# An $R$-stable ANR which is not $FR$-stable

by

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**Abstract.** THEOREM. *There exists an* ANR $X$ *such that* $X$ *does not contain any deformation retract different from itself, however,* $X$ *contains a fundamental deformation retract* $Y$ *different from* $X$. This provides an answer to Problem 2.14 in Borsuk [8]. THEOREM. *Let* $G$ *be an upper semicontinuous decomposition of an* ANR $X$ *such that the nondegenerate elements are contained in* $X-A$, *where* $A$ *is a closed subset of* $X$. *If* $p\colon X\to X/G$ *is a homotopy equivalence and* $A$ *is a fundamental deformation retract of* $X$ *then* $P(A)$ *is a fundamental deformation retract of* $X/G$.

**1. Introduction and terminology.** By a *space* we mean a separable metric space unless otherwise so stated. If a compactum $X$ does not contain any deformation retract of $X$ which is different from $X$ then $X$ is said to be *$R$-stable*. A closed subset $Y\subset X$ is said to be a *fundamental deformation retract of* $X$ if there exists a fundamental sequence $\underline{r}=\{r_k, X, Y\}_{M,M}$ such that $r_{k|Y}=i|_Y$ and that $\{r_k, X, X\}_{M,M}$ homotopic to the identity fundamental sequence $\underline{i}_{X,M}$ or $\{r_k, X, X\}_{M,M}\simeq\underline{i}_{X,M}$. A space $X$ is *fundamentally $R$-stable* (*or $FR$-stable*) if $X$ does not contain any fundamental deformation retract which is different from $X$. The following question appears in Borsuk [8] (as Problem 2.14 on page 264):

PROBLEM. Does there exist an $R$-stable ANR-space which is not $FR$-stable?

We shall prove the following:

THEOREM. *There exists an $R$-stable* ANR-*space* $X$ *of dimension three containing a set* $Y$ *such that* (1) $Y$ *has the shape of* 2-*sphere* $S^2$ *and* (2) $Y$ *is a fundamental deformation retract of* $X$. *Hence* $X$ *is not $FR$-stable.*

For notation and related terminology concerning shape one may consult Borsuk [6] and [8]. By AR, ANR, FAR and FANR we mean absolute retract, absolute neighborhood retract, fundamental absolute retract and fundamental absolute neighborhood retract, respectively. We use this notation only for compact spaces. If $G$ is an u.s.c. decomposition of a space $X$ ("an upper semicontinuous decomposition of a space $X$") then we denote by $X/G$ the associated decomposition space and by $p\colon X\to X/G$ the canonical projection map unless otherwise stated. For additional information concerning decomposition spaces one may consult [3] where other references can be found. Let $Q$ denote the Hilbert cube.

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**2. Construction of an** ANR. Let $E^1$ denote the 1-dimensional Euclidean space and let $E^n$ denote the $n$-fold product of $E^1$ with itself, where $n = 2, 3, 4, \ldots$ For each positive number $\varepsilon$, define $B^n(\overline{0}, \varepsilon) = \{(x_1, x_2, \ldots, x_n) \in E^n\colon x_1^2+x_2^2+\ldots+x_n^2 \leqslant \varepsilon\}$ and $S^{(n-1)}(\overline{0}, \varepsilon) = \{(x_1, x_2, \ldots, x_n) \in E^n\colon x_1^2+x_2^2+\ldots+x_n^2 = \varepsilon\}$, where $n = 1, 2, 3, \ldots$ The set $B^n(\overline{0}, \varepsilon)$ is a closed ball of radius $\varepsilon$ with origin $\overline{0} = (0, 0, \ldots, 0) \in E^n$ as its center and $S^{(n-1)}(\overline{0}, \varepsilon)$ is its boundary $(n-1)$-sphere. The subset $B^3(\overline{0}, 1) \cap [E^2 \times \{0\}]$ of $E^3$ is the set $B^2(\overline{0}, 1) \times \{0\}$ which we may identify with $B^2(\overline{0}, 1)$ whenever appropriate. Let $W$ be the following well-known 1-dimensional continuum in $B^2(\overline{0}, 1) \times \{0\}$ with the shape of a circle as shown in the figure below and such that $W \cap \{(x_1, x_2, 0) \in B^2(\overline{0}, 1) \times \{0\}\colon x_1 = 0\}$ is the set $\{(0, x_2, 0) \in B^3(\overline{0}, 1)\colon \frac{2}{4} \leqslant x_2 \leqslant \frac{3}{4}\}$. The suspension

