Countable-points compactifications * for metric spaces

by

Dedicated to Professor Kiiti Morita on his 60th birthday

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Abstract. K. Morita posed a problem to characterize spaces which have a compactification by adding a countable number of points. In this paper we shall establish a theorem that a metrizable space X has such a compactification iff X is Čech-complete semi-compact and $R(X)$ (= the set of all points having no compact nbd) is Lindelöf.

§ 1. Introduction. All spaces are assumed to be completely regular and T_1.

Let αX be a compactification of a space X. Then αX is called a *countable-points compactification* if the remainder $\alpha X - X$ consists of at most a countable number of points. This notion is due to K. Morita, and concerning this he posed the following problem in [1]: Characterize those spaces which have a countable-points compactification.

As he pointed out there, if a space X has a countable-points compactification then X must be necessarily Čech-complete and semi-compact, and in case X is separable metrizable the converse is also true by a theorem of Zippin [7], to which K. Morita gave a proof in [3] based on his results on uniformities [2]. However, even in case X is Čech-complete semi-compact metrizable it does not have a countable-points compactification in general as will be shown by Example 4.2 below. Thus K. Morita suggested the author to find a necessary and sufficient condition for metrizable spaces to have such a compactification. Namely the purpose of this paper is to give an answer to his suggestion above and to establish the following theorems. In the sequel for a space X $R(X)$ denotes the set of all points having no compact neighborhood.

THEOREM 1. *Let X be a Čech-complete semi-compact space. If $R(X)$ is separable metrizable then X always has a countable-points compactification.*

THEOREM 2. *A metrizable space X has a countable-points compactification if and only if X is Čech-complete semi-compact and $R(X)$ Lindelöf.*

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* The contents of this paper were announced in [8].

§ 2. **Preliminaries.** Throughout sections, for any space X by a uniformity of X agreeing with the topology we shall mean a family Ψ of open covers of X satisfying conditions (i) to (iii) below:

(i) If $\mathfrak{U}, \mathfrak{V} \in \Psi$, there exists $\mathfrak{W} \in \Psi$ such that $\mathfrak{W}$ is a refinement of $\mathfrak{U}$ and $\mathfrak{V}$.

(ii) If $\mathfrak{U} \in \Psi$, there is $\mathfrak{V} \in \Psi$ which is a star-refinement of $\mathfrak{U}$.

(iii) $\{\mathrm{St}(x, \mathfrak{U})|\ \mathfrak{U} \in \Psi\}$ is a local basis at each point x of X.

In this section we shall begin with several definitions and theorems given in Morita [3], which will be essential to our proof of Theorem 1.

An open set U of a space X is called *γ-open* if $\mathrm{Bd}_X U$ is compact, where for a subset A of a space R $\mathrm{Bd}_R A$ denotes the boundary of A in R. A finite open cover consisting of γ-open sets is called a *γ-open cover*. As is known a space X is called *semi-compact* if each point of X has arbitrarily small γ-open sets as its neighborhoods. For this notion the following theorem of Morita [3, Theorem 1] is well-known.

THEOREM 2.1. *The totality Γ of all γ-open covers of a semi-compact space X is a uniformity of X agreeing with the topology, and the completion γX of X with respect to Γ is the maximum of all compactifications θX's of X with the following property*:

(*) *For any point p of θX and any neighborhood U of p there exists an open set V of θX such that $p \in V \subset U$ and $\mathrm{Bd}_{\theta X} V \subset X$.*

K. Morita called the above γX the Freudenthal compactification (cf. [5]).

As was mentioned in the introduction K. Morita gave his own proof to Zippin's theorem above in the following way [3, Theorems 10 and 11]. In the sequel N denotes the set of all positive integers.

THEOREM 2.2. *Let X be a semi-compact separable metrizable space. Then X has a uniformity $\Gamma_0 = \{\mathfrak{U}_n|\ n \in N\}$ of γ-open covers agreeing with the topology with the following properties*:

(a) *$\mathfrak{U}_n$ is a star-refinement of $\mathfrak{U}_{n-1}$.*

(b) *$\mathfrak{U}_n$ is a refinement of the covers*

$$\{U_{ij}, X-\mathrm{Cl}_X U_{ij}, U|\ U \cap \mathrm{Bd}_X U_{ij} \neq \emptyset,\ U \in \mathfrak{U}_{n-1}\}$$

for $i = 1, \ldots, n-1$; $j = 1, \ldots, r_i$, where $\mathfrak{U}_i = \{U_{ij}|\ j = 1, \ldots, r_i\}$.

