The superextension of the closed unit interval is homeomorphic to the Hilbert cube

by

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Abstract. Let X be a compact metric space and let λX be the superextension of X. For the closed unit interval I we show that λI is homeomorphic to the Hilbert cube, thus answering a question of J. de Groot.

1. Introduction. One of the unsolved problems in the theory of superextensions is to determine the superextension of the closed unit interval λI. De Groot [13], conjectured that λI is homeomorphic to the Hilbert cube. This paper contains a proof of this conjecture. Infinite dimensional techniques are very important in this work. We will represent λI as an inverse limit of a sequence of Hilbert cubes, such that the bonding maps are nearhomeomorphisms. An approximation theorem for inverse limits of Brown ([16]) then is applicable, which gives us the desired result. The class of Hilbert cube factors, a subclass of the compact metric absolute retracts, has been investigated by several authors during the last years ([1], [23], [24], [25], [26]). Several of the common types of absolute retracts, have been shown to be Hilbert cube factors, e.g., contractible polyhedra [23], dendra [23], contractible cell complexes [24], and hyperspaces ([11], [19], [25]). These results and Chapman's results concerning Q-manifolds ([7], [8], [9]) will be of great importance for us. This paper is organized as follows: the second section recalls the definitions of supercompactness and superextensions and contains some theorems which have interest in their own rights and which will be, in the fourth section, the tools in proving our main result. The third section contains a proof that the Hilbert cube is a superextension of I, relative a specially chosen nice subbase. This result we need as a first step in our inverse limit construction.

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2. Superextensions. In [13], De Groot defined a space X to be *supercompact* provided that it possesses an open subbase $\mathcal{U}$ such that each covering of X by elements of $\mathcal{U}$ contains a subcover of two elements of $\mathcal{U}$. Such a subbase is called *binary*. Clearly, according to the lemma of Alexander, every supercompact space is compact. The class of supercompact spaces contains the compact metric spaces (Strok and Szymański [20]), compact orderable spaces and compact tree-like spaces (Brouwer and Schrijver [5] or Van Mill [16]). De Groot conjectured that every compact Hausdorff space is supercompact. This was answered in the negative by Bell [4], who showed that *if X is not pseudocompact then βX is not supercompact*. Moreover there exists a compact separable first countable Hausdorff space which is not supercompact (van Douwen and van Mill [12]).

Let X be a topological space and let $\mathscr{S}$ be a subbase for the closed subsets of X. $\mathscr{S}$ is defined to be

(i) a *T_1-subbase* if for each $x_0 \in X$ and $S \in \mathscr{S}$ with $x_0 \notin S$ there exists a $T \in \mathscr{S}$ with $x_0 \in T$ and $T \cap S = \emptyset$.

(ii) a *normal subbase* if for each $S_0, T_0 \in \mathscr{S}$ with $S_0 \cap T_0 = \emptyset$ there exist $S_1, T_1 \in \mathscr{S}$ with $S_1 \cap T_0 = \emptyset = T_1 \cap S_0$ and $S_1 \cup T_1 = X$.

(iii) a *supernormal subbase* if $\mathscr{S}$ is normal while moreover for all $S \in \mathscr{S}$ and $G = \bar{G} \subset X$ with $S \cap G = \emptyset$ there exists an $S_0 \in \mathscr{S}$ such that $G \subset S_0$ and $S \cap S_0 = \emptyset$.

$\mathscr{S}$ is called *binary* if the corresponding open subbase $\mathscr{U} = \{X \setminus S \mid S \in \mathscr{S}\}$ is binary. A subsystem $\mathscr{M} \subset \mathscr{S}$ is called a *linked system* (ls), if every two of its members meet. A linked system $\mathscr{M} \subset \mathscr{S}$ is called *fixed* if $\bigcap \mathscr{M} \neq \emptyset$ and is called *free* if $\bigcap \mathscr{M} = \emptyset$. If $\mathscr{S}$ is binary then any linked system $\mathscr{M} \subset \mathscr{S}$ is fixed (and conversely). A *maximal linked system* or mls (in $\mathscr{S}$) is a linked system not properly contained in any other linked system. By Zorn's lemma every linked system is contained in at least one maximal linked system. The proofs of the following propositions and the proof of Theorem 1 can be found in [21].

PROPOSITION 1. *Let $\mathscr{M}_0, \mathscr{M}_1$ be mls's in $\mathscr{S}$. Then*

(a) $\emptyset \notin \mathscr{M}_0$.

(b) *If $S \in \mathscr{M}_0$, $T \in \mathscr{S}$ and $S \subset T$ then $T \in \mathscr{M}_0$.*

(c) *If $S \in \mathscr{S} \setminus \mathscr{M}_0$ then $\exists T \in \mathscr{M}_0$: $S \cap T = \emptyset$.*

(d) *$\mathscr{M}_0 \neq \mathscr{M}_1$ iff $\exists S \in \mathscr{M}_0$, $\exists T \in \mathscr{M}_1$: $S \cap T = \emptyset$.*

(e) *If $S, T \in \mathscr{S}$ and $S \cup T = X$ then $S \in \mathscr{M}_0$ or $T \in \mathscr{M}_0$.*

Notation. $\lambda_{\mathscr{S}}(X) = \{\mathscr{M} \subset \mathscr{S} \mid \mathscr{M}$ is an mls in $\mathscr{S}\}$.

If $\mathscr{S}$ is a T_1-subbase then for each $x \in X$ the linked system $\mathscr{M}_x = \{S \in \mathscr{S} \mid x \in S\}$ also is maximal linked; the map

$$\underline{i}\colon X \to \lambda_{\mathscr{S}}(X)$$

defined by $\underline{i}(x) = \mathscr{M}_x$ is 1-1. If A is a subset of X then we define

$$A^+ = \{\mathscr{M} \mid \mathscr{M} \in \lambda_{\mathscr{S}}(X) \text{ and } \exists S \in \mathscr{M}\colon S \subset A\}.$$

PROPOSITION 2. (i) *If $A \subset B \subset X$ then $A^+ \subset B^+$.*

(ii) *If $A, B \subset X$ and $A \cap B = \emptyset$ then $A^+ \cap B^+ = \emptyset$.*

(iii) *If $S, T \in \mathscr{S}$ then $S \cap T = \emptyset$ iff $S^+ \cap T^+ = \emptyset$.*

(iv) *If $S, T \in \mathscr{S}$ then $S \cup T = X$ iff $S^+ \cup T^+ = \lambda_{\mathscr{S}}(X)$.*

(v) *If $S \in \mathscr{S}$ then $S^+ \cup (X \setminus S)^+ = \lambda_{\mathscr{S}}(X)$.*

As a closed subbase for a topology on $\lambda_{\mathscr{S}}(X)$ we take

$$\mathscr{S}^+ = \{S^+ \mid S \in \mathscr{S}\}.$$

With this topology $\lambda_{\mathscr{S}}(X)$ is called the *superextension of X relative the subbase $\mathscr{S}$*. In case $\mathscr{S}$ consists of all the closed subsets of X, $\lambda_{\mathscr{S}}(X)$ is denoted by λX and is called the *superextension of X*.

THEOREM 1. (i) *X is embeddable in $\lambda_{\mathscr{S}}(X)$ if $\mathscr{S}$ is a T_1-subbase.*

(ii) *$\lambda_{\mathscr{S}}(X)$ is T_1.*

(iii) *$\lambda_{\mathscr{S}}(X)$ is T_2 if $\mathscr{S}$ is a normal subbase.*

(iv) *$\lambda_{\mathscr{S}}(X)$ is supercompact. A binary subbase is $\{(X \setminus S)^+ \mid S \in \mathscr{S}\}$.*

(v) *$\forall S \in \mathscr{S}$: $\underline{i}^{-1}[S^+] = S$.*

In case $\underline{i}$ is a topological embedding, we often identify X and $\underline{i}[X]$. An *interval structure* ([5]) on a topological space X is a function $I\colon X \times X \to \mathscr{P}(X)$ such that

(i) $x, y \in I(x, y)$ $(x, y \in X)$,

(ii) $I(x, y) = I(y, x)$ $(x, y \in X)$,

(iii) if $u, v \in I(x, y)$ then $I(u, v) \subset I(x, y)$ $(u, v, x, y \in X)$,

(iv) $I(x, y) \cap I(x, z) \cap I(y, z) \neq \emptyset$ $(x, y, z \in X)$.

A subset $A \subset X$ is called *I-closed* if for all $x, y \in A$ it follows that $I(x, y) \subset A$. If X is a supercompact space with binary closed subbase $\mathscr{S}$, then $I_{\mathscr{S}}\colon X \times X \to \mathscr{P}(X)$ defined by

$$I_{\mathscr{S}}(x, y) = \bigcap \{S \in \mathscr{S} \mid x, y \in S\}$$

defines an interval structure on X. Furthermore it is clear that each $S \in \mathscr{S}$ is $I_{\mathscr{S}}$-closed. The converse is also true. *If a compact space X possesses an interval structure I and a closed subbase $\mathscr{S}$ consisting of I-closed sets, then X is supercompact* ([5], Theorem 1.1). In particular, $\mathscr{S}$ is a binary closed subbase for X.

THEOREM 2. *If $\mathscr{S}$ is a binary normal closed subbase for X, then*

$$I_{\mathscr{S}}(x, y) \cap I_{\mathscr{S}}(x, z) \cap I_{\mathscr{S}}(y, z)$$

is a singleton for each $x, y, z \in X$.

Proof. Choose $x, y, z \in X$ and let $p, q \in I_{\mathscr{S}}(x, y) \cap I_{\mathscr{S}}(x, z) \cap I_{\mathscr{S}}(y, z)$, with $p \neq q$. As $\mathscr{S}$ is a binary normal closed subbase, it is a normal T_1-closed subbase ([16], Lemma 1) and therefore there exist $S_0, S_1 \in \mathscr{S}$ such that $p \in S_0 \setminus S_1$ and $q \in S_1 \setminus S_0$ and $S_0 \cup S_1 = X$. We have to consider two cases:

(i) Suppose first that $x \in S_0$. We again distinguish two subcases:

(a) $y \in S_0$. Then $I_{\mathscr{S}}(x, y) \subset S_0$ and consequently $q \in S_0$, which is a contradiction.

(b) $y \in S_1$. If $z \in S_0$, then we can derive the same contradiction as in (a). If $z \in S_1$, then $I_{\mathscr{S}}(y, z) \subset S_1$ and consequently $p \in S_1$ which is a contradiction.

(ii) Suppose that $x \in S_1$. This can be treated in the same way as case (i). ■

LEMMA 1. *If $\mathcal{S}$ is a binary normal closed subbase for X, then the map $f\colon X\times X\times X\to X$ defined by*

$$\{f(x,y,z)\} = I_{\mathcal{S}}(x,y)\cap I_{\mathcal{S}}(x,z)\cap I_{\mathcal{S}}(y,z)$$

is a continuous surjection.

Proof. As a first step we will prove that $(x,y,z)\notin f^{-1}[S]$ iff $I_{\mathcal{S}}(x,y)\cap S=\emptyset$ or $I_{\mathcal{S}}(x,z)\cap S=\emptyset$ or $I_{\mathcal{S}}(y,z)\cap S=\emptyset$ $(S\in\mathcal{S})$.