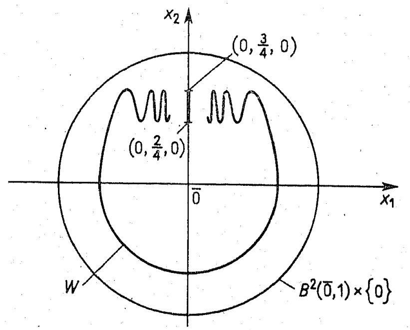


Fig. 1

$\sum B^2(\overline{0}, 1)$ of $B^2(\overline{0}, 1)$ with $(0, 0, \frac{3}{4})$ and $(0, 0, -\frac{3}{4})$ as points of suspension or vertices is contained in $B^3(\overline{0}, 1)$. And hence the suspension $\sum W$ of $W$ with the same vertices is contained in $B^3(\overline{0}, 1)$. There exists a positive number $\varepsilon_0$ sufficiently small such that $B^3(\overline{0}, \varepsilon_0)$ does not intersect $\sum W$. Put

$$X = \{(x_1, x_2, x_3) \in E^3\colon \varepsilon_0 \leqslant x_1^2+x_2^2+x_3^2 \leqslant 1\}.$$

The space $X$ is a compact 3-manifold with boundary and hence an ANR. Also $X$ contains $\sum W$ in its interior. We prove the following for later use:

Lemma 2.1. *The subspace $\sum W$ is a fundamental deformation retract of $X$.*

Proof. There exists a sequence $X_1 \supset X_2 \supset X_3 \ldots$ such that each $X_i$ is a closed annulus contained in $B^2(\overline{0}, 1)$ such that $X_{i+1}$ is a deformation retract $X_i$, for all $i = 1, 2, 3, \ldots$, and $W = \bigcap_{i=1}^{\infty} X_i$. Define $Y_i = \sum X_i$, for $i = 1, 2, 3, \ldots$ and we use the same vertices as for $\sum W$. We have a sequence $Y_1 \supset X_2 \supset Y_3 \supset \ldots$ of ANR's such that $Y_{i+1}$ is a fundamental deformation retract of $Y_i$ for $i = 1, 2, 3, \ldots$, and $\sum W = \bigcap_{i=1}^{\infty} Y_i$. By Theorem 4.1 of Borsuk [8], it follows that $\sum W$ is a fundamental deformation retract of $X$. Since $X$ has the shape of a 2-sphere, $\sum W$ has the shape of a 2-sphere. This finishes the proof of Lemma 2.1.



The set $\sum W$ decomposes $X$ into two disjoint open and connected subsets $X_1$ and $X_2$. More precisely, $X - \sum W = X_1 \cup X_2$, and $X = X_1 \cup X_2 \cup \sum W$, where these are disjoint unions. For each $i = 1$ and 2, describe an u.s.c. decomposition $G_i$ of $X_i$ such that the nondegenerate elements of $G_i$ form a null sequence of arcs such that the decomposition space does not contain any ANR of dimension two or any ANR of dimension three and different from itself. Define an u.s.c. decomposition $G$ of the space $X$ such that $G = G_1 \cup G_2 \cup \{\{x\}\colon x \in \sum W\}$. The decomposition space $X/G$ is 3-dimensional [5] and by a Theorem of Smale [14] mentioned in Borsuk [6] it follows that $X/G$ is an ANR. Let $p\colon X \to X/G$ be the projection map. It is well-known that $p\colon X \to X/G$ is a homotopy equivalence, see [1] and [12].

**3. Homotopy inverses and fundamental deformation retracts.**

Lemma 3.1. *Let $G$ be an u.s.c. decomposition of an* ANR *$X$ such that the non-degenerate elements of $G$ are contained in $X - A$, where $A$ is a closed subset of $X$. If the projection $p\colon X \to X/G$ is a homotopy equivalence then there exists a homotopy inverse $q\colon X/G \to X$ of $p\colon X \to X/G$ satisfying $qp(a) = a$, for each $a \in A$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{p_1} & p(A) & \xrightarrow{p_1^{-1}} & A \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle i}\downarrow \\ X & \xrightarrow{p} & X/G & \xrightarrow{r} & X \end{array}$$

where $i$ and $j$ are inclusions, $p_1$ is the restriction of $p$ and $p_1^{-1}$ is its inverse, and $r$ is a homotopy inverse of $p$. Now $jp_1 = pi$ and hence by composing with $r$ on both sides we have $rjp_1 = rpi$. Composing with $p_1^{-1}$ we get

$$rjp_1p_1^{-1} = rpip_1^{-1} \quad \text{or} \quad rj = (rp)ip_1^{-1} \quad \text{or} \quad rj \cong ip_1^{-1},$$

since $rp$ is homotopic to the identity on $X$ (or shortly $rp \cong i_X$).