Then the completion δX of X with respect to Γ_0 is compact metrizable with property (*) in Theorem 2.1.

THEOREM 2.3. *For any metrizable compactification θX with property (*) in Theorem 2.1 of a Čech-complete semi-compact separable metrizable space X there exist a compactification αX with $\mathrm{Card}(\alpha X - X) \leqslant \aleph_0$ and a continuous map φ from θX onto αX such that $\varphi|X =$ the identity of X.*

We shall need further several lemmas. The condition (b) in Lemma 2.4 below was pointed out by K. Morita.

LEMMA 2.4. *Let X be a Čech-complete semi-compact space and therefore let $X = \bigcap \{G_i|\ i \in N\}$ with an open set G_i of γX for $i \in N$. If a closed set F of X has a countable open basis then there exists a countable collection $\mathfrak{U}$ of γ-open sets of X satisfying the following conditions*:

(a) *For any $p \in F$ and any neighborhood V of p in X*

$$p \in U \subset \mathrm{Cl}_X U \subset U', \quad U' \cap F \subset V$$

hold for some $U, U' \in \mathfrak{U}$.

(b) *For any $p \in F$ and any $i \in N$*

$$p \in U \subset \mathrm{Cl}_X U \subset U' \subset \mathrm{Cl}_{\gamma X} U' \subset G_i$$

holds for some $U, U' \in \mathfrak{U}$.

Proof. Let $\mathfrak{B} = \{B_i|\ i \in N\}$ be a countable open basis of F. Let us put

$$\mathfrak{F} = \{\mathrm{Cl}_{\gamma X} B_i|\ i \in N\} \cup \{\gamma X - G_i|\ i \in N\}.$$

For each pair of sets E, E' in $\mathfrak{F}$ with $E \cap E' = \emptyset$, by Theorem 2.1, we can choose two open sets $V_{EE'}, V_{E'E}$ of γX so that

$$(1) \qquad E \subset V_{EE'}, \quad E' \subset V_{E'E}, \quad \mathrm{Cl}_{\gamma X} V_{EE'} \cap \mathrm{Cl}_{\gamma X} V_{E'E} = \emptyset,$$

and

$$(2) \qquad \mathrm{Bd}_{\gamma X} V_{EE'} \subset X, \quad \mathrm{Bd}_{\gamma X} V_{E'E} \subset X.$$

Let $\mathfrak{V}$ be the collection of those sets chosen above. Then by (2) $\mathfrak{V} \cap X = \{V \cap X|\ V \in \mathfrak{V}\}$ is a countable collection of γ-open sets of X. Let us put

$$\mathfrak{U} = \{V_1' \cap \ldots \cap V_s', X - \mathrm{Cl}_X(V_1' \cup \ldots \cup V_s')|\ V_i' \in \mathfrak{V} \cap X, i = 1, \ldots, s; s \in N\}.$$

Then $\mathfrak{U}$ is also a countable collection of γ-open sets, and satisfies the desired properties of the lemma. To see this, let $p \in F$ and V be open in X with $p \in V$. Choose an open set V' of γX so that $V = V' \cap X$. Let q be any point of $\mathrm{Cl}_{\gamma X} F - V'$. Then the following two cases arise:

Case (i). If $q \in X$ then $q \in F$ and $q \neq p$. Hence for some $B_i, B_{i'} \in \mathfrak{B}$ we have $q \in B_i$, $p \in B_{i'}$ and $\mathrm{Cl}_{\gamma X} B_i \cap \mathrm{Cl}_{\gamma X} B_{i'} = \emptyset$.

Case (ii). If $q \notin X$ then $q \in \gamma X - G_i$ for some $i \in N$ since $X = \bigcap_j G_j$. Then there is $B_k \in \mathfrak{B}$ such that $p \in B_k \subset \mathrm{Cl}_{\gamma X} B_k \subset G_i$. Hence each of the two cases above implies that there are two sets V_q^1, V_q^2 in $\mathfrak{V}$ such that

$$q \in V_q^1, \quad p \in V_q^2, \quad \mathrm{Cl}_{\gamma X} V_q^1 \cap \mathrm{Cl}_{\gamma X} V_q^2 = \emptyset.$$