"$\Rightarrow$" Suppose that $I_{\mathcal{S}}(x,y)\cap S\neq\emptyset$ and $I_{\mathcal{S}}(x,z)\cap S\neq\emptyset$ and $I_{\mathcal{S}}(y,z)\cap S\neq\emptyset$. Then $\mathcal{A}=\{T\in\mathcal{S}|\ x,y\in T$ or $x,z\in T$ or $y,z\in T\}\cup\{S\}$ is a linked system and consequently, since $\mathcal{S}$ is binary, $\bigcap\mathcal{A}\neq\emptyset$. As $\bigcap\mathcal{A}=I_{\mathcal{S}}(x,y)\cap I_{\mathcal{S}}(x,z)\cap\cap I_{\mathcal{S}}(y,z)\cap S$, it would follow that $f(x,y,z)\in S$, which is a contradiction

"$\Rightarrow$" If $I_{\mathcal{S}}(x,y)\cap S=\emptyset$, then $I_{\mathcal{S}}(x,y)\cap I_{\mathcal{S}}(x,z)\cap I_{\mathcal{S}}(y,z)\cap S=\emptyset$ and so $f(x,y,z)\notin S$.

From Theorem 1 it follows that f is well-defined. To prove that f is continuous, choose $S\in\mathcal{S}$ and let $(x,y,z)\notin f^{-1}[S]$. Without loss of generality we may assume that $I_{\mathcal{S}}(x,y)\cap S=\emptyset$. Using the fact that $\mathcal{S}$ is binary and that $I_{\mathcal{S}}(x,y)$ is an intersection of subbase elements it follows that there exists an $S_0\in\mathcal{S}$ such that $I_{\mathcal{S}}(x,y)\subset S_0$ and $S_0\cap S=\emptyset$. The normality of $\mathcal{S}$ implies the existence of $S_0', S'\in\mathcal{S}$, such that $S_0\cap S'=\emptyset$ and $S\cap S_0'=\emptyset$ and $S_0'\cup S'=X$. Then $x,y\in S_0\subset X\setminus S'\subset S_0'$. Define $U=X\setminus S'$. Let Π_i $(i=0,1,2)$ denote the projection maps of the product $X\times X\times X$. Then $(x,y,z)\in\Pi_0^{-1}[U]\cap\Pi_1^{-1}[U]$. Furthermore $\Pi_0^{-1}[U]\cap\Pi_1^{-1}[U]\cap\cap f^{-1}[S]=\emptyset$, for suppose to the contrary that there exists a point $(x_0,y_0,z_0)\in\Pi_0^{-1}[U]\cap\Pi_1^{-1}[U]\cap f^{-1}[S]$. Then $x_0\in U$ and $y_0\in U$ and consequently $I_{\mathcal{S}}(x_0,y_0)\subset S_0'$. Hence it follows that $I_{\mathcal{S}}(x_0,y_0)\cap S=\emptyset$ and consequently $(x_0,y_0,z_0)\notin f^{-1}[S]$, which is a contradiction. Therefore f is continuous. To prove that f is onto, choose $x\in X$. Then

$$\{f(x,x,x)\} = I_{\mathcal{S}}(x,x)\cap I_{\mathcal{S}}(x,x)\cap I_{\mathcal{S}}(x,x) = \{x\}\,. \ \blacksquare$$

LEMMA 2. *$g=f|\ \{x\}\times\{y\}\times X$ is a retraction of X onto $I_{\mathcal{S}}(x,y)$.*

Proof. g is continuous, and furthermore it is clear that

$$g[\{x\}\times\{y\}\times X]\subset I_{\mathcal{S}}(x,y)\,.$$

Choose $z\in I_{\mathcal{S}}(x,y)$. Then

$$\{g(z)\} = \{f(x,y,z)\} = I_{\mathcal{S}}(x,y)\cap I_{\mathcal{S}}(x,z)\cap I_{\mathcal{S}}(y,z) = \{z\}\,,$$

since $z\in I_{\mathcal{S}}(x,y)$ (Theorem 2). This proves that g is a retraction. $\blacksquare$

COROLLARY 1. *If $\mathcal{S}$ is a binary normal closed subbase for the topological space X, then the following properties are equivalent:*

(i) *X is connected.*

(ii) *$\forall x,y\in X$: $I_{\mathcal{S}}(x,y)$ is connected.*

(iii) *Each intersection of elements of $\mathcal{S}$ either is void or is connected.*

Proof. (i)$\Rightarrow$(ii) This is a consequence of Lemma 2.

(ii)$\Rightarrow$(i) Suppose that X is not connected. Then there exist open non empty sets U and V in X such that $U\cup V=X$ and $U\cap V=\emptyset$. Choose $x\in U$ and $y\in V$. Then $I_{\mathcal{S}}(x,y)$ is not connected, which is a contradiction.

(iii)$\Rightarrow$(ii) Obvious.

(ii)$\Rightarrow$(iii) Let $\mathcal{A}$ be a subsystem of $\mathcal{S}$ such that $\bigcap\mathcal{A}\neq\emptyset$ and $\bigcap\mathcal{A}$ is not connected. Choose $x,y\in\bigcap\mathcal{A}$ and open sets U and V such that $x\in U$, $y\in V$ and $\bigcap\mathcal{A}\subset U\cup V$ and $U\cap V\cap\bigcap\mathcal{A}=\emptyset$. Then for each $A\in\mathcal{A}$ the interval $I_{\mathcal{S}}(x,y)$ is contained in A and consequently $I_{\mathcal{S}}(x,y)\subset\bigcap\mathcal{A}$. This is a contradiction. $\blacksquare$

A *mean* m is a continuous map $m\colon X\times X\to X$ such that

(i) $m(x,x)=x$ for all $x\in X$.

(ii) $m(x,y)=m(y,x)$, for all $x,y\in X$.

THEOREM 3. *Any topological space which possesses a binary normal closed subbase, also has a mean.*

Proof. Let $\mathcal{S}$ be a binary normal closed subbase for the topological space X. Let f be defined as in Lemma 1. Choose $p\in X$ and define $m\colon X\times X\to X$ by $m=f|\ \{p\}\times X\times X$. Then m is a continuous map of $X\times X$ onto X. Furthermore

$$\{m(x,x)\} = I_{\mathcal{S}}(x,x)\cap I_{\mathcal{S}}(x,p)\cap I_{\mathcal{S}}(p,x) = \{x\}$$

and

$$\begin{aligned}\{m(x,y)\} &= I_{\mathcal{S}}(x,y)\cap I_{\mathcal{S}}(x,p)\cap I_{\mathcal{S}}(y,p) = I_{\mathcal{S}}(y,x)\cap I_{\mathcal{S}}(y,p)\cap I_{\mathcal{S}}(x,p)\\ &= \{m(y,x)\}\,. \ \blacksquare\end{aligned}$$

Of course there are many spaces which possess a binary normal subbase. Examples are products of compact orderable spaces, products of compact tree-like spaces ([16]) and superextensions of normal spaces. Theorem 3 gives us many easy examples of spaces which are supercompact, but which do not possess a binary *normal* closed subbase. For example the supercompact space

$$Y=\{(0,y)|\ -1\leqslant y\leqslant 1\}\cup\{(x,\sin(1/x))|\ 0<x\leqslant 1\}$$

possesses no binary normal closed subbase, since this space has no mean ([3]). That Y is supercompact is not trivial. To prove this, define for each $n\in\{0,1,2,\ldots\}$

$$x_n=\frac{2}{(2n+1)\pi}\,.$$

Notice that $\sin(1/x_n)=1$ if n is even and that $\sin(1/x_n)=-1$ if n is odd. Let r be a retraction of Y onto $\{0\}\times[-1,1]$ defined by

$$r(x,y)=\begin{cases}(x,y) & \text{if}\quad (x,y)\in\{0\}\times[-1,1]\,,\\ (0,y) & \text{if}\quad (x,y)\notin\{0\}\times[-1,1]\,.\end{cases}$$

It can be shown that

$$\{r^{-1}[\{0\}\times[x,1]]\setminus C|\ -1\leqslant x\leqslant 1 \text{ and } C \text{ is a component of } r^{-1}[\{0\}\times[x,1]]\}\cup\{r^{-1}[\{0\}\times[-1,x]]\setminus C|\ -1\leqslant x\leqslant 1 \text{ and } C \text{ is a component of } r^{-1}[\{0\}\times[-1,x]]\}\cup\{(x,\sin(1/x))|\ x\in[x_n,p], \text{ where } x_n\leqslant p\leqslant x_{n-1};\ n\in\{0,1,2,\ldots\}\}\cup\{(x,\sin(1/x))|\ x\in[p,x_n], \text{ where } x_{n+1}\leqslant p\leqslant x_n;\ n\in\{0,1,2,\ldots\}\}$$

is a binary closed subbase for Y. Moreover it is obvious that this subbase is not normal. That Y possesses no binary normal closed subbase can also be derived from a rather deep theorem of Verbeek [21]; *if a connected space possesses a binary normal closed subbase then this space must be locally connected* ([21], III. 4.1 Corollary). Clearly Y is not locally connected and therefore the result also follows. However, this argument cannot be used in the class of connected and locally connected spaces. Then our theorem applies. It is well-known for example that the n-spheres S_n are supercompact, but do not have a mean ([22]) and consequently they cannot possess a binary normal closed subbase. It is unknown whether there exists a contractible locally connected example. We will prove that in the class of metrizable spaces, each continuum which possesses a binary normal closed subbase must be an AR (absolute retract), a theorem which has wide applications.

For a compact metric space X, let 2^X be the space of all nonempty closed subsets of X with the Vietoris topology, i.e., the topology induced by the Hausdorff metric. This space is called the *hyperspace* of X. A basis for the open sets consists of all sets

$$\langle O_1, O_2, \ldots, O_n\rangle = \{G\in 2^X|\ G\subset \bigcup_{i=1}^{n} O_i \text{ and } G\cap O_i\neq\emptyset \text{ for } i=1,2,\ldots,n\}$$

where $O_1, O_2, \ldots, O_n$ denotes an arbitrary finite sequence of open subsets of X. For many strong results concerning hyperspaces see [11], [19] and [25].

LEMMA 3. *Let $p\in X$ and let G be a closed subset of X. Then $h(G) := \{A\subset X|\ A$ is closed and $G\subset A$ or $p\in A$ and $A\cap G\neq\emptyset\}$ is an mls.*

Proof. Verbeek [21], I. 1.3(d). ■

THEOREM 4. *Let X be a connected metrizable space, which possesses a binary normal closed subbase. Then X is an* AR.

Proof. Fix a point $p\in X$ and define a map $h\colon 2^X\to\lambda X$ by $h\colon G\to h(G)$ where $h(G)$ is defined as in Lemma 3. We will prove that h is continuous. Let O be an open set in X and let $G\in h^{-1}[O^+]$.

Case 1. $p\in O$. Define $U=\langle O, X\rangle = \{H\in 2^X|\ H\cap O\neq\emptyset\}$. As $G\in h^{-1}[O^+]$ it follows that $h(G)\in O^+$ and consequently $G\cap O\neq\emptyset$, since $G\in h(G)$. Therefore $G\in U$.