Now $p(A)$ is a closed subset of $X/G$ and $X$ is an ANR we conclude that $ip_1^{-1}\colon p(A) \to A$ can be extended and this extension $q\colon X/G \to X$ is homotopic to $r$. This can be done since ANR's have the homotopy extension property, see Borsuk [7, p. 6]. The homotopy inverse $q\colon X/G \to X$ has the required property $qp(a) = ip_1^{-1}p(a) = a$, for each $a \in A$. This finishes the proof of Lemma 3.1.

There are known theorems which guarantee that the projection map $p\colon X \to X/G$ is a homotopy equivalence when certain conditions are imposed on the decomposition and the dimension of $X/G$ is finite. For example, if elements of $G$ have trivial shape and $X/G$ is finite dimensional then the projection $p\colon X \to X/G$ is a homotopy equivalence. For more details see [1] and [12]. We prove the following:

THEOREM 3.1. *Let $G$ be an u.s.c. decomposition of an* ANR *$X$ such that all the nondegenerate elements are contained in $X-A$. If the projection $p\colon X\to X/G$ is a homotopy equivalence and $A$ is a fundamental deformation retract of $X$, then $p(A)$ is a fundamental deformation retract of $X/G$.*

Proof. Let $q\colon X/G\to X$ be a homotopy inverse of $p\colon X\to X/G$ such that $qp(a)=a$, for each $a\in A$. This follows from Lemma 3.1. Let $\underline{r}=\{r_k, X, A\}_{Q,Q}$ be a fundamental deformation retraction from $X$ onto $A$, where $Q$ denotes the Hilbert cube. Define a fundamental sequence

$$\underline{s}=\{\hat{p}r_k\hat{q}, X/G, p(A)\}_{Q,Q},$$

where $\hat{p}\colon Q\to Q$ and $\hat{q}\colon Q\to Q$ are extensions of $p\colon X\to X/G$ and $q\colon X/G\to X$. Clearly, $\hat{p}r_k\hat{q}(p(a))=p(a)$ for each point $p(a)\in p(A)$.

Since,

$$\hat{p}\underline{r}\hat{q}\cong\hat{p}i_{X,Q}\hat{q}\cong\hat{p}\hat{q}\cong i_{X/G,Q},$$

it follows that

$$\underline{s}=\{\hat{p}r_k\hat{q}, X/G, X/G\}_{Q,Q}\cong i_{X/G,Q}.$$

This proves that $p(A)$ is a fundamental deformation of $X/G$.

Recall the constructions of Section 2 where we described an u.s.c. decomposition $G$ of a space $X$ which contains the set $A=\sum W$ and the nondegenerate elements of $G$ are contained in $X-A$. We are in a position to prove the following:

THEOREM 3.2. *The decomposition space $X/G$ is an $R$-stable* ANR *(of dimension three) which is not $FR$-stable.*

Proof. It follows from discussion in Section 2 that $X/G$ is a 3-dimensional ANR. From Lemma 2.1, we known that $A=\sum W$ is a fundamental deformation retract of $X$. The set $p(A)$ is homeomorphic to $A$ and $p(A)$ is a fundamental deformation retract of $X/G$. This follows from Theorem 3.1, since $p\colon X\to X/G$ is a homotopy equivalence ([1] and [12]). This proves that the decomposition space $X/G$ is not $FR$-stable. It remains to be proved that $X/G$ is $R$-stable.

It is clear that no 1-dimensional subset of $X/G$ can be a deformation retract of $X/G$. If $Y\subset X/G$ is a deformation retract of $X/G$ then $\dim Y\geqslant 2$ and $Y$ is a retract of $X/G$. Therefore, $Y$ is an ANR and $Y$ has the homotopy type of a 2-sphere. Now $p(A)$ which is homeomorphic to $\sum W$ cannot contain an ANR which has the homotopy type of a 2-sphere, and hence $Y\cap[X/G-p(A)]\neq\emptyset$. We consider the following two cases:

Case I. Assume $Y$ contains a simple closed curve $\alpha$ such that $\alpha\subset[Y-p(A)]$ and $\alpha$ is homotopic to zero in $Y-p(A)$. Recall that $X/G-p(A)=X_1/G_1\cup X_2/G_2$ is a disjoint union of the complementary domains of $p(A)$. Assume without loss of generality that $\alpha\subset X_1/G_1$. By applying a linking argument of [13] we arrive at a contradiction and thus showing that $X/G$ does not contain such an ANR $Y$.