Since $\mathrm{Cl}_{\gamma X} F - V'$ is compact, we can choose q_i, $i = 1, \ldots, n$ so that

$$\mathrm{Cl}_{\gamma X} F - V' \subset V_{q_1}^1 \cup \ldots \cup V_{q_n}^1.$$

Then each of sets

$$U = \bigcap_{i=1}^{n} (V_{q_i}^2 \cap X), \quad U' = X - \bigcup_{i=1}^{n} \mathrm{Cl}_X(V_{q_i}^1 \cap X)$$

is a member of $\mathfrak{U}$, and we easily have

$$p\in U\subset \mathrm{Cl}_X U\subset U', \quad U'\cap F\subset V,$$

which shows (a). Let $p\in F$ and $i\in N$. Then there is $B_k\in\mathfrak{B}$ such that $p\in B_k\subset \subset \mathrm{Cl}_{\gamma X}B_k\subset G_i$. Let V_1, V_2 be the sets in $\mathfrak{B}$ chosen above for the pair $\mathrm{Cl}_{\gamma X}B_k$, $\gamma X-G_i$ with $\mathrm{Cl}_{\gamma X}B_k\subset V_1$ and $\gamma X-G_i\subset V_2$. Then by putting $U_1=V_1\cap X$ and $U_2=X-\mathrm{Cl}_X(V_2\cap X)$ we easily have

$$p\in U_1\subset \mathrm{Cl}_X U_1\subset U_2\subset \mathrm{Cl}_{\gamma X}U_2\subset G_i.$$

Since each of U_1 and U_2 is a member of $\mathfrak{U}$, the condition (b) holds. This proves Lemma 2.4.

For any γ-open cover $\mathfrak{U}$ of a semi-compact space X let $\tilde{\mathfrak{U}}$ denote the collection $\{U|\ \tilde{U}\in\mathfrak{U}\}$, where $\tilde{U}=\gamma X-\mathrm{Cl}_{\gamma X}(X-U)$. By Theorem 2.1 $\tilde{\mathfrak{U}}$ is an open cover of γX.

LEMMA 2.5. *Let X and F be as in Lemma 2.4. Then there exists a sequence $\{\mathfrak{U}_n|\ n\in N\}$ of γ-open covers of X such that for any point p of F*

(c) *$\{\mathrm{St}(p,\mathfrak{U}_n)\cap F|\ n\in N\}$ is a local basis at p in F, and*

(d) *$\bigcap\{\mathrm{St}(p,\tilde{\mathfrak{U}}_n)|\ n\in N\}\subset X$.*

Proof. Let $\mathfrak{U}$ be the collection obtained in Lemma 2.4. Consider a pair of sets U, U' in $\mathfrak{U}$ such that $\mathrm{Cl}_X U\subset U'$. Then $\{U', X-\mathrm{Cl}_X U\}$ is a γ-open cover, and since the set of all such covers is countable, we write them as $\mathfrak{U}_1, \mathfrak{U}_2, \ldots$ Then the collection $\{\mathfrak{U}_n|\ n\in N\}$ satisfies (c), (d). Clearly (c) is satisfied by (a) in Lemma 2.4. Let $p\in F$ and $i\in N$. By (b) in Lemma 2.4, there are U, U' in $\mathfrak{U}$ such that $p\in U\subset \mathrm{Cl}_X U\subset U'\subset \mathrm{Cl}_{\gamma X}U'\subset G_i$. Then $\{U', X-\mathrm{Cl}_X U\}$ is some $\mathfrak{U}_{n_i}$, and we have

$$\mathrm{St}(p,\tilde{\mathfrak{U}}_{n_i})=\gamma X-\mathrm{Cl}_{\gamma X}(X-U')\subset \mathrm{Cl}_{\gamma X}U'\subset G_i.$$

Thus

$$\bigcap_n\{\mathrm{St}(p,\tilde{\mathfrak{U}}_n)|\ n\in N\}\subset \bigcap_i\{\mathrm{St}(p,\tilde{\mathfrak{U}}_{n_i})|\ i\in N\}\subset \bigcap_i G_i=X,$$

which shows (d). This proves Lemma 2.5.