Choose $H\in U$. Then $H\cap O\neq\emptyset$ and so there exists a $q\in H\cap O$. Since $p\in O$, it follows that $\{p,q\}\subset O$. However it is clear that $\{p,q\}\in h(H)$, and consequently $H\in h^{-1}[O^+]$. Therefore U is an open neighborhood of G which is contained in $h^{-1}[O^+]$.

Case 2. $p\notin O$. Define $U=\langle O\rangle = \{H\in 2^X|\ H\subset O\}$ Choose $H\in U$. Then $H\subset O$ and consequently $H\in h^{-1}[O^+]$. Conversely if $H\in h^{-1}[O^+]$, then $H\subset O$, since $p\notin O$. The combination of these two results yields $h^{-1}[O^+]=U$, and therefore $h^{-1}[O^+]$ is an open neighborhood of G.

Now, since X possesses a binary normal closed subbase $\mathcal{S}$ there is a retraction $r\colon\lambda X\to X$ defined by $\{r(\mathcal{M})\}=\bigcap\{S\in\mathcal{S}|\ S\in\mathcal{M}\}$ (Verbeek [21], II. 4.5) and consequently the map $\xi\colon 2^X\to X$ defined by $\xi = r\circ h$ is a continuous surjection. We will show that ξ also is a retraction. Choose $x\in X$. Then

$$\begin{aligned}\xi(\{x\}) &= rh(\{x\}) = r\{A\subset X|\ A \text{ is closed and } x\in A \text{ or } p\in A \text{ and } x\in A\}\\ &= r\{A\subset X|\ A \text{ is closed and } x\in A\}\\ &= r(x)\\ &= x,\end{aligned}$$

since r is a retraction. Now the connectedness of X implies that X is a Peano continuum (Verbeek [21], III. 4.1 Corollary) and consequently 2^X is homeomorphic to the Hilbert cube Q (Curtis and Schori [11]). It now follows that X is an AR. ■

The above theorem may seem to be a deep theorem, since we use the Curtis and Schori result: $2^X\cong Q$ iff X is a Peano continuum. However we only need that 2^X is an AR iff X is a Peano continuum, since a retract of an AR is again an AR, and this was proved by Wojdyslawski [27] in 1939. The superextension λN of any normal space N possesses a binary normal closed subbase and therefore we immediately obtain the following corollary:

COROLLARY 2. *The superextension λM is a strongly infinite dimensional* AR *iff M is a non degenerate metrizable continuum.*

Proof. This immediately follows from Verbeek's theorem ([21], IV. 2.6): λM is a strongly infinite dimensional Peano continuum iff M is a non degenerate metrizable continuum. ■

This answers a question of Verbeek raised in [21].

QUESTION 1. Is the converse of Theorem 4 also true (1)?

A counter example to this question cannot be obtained within the class of one dimensional AR's, since this class consists of dendrites, which possess binary normal subbases ([16]).

A surprising consequence of Theorem 4, for which no direct proof is known, is that the superextension of any metrizable continuum is contractible.

We will now derive some results concerning supernormal subbases.

LEMMA 4. *Let $\mathcal{S}$ be a closed supernormal T_1-subbase for X and let $\mathcal{U}$ be a closed T_1-subbase such that $\mathcal{S}\subset\mathcal{U}$. Then $\forall\mathcal{M}\in\lambda_{\mathcal{U}}(X)$: $\{S\in\mathcal{S}|\ S\in\mathcal{M}\}$ is an mls in $\mathcal{S}$.*

(1) This was answered in the negative recently by A. Szymański.

Proof. Let $\mathscr{M} \in \lambda_{\mathscr{U}}(X)$ and define $P^{\mathscr{M}} = \{S \in \mathscr{S} | \ S \in \mathscr{M}\}$. From the normality of $\mathscr{S}$ it follows that $P^{\mathscr{M}} \neq \emptyset$, and therefore $P^{\mathscr{M}}$ is a linked system. Suppose that $P^{\mathscr{M}}$ is not maximal linked. Then there exists an $S_0 \in \mathscr{S}$ such that $P^{\mathscr{M}} \cup \{S_0\}$ is linked and $S_0 \notin P^{\mathscr{M}}$. Then $S_0 \notin \mathscr{M}$ and so there exists an $M \in \mathscr{M}$ with $M \cap S_0 = \emptyset$. Since $\mathscr{S}$ is supernormal there is an $S^* \in \mathscr{S}$ with $M \subset S^*$ and $S^* \cap S_0 = \emptyset$. This is a contradiction, since $M \in \mathscr{M}$ implies $S^* \in \mathscr{M}$ and therefore $S^* \in P^{\mathscr{M}}$. ∎

THEOREM 5 (G. A. Jensen; cf. [14]). *Let $\mathscr{S}$ be a T_1-subbase for X, let $\mathscr{T}$ be a normal T_1-subbase for Y and let f be a continuous map $f\colon X \to Y$ such that $\forall T \in \mathscr{T}\colon f^{-1}[T] \in \mathscr{S}$. Then f can be extended to a continuous map $\bar{f}\colon \lambda_{\mathscr{S}}(X) \to \lambda_{\mathscr{T}}(Y)$. Moreover, if f is onto then $\bar{f}$ is onto. If f is 1-1 and $\forall S \in \mathscr{S}\colon f[S] \in \mathscr{T}$ then $\bar{f}$ is an embedding.*

The construction of the map $\bar{f}$ is very simple. If $\mathscr{M} \in \lambda_{\mathscr{S}}(X)$, then

$$P_{\mathscr{M}} = \{T \in \mathscr{T} | \ f^{-1}[T] \in \mathscr{M}\}$$

is contained in precisely one mls in $\mathscr{T}$. This mls is denoted by $\underline{P_{\mathscr{M}}}$ and the map $\bar{f}$ is defined by $\bar{f}(\mathscr{M}) = \underline{P_{\mathscr{M}}}$. These mappings will be called *Jensen mappings*.

For another solution of the extension problem see [17].

COROLLARY 3. *Let $\mathscr{S}$ be a closed supernormal T_1-subbase for X and let $\mathscr{U}$ be a closed T_1-subbase such that $\mathscr{S} \subset \mathscr{U}$. Then $\lambda_{\mathscr{S}}(X)$ is a Hausdorff quotient of $\lambda_{\mathscr{U}}(X)$ under the map f defined by $f(\mathscr{M}) = \{S \in \mathscr{S} | \ S \in \mathscr{M}\}$. Moreover: f is the identity on X.*

The definition of subbases which are supernormal seems to be pathological, since in compactification theory a closed subbase almost always fails to have this property. In our construction for λI however, subbases which are supernormal appear in a natural way. Therefore it is worth the trouble to study some elementary properties of these subbases first.

PROPOSITION 3. *Let $\{\mathscr{S}_\alpha | \ \alpha \in I\}$ be a collection of closed T_1-subbases for the topological space X, which all are supernormal. Then $\bigcup_{\alpha \in I} \mathscr{S}_\alpha$ is a closed T_1-subbase which is supernormal. Moreover $\lambda_{\bigcup_{\alpha \in I} \mathscr{S}_\alpha}(X)$ can be embedded in $\prod_{\alpha \in I} \lambda_{\mathscr{S}_\alpha}(X)$.*

Proof. That $\bigcup_{\alpha \in I} \mathscr{S}_\alpha$ is a closed T_1-subbase which is supernormal is obvious. To prove the embedding property, for all $\alpha \in I$ let

$$f_\alpha\colon \lambda_{\bigcup_{\alpha \in I} \mathscr{S}_\alpha}(X) \to \lambda_{\mathscr{S}_\alpha}(X)$$

be the Jensen mapping. Note that these mappings exist. Let

$$e\colon \lambda_{\bigcup_{\alpha \in I} \mathscr{S}_\alpha}(X) \to \prod_{\alpha \in I} \lambda_{\mathscr{S}_\alpha}(X)$$

be the evaluation map defined by $(e(x))_\alpha = f_\alpha(x)$. We will show that e is an embedding and for this it suffices to show that e is one to one. Choose $\mathscr{M}_0, \mathscr{M}_1 \in \lambda_{\bigcup_{\alpha \in I} \mathscr{S}_\alpha}(X)$ such that $\mathscr{M}_0 \neq \mathscr{M}_1$. Then there exist $M_i \in \bigcup_{\alpha \in I} \mathscr{S}_\alpha$ $(i = 1, 0)$ such that $M_i \in \mathscr{M}_i$ $(i = 0, 1)$ and $M_0 \cap M_1 = \emptyset$. Choose $\alpha_0 \in I$ such that $M_0 \in \mathscr{S}_{\alpha_0}$. Then, since $\mathscr{S}_{\alpha_0}$ is supernormal, we may assume without loss of generality that also $M_1 \in \mathscr{S}_{\alpha_0}$. However, Corollary 3 shows that $M_i \in f_{\alpha_0}(\mathscr{M}_i)$ $(i = 0, 1)$ and consequently $f_{\alpha_0}(\mathscr{M}_0) \neq f_{\alpha_0}(\mathscr{M}_1)$. ∎

If the conditions of Proposition 3 are satisfied, then we will always identify $\lambda_{\bigcup_{\alpha \in I} \mathscr{S}_\alpha}(X)$ and $e[\lambda_{\bigcup_{\alpha \in I} \mathscr{S}_\alpha}(X)]$. It then is usefull to characterize those points of $\prod_{\alpha \in I} \lambda_{\mathscr{S}_\alpha}(X)$ which belong to $\lambda_{\bigcup_{\alpha \in I} \mathscr{S}_\alpha}(X)$. Note that a point $x = (x_\alpha)_{\alpha \in I}$ of $\prod_{\alpha \in I} \lambda_{\mathscr{S}_\alpha}(X)$ is a point of which the coordinates are maximal linked systems so that we can speak of $\bigcup_{\alpha \in I} x_\alpha$.

LEMMA 5. *Let $\{\mathscr{S}_\alpha | \ \alpha \in I\}$ be a collection of closed T_1-subbases for the topological space X, which all are supernormal. Then $x \in \prod_{\alpha \in I} \lambda_{\mathscr{S}_\alpha}(X)$ is an element of $\lambda_{\bigcup_{\alpha \in I} \mathscr{S}_\alpha}(X)$ if and only if $\bigcup_{\alpha \in I} x_\alpha$ is linked.*

Proof. If $x \in \lambda_{\bigcup_{\alpha \in I} \mathscr{S}_\alpha}(X)$, then $x = \bigcup_{\alpha \in I} x_\alpha$, so that $\bigcup_{\alpha \in I} x_\alpha$ is linked. If $\bigcup_{\alpha \in I} x_\alpha$ is linked, then it also is maximal linked (in $\bigcup_{\alpha \in I} \mathscr{S}_\alpha$) for suppose to the contrary that there exists an $S \in \mathscr{S}_\alpha$ such that $\bigcup_{\alpha \in I} x_\alpha \cup \{S\}$ is linked, but $S \notin \bigcup_{\alpha \in I} x_\alpha$. Choose $\alpha_0 \in I$ such that $S \in \mathscr{S}_{\alpha_0}$. It then follows that $x_{\alpha_0} \cup \{S\}$ is linked and consequently, since x_{α_0} is a maximal linked system, $S \in x_{\alpha_0} \subset \bigcup_{\alpha \in I} x_\alpha$. This is a contradiction. Hence $\bigcup_{\alpha \in I} x_\alpha \in \lambda_{\bigcup_{\alpha \in I} \mathscr{S}_\alpha}(X)$ and now it is not hard to see that $e[\bigcup_{\alpha \in I} x_\alpha] = x$. ∎

The importance of Proposition 3 and Lemma 5 is that one can study the behaviour of a superextension relative the union of certain subbases, in a product of superextensions. We will demonstrate this by an example. Let I denote the real number interval $[0, 1]$ and let I be embedded in I^2 as indicated in Figure 1. Define

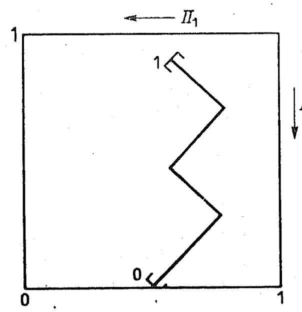

Fig. 1

$$\mathscr{T} = \{A \subset I^2 | \ A = \Pi_i^{-1}[0, x] \vee A = \Pi_i^{-1}[x, 1] \ (i \in \{0, 1\}), x \in I\}.$$

Then $\mathscr{T}$ is a binary normal closed subbase for I^2. We are interested in $\lambda_{\mathscr{T}_0}(I)$ where $\mathscr{T}_0$ is defined by

$$\mathscr{T}_0 = \{T \cap I \mid T \in \mathscr{T}\}.$$

(Here I denotes the embedded copy of I in I^2.)