Case II. If there exists a point $y_0\in Y-p(A)$ such that $Y$ has dimension at least two at $y_0$, then every neighborhood of $y_0$ contains a simple closed curve. By taking a small neighborhood $V$ of $y_0$ we find a simple closed curve $\alpha\subset V$ and $\alpha$ is homotopic to zero in $Y-p(A)$.



Since this reduces to Case I, we assume from now on that $\dim[Y-p(A)]\leqslant 1$. It is also clear from our construction that $p(A)\cap Y\neq\emptyset$. Now, there are two possible cases (i) $p(A)\subset Y$, or (ii) $p(A)\not\subset Y$. Our final goal is to show that each of these two cases leads to a contradiction. We assume that $p(A)$ is a subset of $Y$ in the following paragraphs, unless, otherwise so stated.

Clearly, the boundary $F(C)$ of a component $C$ of the set $[Y-p(A)]$ is a nonempty subset of $p(A)$. By Theorem 7 of [11, p. 266], there exists a dense subset $D$ of $F(C)$ such that each point of $D$ is arcwise accessible from $C$. We shall prove, in the next few paragraphs, that $D$ contains exactly one point.

It is easy to see that the set $p(A)$ is shape equivalent to the 2-sphere $S^2$. Hence, $p(A)$ has the property $\mathrm{UV}^1$ inside $Y$, see [1], [2], [4], [6] and [9]. Let $q\colon Y\to Y/H$ be the projection onto the decomposition space $Y/H$ associated with u.s.c. decomposition $H$ of $Y$ such that $p(A)$ is the only nondegenerate element of $H$. Then the induced map $q_*\colon\pi_1(Y)\to\pi_1(Y/H)$ on the fundamental groups (with suitably chosen base points) is an isomorphism [4, Theorem 6.1]. This proves that $Y/H$ is simply connected. Also, $Y/H$ is 1-dimensional since $[Y-p(A)]$ is 1-dimensional and $Y/H$ is a Peano Continuum. Now, it can be easily seen from Theorem VIII 3′ [10, p. 151] that $Y/H$ is a dendrite, see [11, p. 300] for a definition of "dendrite."

Let $x$ and $y$ be two distinct points in $D$ and $z$ be a point in $C$. Since $x$ and $y$ are accessible from $z$, there exists arcs $[z,x]\subset(C\cup\{x\})$ and $[z,y]\subset(C\cup\{y\}$ with endpoints $z$, $x$ and $z$, $y$, respectively. If $[z,x]\cap[z,y]=\{z\}$, then the image $(q[z,x]\cup[z,y])$ is a simple closed curve. This leads to a contradiction since $Y/H$ is a dendrite. Now, we may assume that $[z,x]\cap[z,y]\neq\{z\}$. It is easy to see that $[z,x]\cap[z,y]\neq\{z\}$ implies that there exists a simple closed curve $\alpha\subset([z,x]\cup[z,y])$ such that $\alpha$ is homeomorphic to $q(\alpha)$. Since each of these two cases lead to a contradiction, the set $D$ cannot contain two distinct points. This proves that the set $F(C)$ contains exactly one point. The closure $\overline{C}$ of $C$ in $Y$ is a Peano Continuum, see a theorem of R. L. Moore as quoted in [11, p. 247]. The set $\overline{C}$ is a dendrite, and therefore, embeddable in the plane [11, p. 305]. By Theorem 7.1 of [6, p. 221], the set $\overline{C}$ is shape equivalent to a point ($\mathrm{UV}^\infty$ [2], or cell-like [12]).

Let $\{C_i\colon i\in I\}$ denote the set of components of $[Y-p(A)]$. For each $i$ and $j$ in $I$ such that $i\neq j$, the set $(\overline{C}_i\cap\overline{C}_j)$ is empty or contains exactly one point. Therefore, the components of the set $\bigcup_{i\in I}\overline{C}_i$ are sets of the form $\overline{C}_i$ or the wedges of the sets of the form $\overline{C}_i$. Let $H_1$ be a decomposition of $Y$ such that the nondegenerate elements of $H_1$ are the components of $\bigcup_{i\in I}\overline{C}_i$. Note that the set $I$ is countable since $C_i$'s are open subsets of $Y$. It is easy to see that $H_1$ is an u.s.c. decomposition of $Y$ into closed sets of trivial shape. In a proof of this statement, one relies heavily on the geometry concerning the components $C_i$'s and their closures $\overline{C}_i$'s. Now, the decomposition space $Y/H_1$ is an ANR [2]. This leads to a contradiction since $Y/H_1$ is homeomorphic to $p(A)$. This proves that $p(A)$ cannot be a subset $Y$.