LEMMA 2.6. *Let F be a closed set of a semi-compact space X. Then for any γ-open set G in X and any γ-open cover $\mathfrak{B}$ of X there exists a γ-open cover $\mathfrak{U}$ of X such that $\mathfrak{U}\cap F$ is a refinement of*

$$\{F\cap G, F-\mathrm{Cl}_X(F\cap G), F\cap V|\ V\cap \mathrm{Bd}_F(F\cap G)\neq\emptyset, V\in\mathfrak{B}\}.$$

Proof. Let G be a γ-open set and $\mathfrak{B}$ a γ-open cover of X. Let us put

$$K=F\cap \mathrm{Bd}_X G-\mathrm{St}(\mathrm{Bd}_F(F\cap G),\mathfrak{B}).$$

Then we have $K\subset G\cup(X-\mathrm{Cl}_X(F\cap G))$, and since K is compact there exist two γ-open sets U_1, U_2 of X such that

$$K\subset U_1\cup U_2, \quad U_1\subset G, \quad U_2\subset X-\mathrm{Cl}_X(F\cap G).$$

Let us put

$$H=U_1\cup U_2\cup \mathrm{St}(\mathrm{Bd}_F(F\cap G),\mathfrak{B}).$$

Then H is a γ-open set of X, and we have $(\mathrm{Gd}_X G-H)\cap F=\emptyset$. Since $\mathrm{Bd}_X G-H$ is compact and F closed, there exists a γ-open set U_3 of X such that

$$\mathrm{Bd}_X G-H\subset U_3 \quad\text{and}\quad U_3\cap F=\emptyset.$$

Now let us put

$$\mathfrak{U}=\{U_1, U_2, U_3, G, X-\mathrm{Cl}_X G, V|\ \mathrm{Bd}_F(F\cap G)\cap V\neq\emptyset,\ V\in\mathfrak{B}\}.$$

Then the construction above shows that $\mathfrak{U}$ is a γ-open cover of X and satisfies the desired property of the lemma. This completes the proof.

In concluding this section we shall state the following lemma given in Morita [2, IV, Theorem 2].

LEMMA 2.7. *Let Y be a space which is complete with respect to a uniformity $\{\mathfrak{U}_\alpha\}$ agreeing with the topology, and X a dense subspace of Y. Then there exists a homeomorphism φ from Y onto the completion X^* of X with respect to $\{\mathfrak{U}_\alpha\cap X\}$ such that $\varphi|X=$ the identity of X.*

§ 3. Proof of Theorem 1.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Assume that X is Čech-complete semi-compact and $R(X)$ separable metrizable. Note that $R(X)$ is closed in X. Hence, by Lemma 2.5, there exists a sequence $\{\mathfrak{U}_n|\ n\in N\}$ of γ-open covers of X such that for any point p of $R(X)$

(3) $\{\mathrm{St}(p,\mathfrak{U}_n)\cap R(X)|\ n\in N\}$ is a local basis at p in $R(X)$,

(4) $\bigcap\{\mathrm{St}(p,\tilde{\mathfrak{U}}_n)|\ n\in N\}\subset X$.

Moreover, by induction with the aid of Theorem 2.1 and Lemma 2.6, we can construct a sequence $\{\mathfrak{B}_n|\ n\in N\}$ of γ-open covers of X satisfying the following properties:

(5) $\mathfrak{B}_n$ is a star-refinement of $\mathfrak{B}_{n-1}$.

(6) $\mathfrak{B}_n$ is a refinement of $\mathfrak{U}_n$.

(7) $\mathfrak{B}_n\cap R(X)$ is a refinement of the families

$$\{V_{ij}\cap R(X),\ R(X)-\mathrm{Cl}_X(R(X)\cap V_{ij}),\ V\cap R(X)|\ V\cap \mathrm{Bd}_{R(X)}(R(X)\cap V_{ij})\neq\emptyset,\ V\in\mathfrak{B}_{n-1}\}$$

for $i=1,\ldots,n-1$; $j=1,\ldots,r_i$, where $\mathfrak{B}_i=\{V_{ij}|\ j=1,\ldots,r_i\}$.

Let us put

$$\mathfrak{A}_n=\{\tilde{V}\cap(\gamma X-X)|\ V\cap R(X)\neq\emptyset,\ V\in\mathfrak{B}_n\}.$$

By Theorem 2.1 each member of $\mathfrak{A}_n$ is open and closed in $\gamma X-X$. Let us put $A_n=(\gamma X-X)\cap \mathrm{St}(R(X),\tilde{\mathfrak{B}}_n)$. Then $(\gamma X-X)-A_n$ is open in $\gamma X-X$, and is compact since $X-R(X)$ is open in γX and $X\cup A_n=(X-R(X))\cup \mathrm{St}(R(X),\tilde{\mathfrak{V}}_n)$. Let us denote the members of $\mathfrak{A}_n$ by A_{ni}, $i=1,\ldots,s_n$. Let us put $D_0=(\gamma X-X)-A_1$ and for $n\in N$

$$D_{n1}=A_{n1}-A_{n+1},\qquad D_{ni}=(A_{ni}-\bigcup_{j<i}A_{nj})-A_{n+1},\qquad 1<i\leqslant s_n .$$