It is easy to see that $\mathscr{T}_0$ is a subbase which is supernormal. We assert that $\lambda_{\mathscr{T}_0}(I)$ is homeomorphic to the space X indicated in Figure 2. To prove this define an interval structure I_X on X by

$$I_X(x, y) = \bigcap \{T \in \mathscr{T} \mid x, y \in T\} \cap X.$$

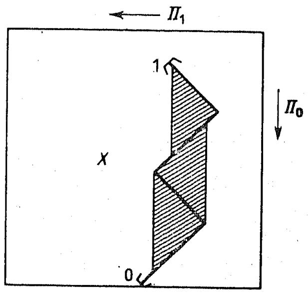

Fig. 2

The verification that I_X indeed is an interval structure is routine and follows immediately from Figure 2, since for all $x, y, z \in X$ we have

$$I_{\mathscr{T}}(x, y) \cap I_{\mathscr{T}}(x, z) \cap I_{\mathscr{T}}(y, z) \subset X.$$

Consequently, each element of $\mathscr{T} \cap X = \{T \cap X \mid T \in \mathscr{T}\}$ is I_X-closed and therefore $\mathscr{T} \cap X$ is a binary closed subbase for X. We now take recourse to the following theorem.

THEOREM 6 ([17]). *Let X be a subspace of the topological space Y. Then Y is homeomorphic to a superextension of X iff Y possesses a binary closed subbase $\mathscr{T}$ such that for all $T_0, T_1 \in \mathscr{T}$ with $T_0 \cap T_1 \neq \emptyset$ we have that $T_0 \cap T_1 \cap X \neq \emptyset$.*

In particular, under these conditions $Y \cong \lambda_{\mathscr{T} \cap X}(X)$. As an application of this theorem it follows that in the example under discussion $X \cong \lambda_{\mathscr{T}_0}(I)$, since for all $T_0, T_1 \in \mathscr{T}$ with $T_0 \cap T_1 \cap X \neq \emptyset$ we have that $T_0 \cap T_1 \cap I \neq \emptyset$, as can easily be seen. The homeomorphism between X and $\lambda_{\mathscr{T}_0}(I) = \lambda_{\mathscr{T} \cap I}(I)$ is very "direct". For instance the point p in Figure 3 represents the $\mathscr{T} \cap I$ mls $\mathscr{M}$ for which

$$\{[0, e], [e, 1], [a, b] \cup [c, d], [0, a] \cup [b, c] \cup [d, 1]\}$$

is a pre-mls (A pre-mls is a linked system which is contained in precisely one mls).

Now, if one takes two different embeddings of I in I^2 of the above type, then there arise two different superextensions $\lambda_{\mathscr{T}_0}(I)$ and $\lambda_{\mathscr{T}_1}(I)$. What about $\lambda_{\mathscr{T}_0 \cup \mathscr{T}_1}(I)$?

Proposition 3 shows that $\lambda_{\mathscr{T}_0 \cup \mathscr{T}_1}(I)$ can be embedded in $\lambda_{\mathscr{T}_0}(I) \times \lambda_{\mathscr{T}_1}(I)$, which is contained in $I^2 \times I^2$, so that in any case $\lambda_{\mathscr{T}_0 \cup \mathscr{T}_1}(I)$ is *finite dimensional*. The normality of $\mathscr{T}_0 \cup \mathscr{T}_1$ implies that $\lambda_{\mathscr{T}_0 \cup \mathscr{T}_1}(I)$ is connected (Verbeek [21], III. 4.1 Corollary)

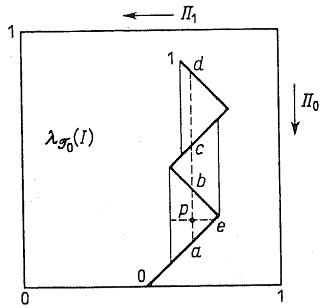

Fig. 3

and consequently $\lambda_{\mathscr{T}_0 \cup \mathscr{T}_1}(I)$ is an AR (Theorem 4). Moreover Lemma 5 shows that the points of $I^2 \times I^2$ which belong to $\lambda_{\mathscr{T}_0 \cup \mathscr{T}_1}(I)$ are completely characterized in a simple way. We will see that there is much more to say about $\lambda_{\mathscr{T}_0 \cup \mathscr{T}_1}(I)$.

3. The Hilbert cube as a superextension of *I*. The Hilbert cube Q is the topological product of infinitely many copies of I. A Hilbert cube is a topological space which is homeomorphic to Q. In [17] it was shown that Q belongs to the class of superextensions of I, however this was not a satisfying result because we could not describe the defining subbase well. We will present another subbase $\mathscr{S}$ for I such that $\lambda_{\mathscr{S}}(I) \cong Q$.

As in Section 2, let $\mathscr{T}$ be the canonical binary subbase for I^2,

$$\mathscr{T} = \{A \subset I^2 \mid A = \Pi_i^{-1}[0, x] \vee A = \Pi_i^{-1}[x, 1] \ (i \in \{0, 1\}); x \in I\}.$$

Define

$$\boldsymbol{E} = \{-2.3^k \mid k = 0, 1, 2, \ldots\}$$

and for each $n \in \boldsymbol{E}$ let I be embedded in I^2, preserving arc-length, as indicated in Figure 4.

All angles are $\frac{1}{2}\pi$ except the one at $(\frac{1}{2}, 0)$ which is $\frac{1}{4}\pi$. Define $\mathscr{A}_n$ by

$$\mathscr{A}_n = \{T \cap I \mid T \in \mathscr{T}\}.$$

Then $\lambda_{\mathscr{A}_n}(I)$ is the convex-hull of the embedded copy of I in I^2. We will show that $\lambda_{\bigcup_{i \in E} \mathscr{A}_i}(I)$ is homeomorphic to Q.

LEMMA 6. *$\lambda_{\bigcup_{i \in E} \mathscr{A}_i}(I)$ is a convex subspace of $\prod_{i \in E} \lambda_{\mathscr{A}_i}(I)$.*

Proof. Suppose that $\lambda_{\bigcup_{i \in E} \mathscr{A}_i}(I)$ is not a convex subspace of $\prod_{i \in E} \lambda_{\mathscr{A}_i}(I)$. Then there exist $x, y \in \lambda_{\bigcup_{i \in E} \mathscr{A}_i}(I)$ and $\alpha, \beta \in \boldsymbol{R}$ with $\alpha + \beta = 1$, $\alpha \geqslant 0$, $\beta \geqslant 0$ such that $\alpha x + \beta y \notin \lambda_{\bigcup_{i \in E} \mathscr{A}_i}(I)$.

Since for all $i \in E$ the point $\alpha x_i+\beta y_i \in \lambda_{\mathscr{A}_i}(I)$ it follows that $\bigcup_{i\in E}(\alpha x_i+\beta y_i)$ is not linked (Lemma 5). Note that we identify $\alpha x_i+\beta y_i$ and the mls which is represented by $\alpha x_i+\beta y_i$ $(i \in E)$. Now, there exist indices i_0 and i_1 such that $(\alpha x_{i_0}+\beta y_{i_0}) \cup \cup(\alpha x_{i_1}+\beta y_{i_1})$ is not linked. Hence there exist an $M \in \alpha x_{i_0}+\beta y_{i_0}$ and an $N \in \alpha x_{i_1}+\beta y_{i_1}$ such that $M \cap N = \emptyset$. If in the copy of I^2 corresponding to i_0 we draw a horizontal

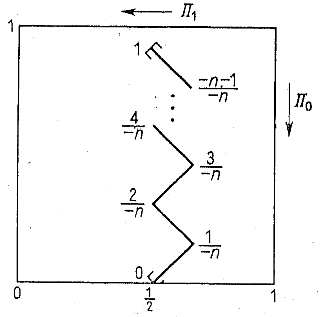

Fig. 4

line through x_{i_0} and determine its intersection p_0 with the embedded copy of I, and we do the same in the copy of I^2 corresponding to i_1, thus obtaining p_1, then p_0 and p_1 are derived from the same point of I; for it not, then it is easy to see that $x_{i_0} \cup x_{i_1}$ is not linked. In the same way, straight horizontal lines through y_{i_0} and y_{i_1} also must determine the same point on the embedded copies of I and consequently the same is true for horizontal lines through $\alpha x_{i_0}+\beta y_{i_0}$ and $\alpha x_{i_1}+\beta y_{i_1}$ because of the specially chosen embeddings of I. Hence it follows that the situation drawn in Figure 5 is the only possibility (except for interchanging i_0 and i_1).

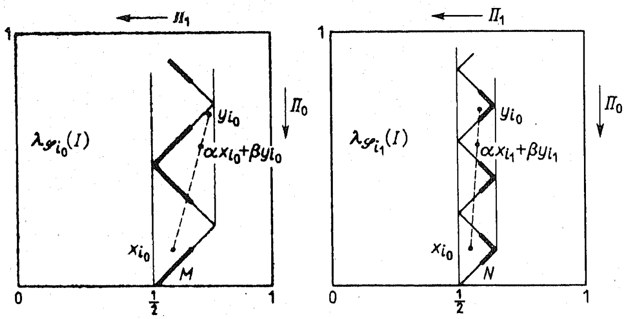

Fig. 5

Remarks. (i) M meets any set of the form $\Pi_0^{-1}[\frac{1}{2}, x] \cap I$ with $x \geqslant \Pi_0(\alpha x_{i_1}+\beta y_{i_1})$ in the point O of the embedded copy of I.

(ii) N meets any set of the form $\Pi_0^{-1}[x, 1] \cap I$ with $x \leqslant \Pi_0(\alpha x_{i_0}+\beta y_{i_0})$ in the point $\frac{1}{2}$ of the embedded copy of I.

(iii) It is possible that an element of $\alpha x_{i_0}+\beta y_{i_0}$, containing M, and an element of $\alpha x_{i_1}+\beta y_{i_1}$, containing N, have a void intersection. In that case of course the sets M and N also have a void intersection.