We may now assume that $A' = [p(A) \cap Y] \neq p(A)$. Let $H^n(X)$ denote the $n$th Čech cohomology group of a space $X$ with integers as the coefficient group. Since $A'$ is a proper closed subset of the suspension over the Polish circle $p(A)$, it follows that $A'$ cannot separate $E^3$, and hence, by Alexander duality [10, p. 150] the cohomology group $H^2(A')$ is zero. Let $Y/A'$ denote the space obtained from $Y$ by identifying $A'$ to a point. It follows from $\dim(Y-A') \leqslant 1$ that $\dim(Y/A') \leqslant 1$ [10, p. 32]. By Theorem VIII 4 of [10, p. 152], it follows that $H^2(Y/A')$ is zero. It can be easily shown by the continuity of the Čech cohomology theory that $H^2(Y, A')$ is isomorphic to $H^2(Y/A')$. By the long exact sequence of the pair $(Y, A')$, the sequence $0 = H^2(Y, A') \to H^2(Y) \to H^2(A') = 0$ is exact. This is a contradiction.

This finishes our proof that the decomposition space $X/G$ is $R$-stable.

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# An expansion of an $\aleph_0$-categorical model

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**Abstract.** We show the existence of an $\aleph_0$-categorical model $M$ having an expansion $M^*$ with the "same" elementary submodels, but which is not $\aleph_0$-categorical. In addition, (1) $M$ contains $\aleph_0$ disjoint sets intersected by every elementary submodel; (2) for every countable $N^* \equiv M^*$, $\prec(N^*) \cong \prec(M^*)$, where $\prec(N^*)$ is the set of elementary submodels of $N^*$ partially ordered by $\prec$.

**Introduction.** An expansion $M^*$ of $M$ is said to be *elementary* if the universe of every elementary submodel of $M$ is the universe of an elementary submodel of $M^*$. This concept was introduced in [2] where it was shown that if $M$, $N$ are countable, not isomorphic, $N$ is not saturated, and $M$ is prime, then there is an elementary expansion $M^*$ of $M$ such that there is no expansion $N^*$ of $N$ with $N^* \equiv M^*$. Of course, the interesting case is when $M \equiv N$. If in addition $M$ is $\aleph_0$-categorical then $N \cong M$ and the above theorem does not apply. Nevertheless the properties of elementary expansions of $\aleph_0$-categorical models are worthy of investigation. Here we show (Theorem 1) the existence of an $\aleph_0$-categorical model $M$ having a non-$\aleph_0$-categorical elementary expansion $M^* = (M, P_i)_{i<\omega}$, where the $P_i$ are unary relation symbols interpreted as disjoint sets. Thus $M$ contains $\aleph_0$ disjoint sets which are intersected by every elementary submodel. In a sense, this is as close as an $\aleph_0$-categorical model can get to being a minimal model (a model with no proper elementary submodels).

In addition there is a theory $T_1^* \subseteq \mathrm{Th}(M^*)$ such that every model of $T_1^*$ can be realized as an elementary expansion of a model of $\mathrm{Th}(M)$. In particular, then, for all countable models elementary equivalent to $M^*$, the partially ordered sets of their elementary submodels are isomorphic.

**Notation and definitions.** We deal here with models $M$, $N$, etc. in languages $L(M)$, $L(N)$, etc. Most of the terminology and notation is standard. Anything not defined below can be found in Chang–Keisler [1]. We use the term language to mean a set of relation symbols. If $L$ is a language, $\bar{L}$ is the set of first order formulas built up from $L$ and the finitary connectives and quantifiers. An $L$-$m$-diagram is any subset of $\{R(x_{i_0}, \ldots, x_{i_{k-1}})\colon R \in L, i_j < m\}$. If $\bar{a} = \langle a_0, \ldots, a_{m-1}\rangle$ is a sequence of elements in an $L$-structure, $M$, written $\bar{a} \in M$ or $\bar{a} \subseteq M$, then the $L$-diagram of $\bar{a}$ is