Then $\{D_0, D_{ni}|\ n\in N,\ 1\leqslant i\leqslant s_n\}$ is a disjoint family of open compact subsets of $\gamma X-X$. Moreover, since

$$D_{ni}\cap \mathrm{St}(R(X),\tilde{\mathfrak{V}}_{n+1})=\emptyset \quad\text{and}\quad \mathrm{Cl}_{\gamma X}R(X)\subset \mathrm{St}(R(X),\tilde{\mathfrak{V}}_{n+1}),$$

D_{ni} is open also in $(\gamma X-X)\cup R(X)$. Similarly D_0 is open in $(\gamma X-X)\cup R(X)$.

Let us put

$$Z=(\gamma X-X)\cup R(X)$$

and

$$\mathfrak{W}_1=\{D_0,\tilde{V}\cap Z|\ V\cap R(X)\neq\emptyset,\ V\in\mathfrak{V}_1\},$$
$$\mathfrak{W}_n=\{D_0, D_{kj}|\ 1\leqslant k<n,\ j=1,\ldots,s_k\}\cup$$
$$\cup\{\tilde{V}\cap Z|\ V\cap R(X)\neq\emptyset,\ V\in\mathfrak{V}_n\}\quad \text{for}\quad n>1.$$

Then Z is a compact subspace of γX and by the construction above each $\mathfrak{W}_n$ is an open cover of Z for $n\in N$. Since $\tilde{\mathfrak{V}}_{n+1}$ is a star-refinement of $\tilde{\mathfrak{V}}_n$ by (5), we have

(8) $\mathfrak{W}_{n+1}$ is a star-refinement of $\mathfrak{W}_n$.

Therefore $\Phi=\{\mathfrak{W}_n|\ n\in N\}$ is a normal sequence of open covers of Z.

Here we note that the following properties are satisfied.

(9) $\bigcap_n \mathrm{St}(R(X),\mathfrak{W}_n)=Z-D_0\cup\bigcup\{D_{ni}|\ n\in N,\ 1\leqslant i\leqslant s_n\}$ and $\mathrm{St}(D,\mathfrak{W}_n)=D$, where $D=D_0$ or D_{kj}, for $n\in N$.

(10) $\mathrm{St}(p,\mathfrak{W}_n)=\mathrm{St}(p,\tilde{\mathfrak{V}}_n)\cap Z$ for each point p of $\mathrm{Cl}_{\gamma X}R(X)$.

(11) $\{\mathrm{St}(p,\mathfrak{W}_n)|\ n\in N\}$ forms a local basis at p in Z for any point p of $R(X)$.

(12) $\bigcap_n \mathrm{St}(R(X),\mathfrak{W}_n)=\bigcap_n \mathrm{St}(R(X),\tilde{\mathfrak{V}}_n)\cap Z=\bigcup\{\bigcap_n \mathrm{St}(p,\mathfrak{W}_n)|\ p\in \mathrm{Cl}_{\gamma X}R(X)\}$.

Indeed, (9) and (10) are obvious. Since Z is compact, in view of (3), (4), (6) and (10) we can verify that (11) is satisfied. For (12) it should be noted further that

$$\bigcap_n \mathrm{St}(R(X),\tilde{\mathfrak{V}}_n)=\bigcup\{\bigcap_n \mathrm{St}(p,\tilde{\mathfrak{V}}_n)|\ p\in \mathrm{Cl}_{\gamma X}R(X)\}.$$

Now we shall apply the arguments in Morita [4] to Z and Φ: Let (Z,Φ) be a topological space obtained from Z by taking $\{\mathrm{St}(p,\mathfrak{W}_n)|\ n\in N\}$ as a local basis at each point p of Z, and Z/Φ the quotient space obtained from (Z,Φ) by identifying two points p and q such that $q\in \mathrm{St}(p,\mathfrak{W}_n)$ for each $n\in N$. Let us denote by φ the

composite of the identity map from Z onto (Z,Φ) and the quotient map from (Z,Φ) onto Z/Φ. Then $\varphi\colon Z\to Z/\Phi$ is continuous. For any set A of Z let us put

$$\mathrm{Int}(A;\Phi)=\{p\in Z|\ \mathrm{St}(p,\mathfrak{W}_n)\subset A \text{ for some } n\},$$
$$\mathrm{Int}(\mathfrak{W}_n;\Phi)=\{\mathrm{Int}(W;\Phi)|\ W\in\mathfrak{W}_n\}.$$

Then we have (13) and (14) below.