(iv) In Figure 5 we have drawn the points $x_{i_0}, y_{i_0}, x_{i_1}, y_{i_1}$ in such a way that $\Pi_0 x_{i_0} < \Pi_0 y_{i_0}$ and $\Pi_0 x_{i_1} < \Pi_0 y_{i_1}$. This is done because in the cases $\Pi_0 x_{i_0} = \Pi_0 y_{i_0}$ or $\Pi_0 x_{i_1} = \Pi_0 y_{i_1}$ or $(\Pi_0 x_{i_0} < \Pi_0 y_{i_0}$ and $\Pi_0 x_{i_1} > \Pi_0 y_{i_1})$ or $(\Pi_0 x_{i_0} > \Pi_0 y_{i_0}$ and $\Pi_0 x_{i_1} < \Pi_0 y_{i_1})$ it is easy to see that $(\alpha x_{i_0}+\beta y_{i_0}) \cup (\alpha x_{i_1}+\beta y_{i_1})$ is linked, as the reader can easily verify.

Without loss of generality we may assume that $\Pi_0 y_{i_1} - \Pi_0 x_{i_1} \leqslant \Pi_0 y_{i_0} - \Pi_0 x_{i_0}$. It then follows that

$$\Pi_0^{-1}[\Pi_0 x_{i_1}, 1] \cap I \subset \Pi_0^{-1}(\Pi_0 x_{i_0}, 1] \cap I$$

since $N \subset I \setminus M$ and since

$$\Pi_0(\alpha x_{i_1}+\beta y_{i_1}) - \Pi_0 x_{i_1} \leqslant \Pi_0(\alpha x_{i_0}+\beta y_{i_0}) - \Pi_0 x_{i_0}.$$

However, this is a contradiction since $x_{i_0} \cup x_{i_1}$ is linked. ■

LEMMA 7. *$\lambda_{\bigcup_{i\in E}\mathscr{A}_i}(I)$ is infinite dimensional.*

Proof. We will show that $\lambda_{\bigcup_{i\in E}\mathscr{A}_i}(I)$ contains a copy of the Hilbert cube. For each $n \in E$, let I_n be defined by

$$I_n := \left[\frac{1}{2}+\frac{1}{3\sqrt{2}.-n}, \frac{1}{2}+\frac{1}{3\sqrt{2}.-n}\right].$$

Define a map $\varphi\colon \prod_{n\in E} I_n \to \prod_{n\in E} I^2$ by

$$(\varphi(x))_i = (x_i, \tfrac{1}{4}\sqrt{2}).$$

Note that for each $i \in E$, $(\varphi(x))_i$ is an element of $\lambda_{\mathscr{A}_i}(I)$ for all $x \in \prod_{n\in E} I_n$. Furthermore it is obvious that φ is an embedding. It suffices to show that the image of $\prod_{n\in E} I_n$ is contained in $\lambda_{\bigcup_{n\in E}\mathscr{A}_n}(I)$ and for this it suffices to show that $\bigcup_{n\in E}(\varphi(x))_n$ is linked for all $x \in \prod_{n\in E} I_n$ (Lemma 5). Assume to the contrary that for some $x \in \prod_{n\in E} I_n$, $\bigcup_{n\in E}(\varphi(x))_n$ were not linked. Then there exist indices n_0 and n_1 such that $(\varphi(x))_{n_0} \cup (\varphi(x))_{n_1}$ is not linked and therefore there exists an $M \in (\varphi(x))_{n_0}$ and an $N \in (\varphi(x))_{n_1}$ such that $M \cap N = \emptyset$. Then there are two possibilities drawn in Figure 6 and Figure 7. Without loss of generality we may assume that $n_1 < n_0$. This shows that

$$\Pi_0^{-1}[\tfrac{1}{2}, \Pi_0(\varphi(x))_{n_0}] \cap I \subset \Pi_0^{-1}[\tfrac{1}{2}, \Pi_0(\varphi(x))_{n_1}) \cap I.$$

Since $n_1 < n_0$ it follows that

$$\sqrt{2}\left(\Pi_0(\varphi(x))_{n_1}-\tfrac{1}{2}\right) < \frac{1}{-n_1} \leqslant \frac{1}{-3n_0} \leqslant \sqrt{2}\left(\Pi_0(\varphi(x))_{n_0}-\tfrac{1}{2}\right)$$

and therefore

$$\sqrt{2}\big(\Pi_0(\varphi(x))_{n_1}-\tfrac{1}{2}\big)<\sqrt{2}\big(\Pi_0(\varphi(x))_{n_0}-\tfrac{1}{2}\big)$$

which shows that the component containing 0 of $\Pi_0^{-1}[\frac{1}{2}, \Pi_0(\varphi(x))_{n_0}] \cap I$ cannot be contained in the component containing 0 of $\Pi_0^{-1}[\frac{1}{2}, \Pi_0(\varphi(x))_{n_1}] \cap I$, contradiction.

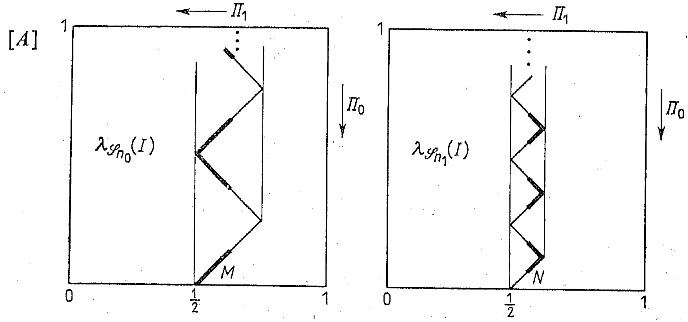

Fig. 6

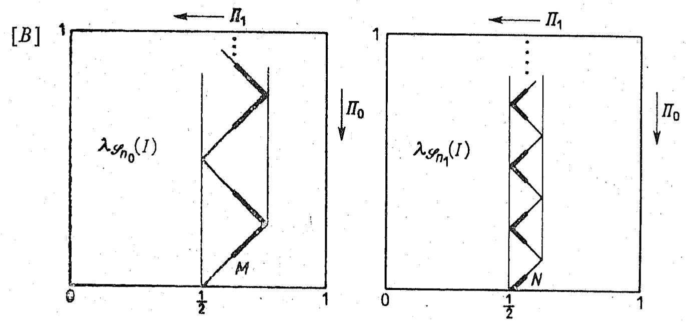

Fig. 7

Now,

$$\Pi_0^{-1}[\Pi_0(\varphi(x))_{n_0}, 1] \cap I \subset \Pi_0^{-1}(\Pi_0(\varphi(x))_{n_1}, 1] \cap I .$$

Since $-n_0 < -n_1$ it follows that the component containing $\frac{1}{2}$ of

$$\Pi_0^{-1}[\Pi_0(\varphi(x))_{n_0}, 1] \cap I$$

cannot be contained in the component containing $\frac{1}{2}$ of $\Pi_0^{-1}(\Pi_0(\varphi(x))_{n_1}, 1] \cap I$, contradiction. ■

Lemma 6 and Lemma 7 now give the following:

THEOREM 7. *$\lambda_{\bigcup_{i\in E}\mathcal{A}_i}(I)$ is a Hilbert cube.*

Proof. According to a theorem of Keller [15] each infinite dimensional compact convex subspace of the Hilbert space is homeomorphic to the Hilbert cube. ■

As noted in the introduction, we need $\lambda_{\bigcup_{n\in E}\mathcal{A}_n}(I)$ as first step in an inverse limit representation of λI. This will be demonstrated in the next section.

4. λI is a Hilbert cube.

DEFINITION. Let $\mathcal{S}$ and $\mathcal{T}$ be two families of closed sets in X. Then $\mathcal{S}$ *separates* $\mathcal{T}$ if for any $T_0, T_1 \in \mathcal{T}$ with $T_0 \cap T_1 = \emptyset$, there exist $S_0, S_1 \in \mathcal{S}$ such that $T_i \subset S_i$ $(i = 0, 1)$ and $S_0 \cap S_1 = \emptyset$.

Notation. $\mathcal{T} \sqsubset \mathcal{S}$.

For the closed unit interval I, define

$$\mathcal{S} = \{G \subset I \mid G \text{ is the union of a finite number of closed intervals with rational endpoints}\} .$$

It is clear that $\mathcal{S}$ separates the collection of closed subsets of I so that each mls $\mathcal{M} \in \lambda I$ is completely determined by its trace on $\mathcal{S}$. In fact it can be proved that λI and $\lambda_{\mathcal{S}}(I)$ are *equivalent* ([17]), which means, homeomorphic under a homeomorphism which on I is the identity. Furthermore it should be noticed that $\mathcal{S}$ is a countable subbase. Define

$$\mathcal{F} = \{(S_0, S_1) \mid S_i \in \mathcal{S}\ (i = 0, 1) \text{ and } S_0 \cap S_1 = \emptyset\} .$$

Then $\mathcal{F}$ again is countable; we enumerate $\mathcal{F}$, using a bijection of $N \setminus \{1\}$ onto $\mathcal{F}$. If $(S_0, S_1) \in \mathcal{F}$, then $\varepsilon = d(S_0, S_1) > 0$ and also $\delta = \frac{1}{2}\varepsilon\sqrt{2} > 0$. Consider the following embedding, depending on (S_0, S_1), of I in I^2 (see Fig. 8). All angles are $\frac{1}{2}\pi$ except

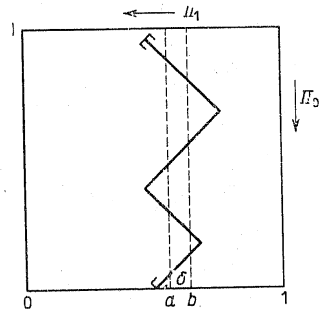

Fig. 8

the one at $(\frac{1}{2}, 0)$ which is $\frac{1}{4}\pi$. Furthermore $b - a = \delta$ and $S_0 \subset \Pi_0^{-1}[0, a] \cap I$ and $S_1 \subset \Pi_0^{-1}[b, 1] \cap I$. Since S_0 and S_1 are finite unions of intervals, such an embedding always is possible. In the embedding of I in I^2 we will not use more angles than necessary. As in Section 2 define

$$\mathcal{T} = \{A \subset I^2 \mid A = \Pi_i^{-1}[0, x] \vee A = \Pi_i^{-1}[x, 1]\ (i \in \{0, 1\}, x \in I\} .$$

This time, $\lambda_{\mathcal{F}\cap I}(I)$ is not the convex hull of the embedded copy of I in I^2, but it is the space designed in Figure 9.

If (S_0, S_1) is the nth element of $\mathcal{F}$, let $\lambda_{\mathcal{S}_n}(I)$ be the superextension of I, as indicated in Figure 9. In addition, put $\mathcal{S}_1 = \bigcup_{i\in E} \mathcal{A}_i$, where the $\mathcal{A}_i$'s are defined as is Section 3.

