Obviously $\mathscr{F}_i(Y_i)$ satisfies (1), and $\mathscr{F}^*_i(Y_i)$ satisfies (1) and (3).

It remains to define Ψ^*_0 on $\mathscr{F}_i(Y_i)$ so that (3) will hold. It is sufficient to define Ψ^*_0 on $g(Y_i)$ for an arbitrary single $g \in \mathscr{F}_i$. This is because if $g_1 \neq g_2$ then $(g_1(Y_i) - Y_i) \cap (g_2(Y_i) - Y_i) = \emptyset$.

Moreover it suffices to define Ψ^*_0 on $g(X)$ for an arbitrary finite $X \subseteq Y_i$.

Case 1. There is $h \in \mathscr{F}_{i-1} \cup \mathscr{F}^*_{i-1}$ such that $X \subseteq h(X_{i-1})$: Assume $g = f^\varepsilon_{i,\langle \bar{a}, \bar{b}\rangle}$, $\bar{a} \in X$ if $\varepsilon = 1$ and $\bar{b} \in X$ if $\varepsilon = -1$. There is $\bar{b}_0 \in Y_0$ such that $\bar{b}$ and $\bar{b}_0$ are Ψ^*-isomorphic. Obviously $\bar{a}$ and $\bar{b}_0$ are Ψ-isomorphic since $\bar{a}$ and $\bar{b}$ are. So $h^{-1}(\bar{a})$ and $\bar{b}_0$ are Ψ-isomorphic. Thus, the function $f = f^\varepsilon_{i-1,\langle h^{-1}(\bar{a}), \bar{b}_0\rangle}$ is in $\mathscr{F}_{i-1} \cup \mathscr{F}^*_{i-1}$. See Fig. 1.

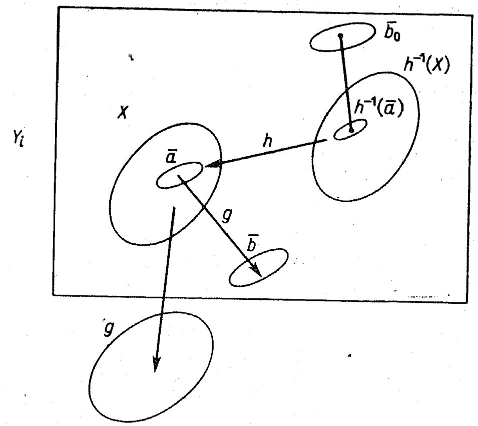

Fig. 1

We can thus define Ψ^*_0 on $g(X)$ the same way we defined Ψ^*_0 on $f(h^{-1}(X))$, i.e., $P_k(g(x)) \leftrightarrow P_k(f(h^{-1}(x)))$.

Case 2. There are $h_1, \ldots, h_m \in \mathscr{F}_{i-1} \cup \mathscr{F}^*_{i-1}$, $m > 1$, such that

$$X \subseteq h_1(X_{i-1}) \cup \ldots \cup h_m(X_{i-1}),$$

m is minimal. For simplicity consider the case $m = 2$. Write $X = H_1 \cup H_2$ where $H_1 \cap H_2 = \emptyset$, $H_1 = h_1(X_{i-1}) \cap X$, $H_2 = X - H_1 \subseteq h_2(X_{i-1}) \cap X$. By Case 1 we can define Ψ^*_0 on $g(H_1)$ and $g(H_2)$ so that both satisfy (3). We leave it to the reader to see that $g(H_1) \cup g(H_2)$ then satisfies (3).

Notice in Case 1 and Case 2 $X \subseteq X_i$.

Case 3. $X \cap (Y_i - X_i) \neq \emptyset$. This presents no new difficulties.

Now define $Y_{i+1} \supseteq X_{i+1}$ so that (1) and (3) are satisfied and also so that if $\bar{y} \subseteq X_{i+1}$, $\delta \in \Delta^*$, then $Y_{i+1} \models \delta(\bar{y}) \to (\exists \bar{x} \supseteq \bar{y}) R_\delta(\bar{x})$.

Now we take $M^* = \bigcup_{i<\omega} X_i = \bigcup_{i<\omega} Y_i$. Clearly (1)-(3) hold. If $\bar{a}$, $\bar{b}$ are Ψ-isomorphic n-tuples, then the Ψ-automorphism of $M^* \restriction \Psi$ taking $\bar{a}$ on $\bar{b}$ is

$$\bigcup_{i_0 < i < \omega} f^1_{i,\langle \bar{a}, \bar{b}\rangle} \cup \bigcup_{i_0 < i < \omega} (f^{-1}_{i,\langle \bar{a}, \bar{b}\rangle})^{-1}$$

where $\bar{a}, \bar{b} \in Y_{i_0}$. This is (4) and (5) is proved similarly.

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Homomorphisms of direct powers of algebras

by

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"Wieleż lat czekać trzeba, nim się przedmiot świeży
Jak figa ucukruje, jak tytuń uleży?" [13]

0. Abstract. Given algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type, a set X and a homomorphism $h: \mathfrak{A}^X \to \mathfrak{B}$ we study the collection of all supports of h, i.e., sets $Y \subseteq X$ such that for all $f, g \in \mathfrak{A}^X$ if $f \restriction Y = g \restriction Y$ then $h(f) = h(g)$.

1. Terminology and generalities. We identify every ordinal number ξ with the set of ordinal numbers smaller than ξ, e.g., $n = \{0, 1, \ldots, n-1\}$ and $\omega = \{0, 1, \ldots\}$. Cardinal numbers are the initial ordinals. α and β denote cardinals. If X is a set then $|X|$ denotes the cardinal of X. α^+ denotes the cardinal successor of α. A filter F of subsets of X is called *α-complete* iff for every $G \subseteq F$ with $|G| < \alpha$ we have $\bigcup G \in F$, and F is called an *ultrafilter* if from any two complementary sets in X at least one is in F. We shall use the following surprising characterisation of α-complete ultrafilters.

1.1 (Galvin and Horn [9]). Let F be a family of subsets of X and α be a cardinal $\geqslant 4$. Then the following two conditions are equivalent

(i) F is an α-complete ultrafilter and $\emptyset \notin F$.

(ii) For every partition P of X with $|P| < \alpha$ we have $|F \cap P| = 1$.

For any cardinal α we denote by $\mu(\alpha)$ the least cardinal such that there exists a nonprincipal α^+-complete ultrafilter of subsets of $\mu(\alpha)$. We recall that $\mu(n) = \omega$ for $2 \leqslant n < \omega$, $\mu(\alpha)$ is a measurable cardinal and

1.2. Every α^+-complete ultrafilter of subsets of $\mu(\alpha)$ is $\mu(\alpha)$-complete.

In fact $\mu(\alpha)$ has many other "closure properties", see [11]. Even the existence of $\mu(\omega)$ does not follow from the Zermelo–Fraenkel axioms of set theory, but the main results of this paper could be easily reformulated so as to avoid the assumptions of the existence of $\mu(\alpha)$ for any infinite α. On the other hand the existence of $\mu(\alpha)$ for every cardinal α is already a well established axiom of set theory, see e.g. [24] p. 47, 48 or [29] p. 675.