(13) $$\varphi^{-1}\varphi(\mathrm{Int}(A;\Phi))=\mathrm{Int}(A;\Phi).$$

(14) $\Psi=\{\varphi(\mathrm{Int}(\mathfrak{W}_n;\Phi))|\ n\in N\}$ is a normal sequence of open covers of Z/Φ, which defines a uniformity of Z/Φ agreeing with the topology.

Therefore Z/Φ is metrizable. These results are proved in [4].

By the construction of Z/Φ and (11), we readily have that the map $\varphi|R(X)\colon R(X)\to\varphi(R(X))$ is one-to-one, and

$$\varphi^{-1}\varphi(R(X))=R(X),\qquad \mathrm{Int}(\mathfrak{W}_n;\Phi)\cap R(X)=\mathfrak{V}_n\cap R(X).$$

Therefore by (13) we have

(15) $$\varphi^{-1}(\varphi(\mathrm{Int}(\mathfrak{W}_n;\Phi))\cap\varphi(R(X)))=\mathfrak{V}_n\cap R(X).$$

Since $\{\mathfrak{V}_n\cap R(X)|\ n\in N\}$ defines a uniformity of $R(X)$ by (3), (5) and (6), by (15) the map $\varphi|R(X)$ is a uniformly homeomorphism between $R(X)$ and $\varphi(R(X))$ when we regard $R(X)$ as a uniform space with the uniformity $\{\mathfrak{V}_n\cap R(X)|\ n\in N\}$ and $\varphi(R(X))$ as a uniform space with the uniformity obtained by restricting Φ to $\varphi(R(X))$. Since Z/Φ is compact, $\mathrm{Cl}_{Z/\Phi}\varphi(R(X))$ is also compact. Hence, in view of Lemma 2.7, $\varphi|R(X)$ is extended to a homeomorphism h from $\delta R(X)$ onto $\mathrm{Cl}_{Z/\Phi}\varphi(R(X))$, where $\delta R(X)$ denotes the completion of $R(X)$ with respect to $\{\mathfrak{V}_n\cap R(X)|\ n\in N\}$.

On the other hand, in view of (7), $\{\mathfrak{V}_n\cap R(X)|\ n\in N\}$ is a uniformity of γ-open covers of $R(X)$ satisfying the same properties as those of Γ_0 in Theorem 2.2. Therefore $\delta R(X)$ is a metrizable compactification of $R(X)$ having the property (*) in Theorem 2.1, where X beeing replaced by $R(X)$. Hence by Theorem 2.3 there exist a compactification $\alpha R(X)$ of $R(X)$ with $\mathrm{Card}(\alpha R(X)-R(X))\leqslant\aleph_0$ and a continuous map η from $\delta R(X)$ onto $\alpha R(X)$ such that

$$\eta|R(X)=\text{the identity of } R(X).$$

Let us put

$$\xi=\eta\circ h^{-1}\colon\ \mathrm{Cl}_{Z/\Phi}\varphi(R(X))\to\delta R(X)\to\alpha R(X).$$

Then we have

$$\xi|\varphi(R(X))=(\varphi|R(X))^{-1}\colon\ \varphi(R(X))\to R(X),$$
$$\xi^{-1}(R(X))=\varphi(R(X)).$$

Let us put

$$\mathfrak{D}_1=\{\{q\}|\ q\in Z/\Phi-\mathrm{Cl}_{Z/\Phi}\varphi(R(X))\}\cup\{\xi^{-1}(p)|\ p\in\alpha R(X)\}.$$

Then $\mathfrak{D}_1$ is an upper semi-continuous decomposition of Z/Φ since ξ is a closed map and $\mathrm{Cl}_{Z/\Phi}\varphi(R(X))$ is closed in Z/Φ. Let S be the quotient space and $\tilde{\xi}\colon Z/\Phi\to S$

the quotient map. That is, S is the adjunction space $Z/\Phi \bigcup_{\xi} \alpha R(X)$. Then as is easily seen $\alpha R(X)$ is embedded as a closed subspace in S and $\tilde{\xi}$ is an extension of ξ such that $\tilde{\xi}$ yields a homeomorphism between $Z/\Phi - \mathrm{Cl}_{Z/\Phi}\varphi(R(X))$ and $S-\alpha R(X)$, and $\tilde{\xi}^{-1}(\alpha R(X)) = \mathrm{Cl}_{Z/\Phi}\varphi(R(X))$ holds.