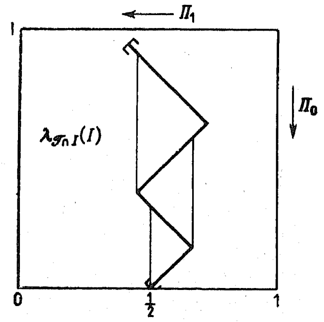

Fig. 9

A *Q-factor* is a space the product of which with the Hilbert cube is a Hilbert cube. A *Q-manifold* is paracompact Hausdorff space modelled on Q, i.e., a space which admits an open cover by sets homeomorphic to open subsets of Q. Q-manifolds are locally compact and metrizable. The hardest part of our program is to show that for each $n\in N$ the superextension $\lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I)$ is a Q-manifold, the proof of which will be postponed till Section 5.

LEMMA 8. *For each $n\in N$, $\lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I)$ is a Q-manifold.*

Now, an interesting theorem of Chapman is applicable to show that $\lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I)$ is even a Hilbert cube.

PROPOSITION 4. *For each $n\in N$, $\lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I)$ is a Hilbert cube.*

Proof. The normality of $\bigcup_{i=1}^{n}\mathcal{S}_i$ implies that $\lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I)$ is connected (Verbeek [21], III. 4.1 Corollary) and consequently $\lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I)$ is an AR (Theorem 4). Therefore $\lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I)$ is a compact contractible Q-manifold by Lemma 8. However, a compact contractible Q-manifold is a Hilbert cube, by a theorem of Chapman [7]. ■

Consider the following inverse limit system:

$$\lambda_{\mathcal{S}_1}(I) \overset{g_1}{\leftarrow} \lambda_{\mathcal{S}_1\cup\mathcal{S}_2}(I) \overset{g_2}{\leftarrow} \lambda_{\mathcal{S}_1\cup\mathcal{S}_2\cup\mathcal{S}_3}(I) \overset{g_3}{\leftarrow} \dots$$

where all the bonding maps are the Jensen mappings.

LEMMA 9. *λI is homeomorphic to $\varprojlim \lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I)$.*

Proof. Since all subbases in the inverse limit system are supernormal and therefore are normal there exists for each $n\in N$ a Jensen mapping

$$\xi_n\colon \lambda I \to \lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I)\,.$$

From the definition of the Jensen mappings it follows that for each $n\in N$ the diagram

$$\begin{array}{ccc} & \overset{\xi_n}{\nearrow} & \lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I) \\ \lambda I & & \downarrow g_{n-1} \\ & \underset{\xi_{n-1}}{\searrow} & \lambda_{\bigcup_{i=1}^{n-1}\mathcal{S}_i}(I) \end{array}$$

commutes, and therefore the map

$$e\colon \lambda I \to \varprojlim \lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I)$$

defined by $(e(\mathcal{M}))_n = \xi_n(\mathcal{M})$ $(n\in N)$ is a continuous closed surjection. It remains to show that e is one to one. Choose $\mathcal{M}, \mathcal{N} \in \lambda I$ such that $\mathcal{M} \neq \mathcal{N}$. Then there exist $M\in\mathcal{M}$ and $N\in\mathcal{N}$ such that $M\cap N = \emptyset$. Since $\mathcal{S}$ separates the closed subsets of I, there exist $S_0, S_1 \in \mathcal{S}$ with $M\subset S_0$, $N\subset S_1$ and $S_0\cap S_1 = \emptyset$. Of course it follows that $S_0\in\mathcal{M}$ and $S_1\in\mathcal{N}$. Now, $(S_0, S_1)\in\mathcal{F}$, say the nth element, and therefore S_0 and S_1 are separated by elements of $\mathcal{S}_n$ and consequently $\xi_n(\mathcal{M}) \neq \xi_n(\mathcal{N})$, since $\mathcal{S}_n \subset \bigcup_{i=1}^{n}\mathcal{S}_i$. This proves that e is one to one consequently e is a homeomorphism. ■

An onto map between homeomorphic compact metric spaces is called a *near-homeomorphism* if it is the uniform limit of homeomorphisms. An approximation theorem for inverse limits of Brown [6], often used in infinite dimensional topology, says that *if $Y = \varprojlim X_i$, where $\{X_i\}_i$ denotes an inverse sequence, and the X_i are all homeomorphic to a compact metric space X and each bonding map is a near-homeomorphism, then Y is homeomorphic to X.*

If X and Y are locally compact, then a map $f\colon X\to Y$ is called *proper* if the inverses of compact subsets of Y are compact in X. A proper map f is called *cell-like or cellular* (CE), if f is onto and point inverses have trivial shape. Chapman announced a theorem that characterizes *near-homeomorphisms* between Hilbert cubes as being *those continuous surjections with the property that the inverse image of each point has trivial shape*. This result is a consequence of his papers [8] and [9]. This theorem makes Brown's approximation theorem applicable in our situation.

LEMMA 10. *Let $g_{n-1}\colon \lambda_{\bigcup_{i=1}^{n}\mathcal{S}_i}(I) \to \lambda_{\bigcup_{i=1}^{n-1}\mathcal{S}_i}(I)$ be the Jensen mapping. Then g_{n-1} is monotone.*

Proof. We will show that point inverses of g_{n-1} are closed under the interval structure of $\lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I)$, which suffices to show that g_{n-1} is monotone (Corollary 1), since $\lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I)$ is connected. Choose $\mathscr{N} \in \lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I)$ and let $\mathscr{M}_0, \mathscr{M}_1 \in g_{n-1}^{-1}(\mathscr{N})$. Suppose there exists an $\mathscr{M}_2 \in I_{\bigcup_{i=1}^n \mathscr{S}_i}(\mathscr{M}_0, \mathscr{M}_1) \setminus g_{n-1}^{-1}(\mathscr{N})$. Then $g_{n-1}(\mathscr{M}_2) \neq \mathscr{N}$ and therefore there exist $N_0, N_1 \in \bigcup_{i=1}^{n-1} \mathscr{S}_i$ such that $N_0 \in g_{n-1}(\mathscr{M}_2)$ and $N_1 \in \mathscr{N}$ and $N_0 \cap N_1 = \emptyset$. However $\bigcup_{i=1}^{n-1} \mathscr{S}_i$ is supernormal, and therefore $\mathscr{N} \subset \mathscr{M}_i$ $(i \in \{0, 1\})$ and $g_{n-1}(\mathscr{M}_2) \subset \mathscr{M}_2$ (Corollary 3). This proves that $N_1 \in \mathscr{M}_0$ and $N_1 \in \mathscr{M}_1$ and therefore

$$I_{\bigcup_{i=1}^n \mathscr{S}_i}(\mathscr{M}_0, \mathscr{M}_1) \subset N_1^+ ,$$

which is a contradiction since $\mathscr{M}_2 \in I_{\bigcup_{i=1}^n \mathscr{S}_i}(\mathscr{M}_0, \mathscr{M}_1)$. ■

Lemma 11. *Let $g_{n-1}\colon \lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I) \to \lambda_{\bigcup_{i=1}^{n-1} \mathscr{S}_i}(I)$ be the Jensen mapping. Then each point inverse either is a point or is homeomorphic to an interval. In particular g_{n-1} is cellular.*

Proof. Let f_n be the Jensen mapping of $\lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I)$ onto $\lambda_{\mathscr{S}_n}(I)$. Let $\mathscr{N} \in \lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I)$. Choose $\mathscr{M}_0, \mathscr{M}_1 \in g_{n-1}^{-1}(\mathscr{N})$ such that $\mathscr{M}_0 \neq \mathscr{M}_1$. Then there exists an $M_0 \in \mathscr{M}$ and an $M_1 \in \mathscr{M}_1$ such that $M_0 \cap M_1 = \emptyset$. Since $\bigcup_{i=1}^{n-1} \mathscr{S}_i$ is supernormal, it follows that M_0 and M_1 are not both elements of $\bigcup_{i=1}^{n} \mathscr{S}_i$ (notice that $g_{n-1}(\mathscr{M}_0) = g_{n-1}(\mathscr{M})$!) and consequently without loss of generality $M_0 \in \mathscr{S}_n$. However, $\mathscr{S}_n$ is also supernormal and therefore we may assume that $M_1 \in \mathscr{S}_n$. It now follows that $f_n(\mathscr{M}_0) \neq f_n(\mathscr{M}_1)$, since $\mathscr{S}_n$ is supernormal (Corollary 3). Therefore $g_{n-1}^{-1}(\mathscr{N})$ and $f_n g_{n-1}^{-1}(\mathscr{N})$ are homeomorphic. However this shows that $g_{n-1}^{-1}(\mathscr{N})$ either is a point or is homeomorphic to an interval, since all points of $f_n g_{n-1}^{-1}(\mathscr{N})$ must be elements of a horizontal line through a point of the embedded copy of I, a point which is completely determined by $\mathscr{N}$, and since $g_{n-1}^{-1}(\mathscr{N})$ is connected (Lemma 10). ■

Theorem 7. *The superextension of the closed interval is homeomorphic to the Hilbert cube.*

Proof. As a consequence of Chapman's theorem it follows from Lemma 11 that all bonding maps in the inverse limit system for λI are near-homeomorphisms. All superextensions in the inverse system are Hilbert cubes (Proposition 4) and therefore Lemma 9 and Brown's approximation theorem give the result $\lambda I \cong Q$. ■

5. $\lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I)$ is a Q-manifold.

Lemma 8. *For each $n \in N$, $\lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I)$ is a Q-manifold.*

Proof. Choose $x \in \lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I) \subset \prod_{i=1}^n \lambda_{\mathscr{S}_i}(I) \subset \prod_{i \in E} \lambda_{\mathscr{A}_i}(I) \times \prod_{i=2}^n \lambda_{\mathscr{S}_i}(I)$. Let $\{p_i|\ i \in E\} \cup \cup \{p_i|\ i \in \{2, 3, \ldots, n\}\}$ denote the projection maps of the latter product. For each $i \in \{2, 3, \ldots, n\}$ the projection of $\lambda_{\mathscr{S}_i}(I)$ onto the first coordinate axis of I^2 is an interval, say $[c_i^0, c_i^1]$. Assume that for each $i \in \{2, 3, \ldots, q\}$ where $q \leqslant n$, $\Pi_0 x_i \in (c_i^0, c_i^1)$ and that for $i \in \{q+1, q+2, \ldots, n\}$ we have $\Pi_0 x_i \notin (c_i^0, c_i^1)$. Then define

$$\varepsilon = \min\{d(\Pi_0 x_i, c_i^j)|\ i = 2, 3, \ldots, q;\ j = 0, 1\}.$$

Let A be the finite set $\{2, 3, \ldots, n\}$. If $i \in A$, $M \in x_i$ define

$$M^* = \operatorname{cl}_I \operatorname{int}_I M$$

(here I refers to the copy of $[0, 1]$ embedded in $\lambda_{\mathscr{S}_i}(I) \subset I^2$). Also, for $i \in A$, put

$$\mathscr{F}(x_i) = \{M^*|\ M \in x_j\ (j \in A \setminus \{i\}) \text{ and } (M = \Pi_0^{-1}[0, \Pi_0 x_j] \cap I \text{ or } M = \Pi_0^{-1}[\Pi_0 x_j, 1] \cap I) \text{ and } M^* \cap \Pi_0^{-1}\Pi_0 x_i = \emptyset\}.$$

Notice that $\mathscr{F}(x_i)$ always is finite. If $i \in \{2, 3, \ldots, q\}$ then choose a subinterval (a_i, b_i) of (c_i^0, c_i^1) (an interval is non-degenerate in our terminology) such that

(i) $\Pi_0 x_i \in (a_i, b_i)$,

(ii) $a_i - c_i^0 > \frac{1}{4}\varepsilon$ and $c_i^0 - b_i > \frac{1}{4}\varepsilon$,

(iii) $\Pi_0^{-1}[a_i, b_i] \cap \lambda_{\mathscr{S}_i}(I)$ consists of two closed convex subspaces D_i^0 and D_i^1 such that $\Pi_0 D_i^0 = [a_i, \Pi_0 x_i]$ and $\Pi_0 D_i^1 = [\Pi_0 x_i, b_i]$,

(iv) $\Pi_0^{-1}[a_i, b_i] \cap \bigcup \mathscr{F}(x_i) = \emptyset$,

(v) for each subinterval $[e_1, e_2]$ of $[a_i, \Pi_0 x_i)$ and for each subinterval $[d_1, d_2]$ of $(\Pi_0 x_i, b_i]$ we have that $\Pi_0^{-1}[e_1, e_2] \cap I$ and $\Pi_0^{-1}[d_1, d_2] \cap I$ both have no isolated points.

If $i \in A \setminus \{2, 3, \ldots, q\}$ then choose a subinterval $[a_i, b_i]$ of $[c_i^0, c_i^1]$ such that

(i) $\Pi_0^{-1}[a_i, b_i] \cap \lambda_{\mathscr{S}_i}(I)$ is convex in $\lambda_{\mathscr{S}_i}(I)$,

(ii) x_i is an interior point of $\Pi_0^{-1}[a_i, b_i] \cap \lambda_{\mathscr{S}_i}(I)$ in $\lambda_{\mathscr{S}_i}(I)$,

(iii) $\Pi_0^{-1}[a_i, b_i] \cap \mathscr{F}(x_i) = \emptyset$,

(iv) for each subinterval $[e_1, e_2]$ of $[a_i, b_i]$ we have that $\Pi_0^{-1}[e_1, e_2] \cap I$ has no isolated points.

(One should convince oneself that in all cases suitable a_i, b_i do indeed exist!) We will show that the closed neighborhood

$$B(x) = \bigcap_{i=2}^n p_i^{-1}[\Pi_0^{-1}[a_i, b_i] \cap \lambda_{\mathscr{S}_i}(I)] \cap \lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I)$$

of x is a Q-manifold, which will establish our lemma (there is an open U in $\lambda_{\bigcup_{i=1}^n \mathscr{S}_i}(I)$ such that $x \in U \subset B(x)$ and as $B(x)$ is a compact Q-manifold, there also is an open O

in $\lambda \bigcup_{i=1}^{n} \mathcal{S}_i(I)$ such that $x \in O \subset U \subset B(x)$ and O is homeomorphic to an open subset of Q).

Let us first anatomize $B(x)$. Consider $F = \{0, 1\}^{\{2,3,\dots,q\}}$ and for each $\sigma = (\sigma_i)_i \in F$ define

$$X(\sigma) = \bigcap_{i=2}^{q} p_i^{-1}[D^{\sigma_i}_i] \cap \bigcap_{i=q+1}^{n} p_i^{-1}[\Pi_0^{-1}[a_i, b_i] \cap \lambda_{\mathcal{S}_i}(I)] \cap \lambda \bigcup_{i=1}^{n} \mathcal{S}_i(I) .$$

It then is clear that

$$\bigcup_{\sigma \in F} X(\sigma) = B(x) .$$

A. For each $\sigma \in F$ the set $X(\sigma)$ is closed and convex in $\lambda \bigcup_{i=1}^{n} \mathcal{S}_i(I)$. Assume to the contrary that for some $\sigma \in F$ the set $X(\sigma)$ were not convex. Then there exist $y, z \in X(\sigma)$ and $\alpha, \beta \in \mathbf{R}$ with $\alpha > 0$ and $\beta > 0$ and $\alpha + \beta = 1$ such that $\alpha y + \beta z \notin X(\sigma)$.

We claim that

$$\bigcup_{i \in E} (\alpha y + \beta z)_i \cup \bigcup_{i=2}^{n} (\alpha y + \beta z)_i$$

is not linked, for else it would follow that $\alpha y + \beta z \in \lambda \bigcup_{i=1}^{n} \mathcal{S}_i(I)$, and as $(\alpha y + \beta z)_i = \alpha y_i + \beta z_i$ for each i, it is easily seen that also $\alpha y + \beta z \in X(\sigma)$. Therefore there exist two indices i_0, j_0 such that $(\alpha y + \beta z)_{i_0} \cup (\alpha y + \beta z)_{j_0}$ is not linked and consequently there exists an $M \in (\alpha y + \beta z)_{i_0}$ and an $N \in (\alpha y + \beta z)_{j_0}$ such that $M \cap N = \emptyset$. Now, if i_0 and j_0 are both elements of $E \cup \{q+1, q+2, \dots, n\}$ then, using the same technique as in Lemma 6, this leads to a contradiction, for we have chosen the intervals $[a_i, b_i]$ $(i \in \{q+1, q+2, \dots, n\})$ in such a way that $\Pi_0^{-1}[e_1, e_2]$ has no isolated points for every subinterval $[e_1, e_2]$ of $[a_i, b_i]$. Therefore, let us assume that $i_0 \in \{2, 3, \dots, q\}$. Since straight horizontal lines through $(\alpha y + \beta z)_{i_0}$ and $(\alpha y + \beta z)_{j_0}$ must intersect the embedded copies of I in the same point, the situation sketched in Figure 10 is the only possibility (except for an interchange of the indices i_0 and j_0, which induces a similar situation).

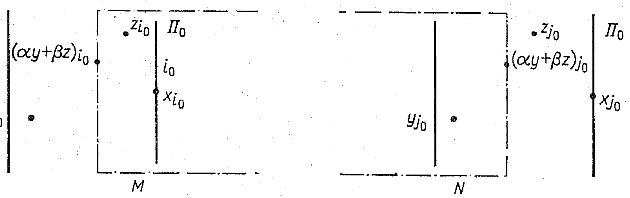

Fig. 10

Remarks. (i) It is possible that an element of $(\alpha y + \beta z)_{i_0}$, containing M, and an element of $(\alpha y + \beta z)_{j_0}$, containing N, have a void intersection. In that case the sets M and N of course also have a void intersection.

(ii) In Figure 10 we have drawn the points $y_{i_0}, z_{i_0}, x_{i_0}, y_{j_0}, z_{j_0}$ and x_{j_0} in such a way that $\Pi_0 y_{i_0} < \Pi_0 z_{i_0} < \Pi_0 x_{i_0}$ and $\Pi_0 y_{j_0} < \Pi_0 z_{j_0} < \Pi_0 x_{j_0}$. This is not the only possible configuration. More generally, we may assume that either $(\Pi_0 y_{i_0} < \Pi_0 z_{i_0} \leqslant \Pi_0 x_{i_0}$ and $\Pi_0 y_{j_0} < \Pi_0 z_{j_0} \leqslant \Pi_0 x_{j_0})$ or $(\Pi_0 x_{i_0} \leqslant \Pi_0 y_{i_0} < \Pi_0 z_{i_0}$ and $\Pi_0 x_{j_0} \leqslant \Pi_0 y_{j_0} < \Pi_0 z_{j_0})$ (these two cases are similar), for in all other cases it is easy to see that $(\alpha y + \beta z)_{i_0} \cup (\alpha y + \beta z)_{j_0}$ is linked. The lack of generality in our diagram will cause no trouble, as will appear from the proof.

We distinguish two subcases:

(a) $\Pi_0 z_{i_0} - \Pi_0 y_{i_0} \leqslant \Pi_0 z_{j_0} - \Pi_0 y_{j_0}$.

Since $M \subset \Pi_0^{-1}(\Pi_0(\alpha y + \beta z)_{j_0}, 1] \cap I$, it follows that $\Pi_0^{-1}[\Pi_0 y_{i_0}, 1] \cap I \subset \subset \Pi_0^{-1}(\Pi_0 y_{j_0}, 1]$, since $\Pi_0^{-1}[\Pi_0 y_{i_0}, 1] \cap I$ has no isolated points and since $\Pi_0(\alpha y + \beta z)_{i_0} - \Pi_0 y_{i_0} \leqslant \Pi_0(\alpha y + \beta z)_{j_0} - \Pi_0 y_{i_0}$. However, this is a contradiction since $y_{i_0} \cup y_{j_0}$ is linked.

(b) $\Pi_0 z_{j_0} - \Pi_0 y_{j_0} \leqslant \Pi_0 z_{i_0} - \Pi_0 y_{i_0}$.

As $N \subset \Pi_0^{-1}[0, \Pi_0(\alpha y + \beta z)_{i_0}) \cap I$ we conclude that $(\Pi_0^{-1}[0, \Pi_0 z_{j_0}] \cap I)^* \subset \subset \Pi_0^{-1}[0, \Pi_0 z_{i_0}) \cap I$, since $\Pi_0(\alpha y + \beta z)_{j_0} - \Pi_0 y_{j_0} \leqslant \Pi_0(\alpha y + \beta z)_{i_0} - \Pi_0 y_{i_0}$. Therefore, if $\Pi_0^{-1} \Pi_0 z_{j_0} \cap I$ contains no isolated point of $\Pi_0^{-1}[0, \Pi_0 z_{j_0}] \cap I$, then this is a contradiction. If $\Pi_0^{-1} \Pi_0 z_{j_0} \cap I$ contains an isolated point of $\Pi_0^{-1}[0, \Pi_0 z_{j_0}] \cap I$, then $\Pi_0 z_{j_0} = \Pi_0 x_{j_0}$, for if not, then $\Pi_0^{-1}[0, \Pi_0 z_{j_0}] \cap I$ is not perfect, which is a contradiction. Now, since

$$(\Pi_0^{-1}[0, \Pi_0 x_{j_0}] \cap I)^* \cap \Pi_0^{-1}[a_{i_0}, \Pi_0 x_{i_0}] = \emptyset ,$$

it follows that also $\Pi_0 y_{j_0} = \Pi_0 x_{j_0}$, for if not, then $y_{i_0} \cup y_{j_0}$ is not linked. However, this implies that also $\Pi_0(\alpha y + \beta z)_{j_0} = \Pi_0 x_{j_0}$ and consequently $N \in z_{j_0}$. This is a contradiction, since $z_{i_0} \cup z_{j_0}$ is linked.

It now follows that the neighborhood $B(x)$ of x is a finite union of closed (and hence compact) convex subspaces. By a theorem of Quinn and Wong ([18], Theorem 3.4) it follows that $B(x)$ is a Q-manifold provided that for all non-void subsets F_0 of F the set $\bigcap_{\sigma \in F_0} X(\sigma)$ either is void or is homeomorphic to Q.

B. Let F_0 be a non-void subset of F. Then $\bigcap_{\sigma \in F_0} X(\sigma)$ either is void or is homeomorphic to Q.

Assume that $\bigcap_{\sigma \in F_0} X(\sigma)$ is non-void. It suffices to show that $\bigcap_{\sigma \in F_0} X(\sigma)$ is infinite dimensional for an infinite dimensional compact convex set of the Hilbert space is homeomorphic to Q (Keller [15]). Choose $y \in \bigcap_{\sigma \in F_0} X(\sigma)$. We again distinguish two subcases:

(a) For each $i \in \{2, 3, \dots, n\}$ the point $\Pi_0 y_i$ is an element of (c_i^0, c_i^1).