Now let us show that $\mathrm{Card}(S-R(X)) \leqslant \aleph_0$. By (12) we have

$$\bigcap_n \mathrm{St}(R(X), \mathfrak{W}_n) = \varphi^{-1}(\mathrm{Cl}_{Z/\Phi}\varphi(R(X)))$$

since

$$\bigcap_n \mathrm{St}(R(X), \mathfrak{W}_n) = \varphi^{-1}\varphi(\mathrm{Cl}_{\gamma X} R(X)) = \varphi^{-1}\varphi(\mathrm{Cl}_Z R(X)) = \varphi^{-1}(\mathrm{Cl}_{Z/\Phi}\varphi(R(X))).$$

On the other hand, by (9) $\varphi(D)$ is a single point of Z/Φ, where $D = D_0$ or D_{kj}, and $Z/\Phi = \varphi(\bigcap_n \mathrm{St}(R(X), \mathfrak{W}_n)) \cup \{\varphi(D)|\ D = D_0 \text{ or } D_{kj}\}$. Hence we have $\mathrm{Card}(Z/\Phi - \mathrm{Cl}_{Z/\Phi}\varphi(R(X)) \leqslant \aleph_0$. Therefore $\mathrm{Card}(S-\alpha R(X)) \leqslant \aleph_0$. Since

$$\mathrm{Card}(\alpha R(X) - R(X)) \leqslant \aleph_0,$$

$$\mathrm{Card}(S-R(X)) = \mathrm{Card}((S-\alpha R(X)) \cup (\alpha R(X) - R(X))) \leqslant \aleph_0.$$

Thus $\mathrm{Card}(S-R(X)) \leqslant \aleph_0$.

Let us put

$$f = \tilde{\xi}\circ\varphi\colon Z \to Z/\Phi \to S.$$

Then we have

$$f/R(X) = \text{the identity of } R(X),$$

and since $\varphi^{-1}\varphi(R(X)) = R(X)$, $\tilde{\xi}^{-1}(\alpha R(X)) = \mathrm{Cl}_{Z/\Phi}\varphi(R(X))$ and $\xi^{-1}R(X) = \varphi(R(X))$, we have

$$f^{-1}R(X) = R(X).$$

Let us put

$$\mathfrak{D}_2 = \{\{x\}\ x \in \gamma X - Z\} \cup \{f^{-1}(s)|\ s \in S\}.$$

Since f is a closed map and Z is closed in γX, applying the same argument as above to $\mathfrak{D}_2$, we can construct a space T containing S as a closed subspace and a continuous map $\tilde{f}$ from γX onto T so that

$$\tilde{f}|Z = f\colon Z \to S, \quad \tilde{f}^{-1}(S) = Z,$$

and $\tilde{f}$ maps $\gamma X - Z = X - R(X)$ homeomorphically onto $T-S$. Then we have

$$\tilde{f}|R(X) = f|R(X) = \text{the identity of } R(X),$$
$$\tilde{f}^{-1}R(X) = f^{-1}R(X) = R(X),$$
$$\tilde{f}^{-1}\tilde{f}(X-R(X)) = X-R(X).$$

Therefore the map $\tilde{f}|X\colon X \to \tilde{f}(X)$ is bijective, and we have $\tilde{f}^{-1}\tilde{f}(X) = X$. Hence $\tilde{f}|X$ is a homeomorphism from X onto $\tilde{f}(X)$, since $\tilde{f}$ is a closed map.

On the other hand, by the continuity of $\tilde{f}$ T is compact Hausdorff and $\tilde{f}(X)$ is dense in T. Therefore we can now regard T as a compactification of X. Moreover, since $T-\varphi(X) = S-R(X)$, we have

$$\mathrm{Card}(T-\varphi(X)) \leqslant \aleph_0.$$

That is, T is a countable-points compactification of X. Thus X admits a countable-points compactification, and our Theorem 1 is now completely proved.

Remark. Theorem 1 was proved by T. Terada [6] in case $R(X)$ is further assumed to be compact. Then his result was extended by K. Morita to the case that $R(X)$ is separable metrizable having the compact boundary in X.

§ 4. Proof of Theorem 2.

LEMMA 4.1. *If a paracompact space X has a countable-points compactification, then $R(X)$ is Lindelöf.*

Proof. Let αX be a countable-points compactification of X. Let $\mathfrak{U}$ be an open cover of $R(X)$. Since X is paracompact and $R(X)$ closed in X, there exists a normal open cover $\mathfrak{V}$ of X such that $\mathfrak{V} \cap R(X)$ refines $\mathfrak{U}$. Since $\mathfrak{V}$ is normal, $\mathfrak{V}$ is refined by a σ-discrete open cover $\mathfrak{B} = \bigcup \mathfrak{B}_n$ of X such that each $\mathfrak{B}_n$ is discrete. Let $\mathfrak{B}'_n = \{B|\ B \cap R(X) \neq \emptyset, B \in \mathfrak{B}_n\}$. Take an open set $\tilde{B}$ of αX so that $\tilde{B} \cap X = B$ for $B \in \mathfrak{B}'_n$. Then since X is dense in αX, each member of $\{\tilde{B}|\ B \in \mathfrak{B}'_n\}$ is mutually disjoint. Moreover, for $B \in \mathfrak{B}'_n$ we have $\tilde{B} - X \neq \emptyset$ since $B \cap R(X) \neq \emptyset$. Therefore the cardinality of $\mathfrak{B}'_n$ is at most countable since $\mathrm{Card}(\alpha X - X) \leqslant \aleph_0$. Since $\bigcup \mathfrak{B}'_n$ covers $R(X)$ and $(\bigcup \mathfrak{B}'_n) \cap R(X)$ refines $\mathfrak{U}$, $\mathfrak{U}$ has a countable subcover. Hence $R(X)$ is Lindelöf. This completes the proof.

Proof of Theorem 2. Theorem 2 now immediately follows from Theorem 1 and Lemma 4.1.

EXAMPLE 4.2. Let S be the topological sum of an uncountable number of copies of the space of irrationals. Then S is Čech-complete semi-compact metrizable. Since $R(S) = S$ is not Lindelöf, Lemma 4.1 shows that S has no countable-points compactification.

Remark. As a necessary condition for a space X to have a countable-points compactification we have that the cardinality of any collection $\{U_\alpha|\ \alpha \in \Omega\}$ of disjoint open sets with $U_\alpha \cap R(X) \neq \emptyset$ for $\alpha \in \Omega$ is at most countable. This is shown in the proof of Lemma 4.1. Clearly the space S above does not satisfy either this condition. This condition was observed for the first time by the author who communicated it to T. Terada together with Example 4.2 above; this condition was utilized also by T. Terada [6] in the construction of his examples.

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Chaînes de théories universelles

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Résumé. Une théorie universelle T (théorie engendrée par des énoncés universels) est dite *irréductible* lorsque T n'est pas l'intersection de deux théories universelles distinctes de T. Nous prouvons que étant données deux théories universelles T et T' d'un langage comportant au plus des prédicats et des constantes, si toute chaîne (pour l'inclusion) de théories universelles irréductibles comprises entre T et T' est au plus dénombrable alors toute suite croissante de telles théories est stationnaire.

Introduction

Divers auteurs ont étudié (ou seulement utilisé) des conditions de chaînes analogues à celles intervenant dans la théorie classique des idéaux, mais concernant certains ensembles d'énoncés ou certaines classes de structures. C'est le cas notamment de A. Robinson (qui les étudie d'un point de vue essentiellement algébrique, voir par exemple les chapitres VII et VIII de [20]) de A. Malcev (qui les utilise de façon implicite, voir le chapitre 33 de [14]) de R. Fraïssé (dont l'intérêt pour ces questions est lié à sa notion d'abritement, voir [5] et le chapitre 3 de [4]) et, pour des études particulières, de G. Higman, [10] J. B. Kruskal [11] C.S^t.J.A. Nash-Williams [16], R. Laver [13].

Il nous a paru intéressant d'entreprendre une étude systématique des conditions de chaînes portant sur les théories universelles (ce cadre d'apparence limité suffisant à exprimer l'essentiel des résultats connus) et plus particulièrement sur celles dont le langage ne comporte pas de fonctions (le cas général nous semblant actuellement trop difficile) avec comme premier objectif une classification de ces théories (en connexion avec le programme suggéré par A. Malcev, voir chapitre 34 § 2 de [14]); Ceci compte tenu d'autres applications, par exemples à des problèmes de décidabilité voir [9] ou d'axiomatisabilité liés à la définissabilité voir [17].

Dans ce texte nous prouvons essentiellement le résultat suivant.

THÉORÈME. *Etant données deux théories universelles T et T' d'un langage comportant au plus des prédicats et des constantes si toute chaîne (pour l'inclusion) de théories universelles irréductibles comprises entre T et T' est au plus dénombrable alors toute suite croissante de telles théories est stationnaire.*