Assume that y is such that for every coordinate y_i $(i \in E \cup \{2, 3, \dots, n\})$ a straight horizontal line through y_i does not intersect I in 0 or 1. (This assumption is justified by the fact that if $y = i(0)$ or $i(1)$, then $\bigcap_{\sigma \in F_0} X(\sigma)$ is the intersection of a finite number of sets, each of which intersects $\underset{\sim}{i}(I)(=I)$ in a neighborhood of y.) This intersection, say f, must be the same point for every coordinate. Define

$$\delta_0 = \min\{|y_i - c_i^0||\ i \in \{2, 3, \dots, n\}\},$$
$$\delta_1 = \min\{|y_i - c_i^1||\ i \in \{2, 3, \dots, n\}\}$$

and choose $n_0 \in E$ such that

$$\frac{1}{n_0} < \tfrac{1}{4}\min\{\delta_0\sqrt{2}, \delta_1\sqrt{2}, f, 1-f\}.$$

For all $j \in \boldsymbol{E}$, let I_j be defined as in Lemma 7.

It is easy to show, using the same technique as in Lemma 7, that for all $j \in \boldsymbol{E}$ with $j \leqslant n_0$ and for each point d of $I_j \times \{f/\sqrt{2}\}$ we have that $\bigcup_{i=2}^{n} y_i \cup d$ is linked (notice that indeed $I_j \times \{f/\sqrt{2}\} \subset \lambda_{\mathscr{A}_j}(I)$).

Now, by induction, for each $k \in \{m \in \boldsymbol{E}|\ n_0 \leqslant m\}$ we will construct a point h_k of $\lambda_{\mathscr{A}_k}(I)$ with the following property: for all $j \in \boldsymbol{E}$ with $j \leqslant n_0$ there exists a (non degenerate) subinterval I_j^k of I_j such that for every point $d_j^k \in I_j^k \times \{f/\sqrt{2}\}$ the system

$$\bigcup_{i=2}^{n} y_i \cup \bigcup_{\substack{j \in E\\ k \leqslant j}} h_j \cup \bigcup_{\substack{j \in E\\ j \leqslant n_0}} d_j^k$$

is linked.

For each $j \in \boldsymbol{E}$ with $j \leqslant n_0$ let $\bar{a}_j$ be the middle of the interval $I_j \times \{f/\sqrt{2}\}$. Then the linked system

$$\bigcup_{i=2}^{n} y_i \cup \bigcup_{\substack{j \in E\\ j \leqslant n_0}} \bar{a}_j$$

is contained in at least one maximal linked system $g_0 \in \lambda_{\bigcup_{i=1}^{n}\mathscr{S}_i}(I)$. Define $h_{-2} := (g_0)_{-2}$. The intervals I_j^{-2} $(j \leqslant n_0)$ now can be found in the following way:

(i) $I_j^{-2} := I_j$ if $\Pi_0 h_{-2} \in I_{-2}$.

(ii) $I_j^{-2} := [\tfrac{1}{2}, \Pi_0 \bar{a}_j] \cap I_j$ if $\Pi_0 h_{-2} \in [\tfrac{1}{2}, \Pi_0 \bar{a}_j] \setminus I_j$.

(iii) $I_j^{-2} := [\Pi_0 \bar{a}_j, 1] \cap I_j$ if $\Pi_0 h_{-2} \in [\Pi_0 \bar{a}_j, 1] \setminus I_j$.

It is easy to verify that the intervals I_j^{-2} $(j \leqslant n_0)$, defined in this way, satisfy our requirements.

Let all points h_k be defined for all $k \geqslant l$ $(l, k \in \{m \in \boldsymbol{E}|\ n_0 \leqslant m\})$. For each $j \in \boldsymbol{E}$, $j \leqslant n_0$ let $\bar{a}_j^l$ be the middle of the interval $I_j^l \times \{f/\sqrt{2}\}$. Then the linked system

$$\bigcup_{i=2}^{n} y_i \cup \bigcup_{\substack{j \in E\\ l \leqslant j}} h_j \cup \bigcup_{\substack{j \in E\\ j \leqslant h_0}} \bar{a}_j^l$$

is contained in at least one maximal linked system $p_0 \in \lambda_{\bigcup_{i=1}^{n}\mathscr{S}_i}(I)$. Define $h_{3l} := (p_0)_{3l}$. The intervals I_j^{3l} $(j \leqslant n_0)$ now can be found in the following way:

(i) $I_j^{3l} := I_j^l$ if $\Pi_0 h_{3l} \in I_{3l}$.

(ii) $I_j^{3l} := [\tfrac{1}{2}, \Pi_0 \bar{a}_j^l] \cap I_j^l$ if $\Pi_0 h_{3l} \in [\tfrac{1}{2}, \Pi_0 \bar{a}_j^l] \setminus I_j$.

(iii) $I_j^{3l} := [\Pi_0 \bar{a}_j^l, 1] \cap I_j^l$ if $\Pi_0 h_{3l} \in [\Pi_0 \bar{a}_j^l, 1] \setminus I_j$.

Again it is easy to verify that the intervals I_j^{3l} $(j \leqslant n_0)$, defined in this way, satisfy our requirements.

Now it is obvious that $\bigcap_{\sigma \in F_0} X(\sigma)$ contains a copy of $\prod_{\substack{j \in E\\ j \leqslant n_0}} I_j^{n_0/3}$, which shows that $\bigcap_{\sigma \in F_0} X(\sigma)$ is infinite dimensional.

(b) There exists a coordinate $i_0 \in \{2, 3, \dots, n\}$ such that $\Pi_0 y_{i_0} \notin (c_{i_0}^0, c_{i_0}^1)$. We will construct a point $g \in \bigcap_{\sigma \in F_0} X(\sigma)$ such that $\Pi_0 g_i \in (c_i^0, c_i^1)$ for all $i \in \{2, 3, \dots, n\}$. Then case (a) is applicable to show that $\bigcap_{\sigma \in F_0} X(\sigma)$ is infinite dimensional. Without loss of generality we may assume that

$$\bigcap_{\sigma \in F_0} X(\sigma) = \bigcap_{i=2}^{n} p_i^{-1}[S_i] \cap \lambda_{\bigcup_{i=1}^{n}\mathscr{S}_i}(I),$$

where each S_i $(2 \leqslant i \leqslant n)$ is convex in $\lambda_{\mathscr{S}_i}(I)$ while, moreover, for each $i > q$ we have $S_i = \Pi_0^{-1}[H_i] \cap \lambda_{\mathscr{S}_i}(I)$ for some (non-degenerate) interval H_i. As in case (a), we may assume that a straight horizontal line through y_i does not intersect I in 0 or 1. Let this intersection be f. Define $V = \{i \in \{2, 3, \dots, n\}|\ \Pi_0 y_i \notin (c_i^0, c_i^1)\}$. Clearly $V \subset \{q+1, q+2, \dots, n\}$. Now, for every $i \in V$ there exists a subinterval L_i of H_i such that $\Pi_0 y_i \in L_i$ and $L_i \times \{f/\sqrt{2}\} \subset \lambda_{\mathscr{S}_i}(I)$. Let δ_i denote the length of this interval $(i \in V)$. Let $\delta = \min\{\delta_i|\ i \in V\}$. Moreover define

$$\varrho_0 = \min\{|\Pi_0 y_i - c_0^j|\,|\,i \in \{2, 3, \dots, n\} \setminus V;\ j \in \{0, 1\}\}$$

and

$$\varrho = \tfrac{1}{4}\min\{\varrho_0, \delta\}.$$

Choose for each $i \in V$ a point $g_i \in L_i \times \{f/\sqrt{2}\} \subset \lambda_{\mathscr{S}_i}(I)$ such that

$$|\Pi_0 y_i - \Pi_0 g_i| = \varrho.$$

Recall that $A = \{2, 3, \dots, n\}$. We will show that

$$\mathscr{L} = \bigcup_{i \in V} g_i \cup \bigcup_{i \in A \setminus V} y_i$$

is linked and consequently each mls $g \in \lambda_{\bigcup_{i=1}^{n}\mathscr{S}_i}(I)$ which contains $\mathscr{L}$ is a point of $\bigcap_{\sigma \in F_0} X(\sigma)$ such that $\Pi_0 g_i \in (c_i^0, c_i^1)$ for all $i \in \{2, 3, \dots, n\}$. Assume that $\mathscr{L}$ were not linked. We again distinguish two subcases:

Case 1. There exist two indices $i_0, j_0 \in V$ such that $g_{i_0} \cup g_{j_0}$ is not linked. Then choose $M \in g_{i_0}$ and $N \in g_{j_0}$ such that $M \cap N = \emptyset$. There are two subcases.

(i) One of the sets M, N contains the corresponding projection of y, say $y_{i_0} \in M$.

Since $N\subset\Pi_0^{-1}[0,\Pi_0 g_{i_0})\cap I$ and since $|\Pi_0 g_{i_0}-\Pi_0 y_{i_0}| = |\Pi_0 g_{j_0}-\Pi_0 y_{j_0}|$ it follows that $\Pi_0^{-1}[0,\Pi_0 y_{j_0}]\cap I\subset\Pi_0^{-1}[0,\Pi_0 y_{i_0})\cap I$ ($\Pi_0^{-1}[0,x]\cap I$ contains no isolated points for $\Pi_0 g_{j_0}\leqslant x\leqslant\Pi_0 y_{j_0}$!). However, this is a contradiction since

$$\Pi_0^{-1}[0,\Pi_0 y_{j_0}]\cap I = I.$$

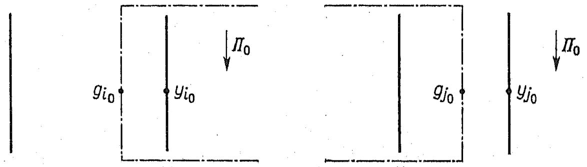

Fig. 11

(ii) None of the sets M, N contains the corresponding projection of y. It now follows that for example $M\subset\Pi_0^{-1}(\Pi_0 g_{j_0},1]\cap I$.

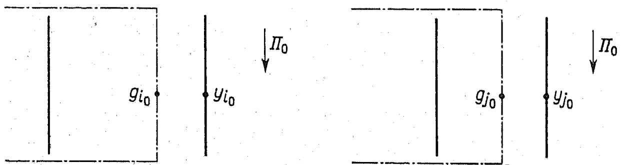

Fig. 12

However, this is a contradiction since M contains a component of length at least $\frac{3}{4}\varrho\sqrt{2}$ while all components of $\Pi_0^{-1}(\Pi_0 g_{j_0},1]\cap I$ have length less than or equal to $\frac{2}{4}\varrho\sqrt{2}$, since $\Pi_0^{-1}[H_{j_0}]\cap I$ contains no isolated points and the same is true for each subinterval of H_{j_0}.

Case 2. There exist indices $i_0\in V$ and $j_0\in A\setminus V$ such that $g_{i_0}\cup y_{j_0}$ is not linked. This can be treated in the same way as Case 1(ii).

This completes the proof of the lemma. ∎

Added in proof. The main result of this paper that $\lambda I\cong Q$ can also be derived by using a recent characterization of the Hilbert cube due to H. Toruńczyk.

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