Define

$$\mathcal{U} = \{Q \cap (\bigcup \mathcal{U}_\delta) |\ \delta \in \Delta\}, \quad \mathcal{V}_\gamma = \{Q \cap V_\gamma(k) |\ k \in \omega\} \quad \text{if} \quad \gamma \in \Gamma,$$
$$\underline{V} = \{\mathcal{V}_\gamma |\ \gamma \in \Gamma\}.$$

We claim that $\mathcal{U}$ and $\underline{V}$ satisfy (a), (b) and (c) of (3).

Check of (a). Let D be a closed discrete subset of Q. D is closed discrete in X. Since X has property D, one can use (α) and (γ) to find $\delta \in \Delta$ with $D \subset \bigcup \mathcal{U}_\delta$.

Check of (b). Let $\mathcal{W}$ be an open (in Q) cover of Q. Then there is a $\gamma \in \Gamma$ such that $\{Q \cap B |\ B \in \mathcal{B}_\gamma\}$ refines $\mathcal{W}$. But then also $\mathcal{V}_\gamma$ refines $\mathcal{W}$.

Check of (c). Let $\gamma \in \Gamma$ and $\delta \in \Delta$ be arbitrary. It follows from (ε) that there is an $n \in \omega$ such that $y_\gamma(k) \notin \bigcup \mathcal{U}_\delta$ for $k \geqslant n$. Fix $k \geqslant n$. We will show that $Q \cap V_\gamma(k) \not\subset \bigcup \mathcal{U}_\delta$. Suppose the contrary. Since $V_\gamma(k) \in \mathcal{B}$, it follows from ($\alpha$) and ($\varepsilon$) that $Q \cap V_\gamma(k) \subset Q \cap U$ for some $U \in \mathcal{U}_\delta$. But $U \in \mathcal{B}$, so $V_\gamma(k) \subset U$, by (β). This leads to the contradiction that $y_\gamma(k) \in \bigcup \mathcal{U}_\delta$. ■

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Effective bounds on Morley rank

by

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To the Memory of Andrzej Mostowski

Abstract. Effective bounds are obtained on the Morley rank of a 1-type, the Morley rank a_T of a totally transcendental theory T, and the Blum density number d_T of a quasi-totally transcendental theory T by means of type-omitting and absoluteness arguments over admissible sets. The above restrictions on T imply that a_T and d_T are ordinals recursive in T. Every theory T is seen to have a universal domain hyperarithmetic in the hyperjump of T.

1. Introduction. This paper might better have been titled: On the Absolute Character of the Morley Derivative. For the bounds given below on Morley rank, and on Blum's density number for quasi-totally transcendental theories, are derived from some absoluteness properties of Morley's analysis of 1-types. Let T be a countable theory of first order logic. Assume T is complete and substructure complete (¹) in order to smooth the application of Morley's rank-and-degree machine to T. (The details of his machine will be reviewed in Section 2. A full account was given in [12].) Suppose $\mathcal{A}$ is a substructure of a model of T, and p is a 1-type over $\mathcal{A}$ (in symbols $\mathcal{A} \in \mathcal{K}(T)$ and $p \in S\mathcal{A}$). If p has a Morley rank, then that rank is denoted by $r_M(p)$, and the existence of that rank is indicated by the inequality: $r_M(p) < \infty$.

Let N be an admissible set as defined in Section 2. Assume T and $\mathcal{A}$ belong to N. One aspect of the absoluteness of the Morley derivative is expressed by:

(1) $$r_M(p) = \beta < \infty \rightarrow p \in N \,\&\, \beta \in N.$$

Another aspect is the fact that the relation

$$p \in S\mathcal{A} \,\&\, r_M(p) \leqslant \beta$$

is Σ_1 over N. (1) implies a bound first obtained by Lachlan [8]:

(2) $$r_M(p) < \infty \rightarrow r_M(p) < \aleph_1,$$

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(¹) Substructure completeness is equivalent to admitting elimination of quantifiers.

since every Morley rank achieved by some p over some $\mathscr{A}$ is also achieved by some p over some countable $\mathscr{A}$, and since each countable $\mathscr{A}$ belongs to some countable N.

The notion of "next" admissible set, as in [1, p. 60], is needed for an effective version of (2). Let $\mathscr{A}^+$ be the least admissible set N such that $\mathscr{A} \in N$. Define $\omega_1^{\mathscr{A}}$ to be the least ordinal not in $\mathscr{A}^+$. An effective version of (2) is:

(3) $$p \in S\mathscr{A} \ \&\ r_M(p) < \infty \to p \in \mathscr{A}^+ \ \&\ r_M(p) < \omega_1^{\mathscr{A}}.$$

If $\mathscr{A}$ is countable, then there exists a real number C such that $\mathscr{A}$ is coded by C and $\omega_1^C = \omega_1^{\mathscr{A}}$. (One such C is in essence a counting of $\mathscr{A}$ generic over $\mathscr{A}^+$.) The conclusion of (3) now becomes: p is (i.e. coded by a real) hyperarithmetic in $\mathscr{A}$, and $r_M(p)$ is an ordinal recursive in $\mathscr{A}$.

The proof of (3), given in Section 3, is in the main an omitting types argument over a countable admissible fragment of $L_{\omega_1,\omega}$. The argument succeeds thanks to some weak absoluteness properties of the Morley derivative extracted in Section 2 from the strong absoluteness of the Cantor–Bendixson derivative. The notion of universal domain also plays a part in the proof of (3). A universal domain $\mathscr{U}$ has the advantage that Morley rank equals Cantor–Bendixson rank for all $p \in S\mathscr{U}$. (3) is applied in Section 5 to prove the existence of a countable universal domain $\mathscr{U}$ for T with most of the effort directed towards holding down the ordinal needed to construct $\mathscr{U}$ from T.

As a further application of (3), a bound on d_T, Blum's density number, is obtained in Section 4. Blum [2] calls T quasi-totally transcendental (q.-t.t.) if the Morley ranked points of $S\mathscr{A}$ are dense in $S\mathscr{A}$ for every $\mathscr{A} \in \mathscr{K}(T)$. If T is q.-t.t., then d_T is defined to be the least β such that for all $\mathscr{A} \in \mathscr{K}(T)$, the points of $S\mathscr{A}$ of Morley rank less than β are dense in $S\mathscr{A}$. It will be shown that $d_T < \aleph_1$ by showing $d_T < \omega_1^T$. (If T is coded by a real, e.g. the set of Gödel numbers of axioms of T, then ω_1^T is the least ordinal not recursive in the code for T.) α_T, the Morley rank of T, defined for all T, is the least β such that for all $\mathscr{A} \in \mathscr{K}(T)$, every Morley ranked point of $S\mathscr{A}$ has rank less than β. L. Harrington has found a quasi-totally transcendental T such that $\alpha_T = \aleph_1$. In Section 4 it is also shown that $\alpha_T < \omega_1^T$ when T is totally transcendental.

A. Kechris [5] has pointed out to the author that some of the results of this paper can be derived from the work of Blass and Cenzer (cf. [3]) on monotonic Π_1^1 inductive definitions of sets of reals. Kechris's approach is a striking example of the power of generalized recursion theory. Yet another approach has been developed by Harnik and Makkai [4] via Vaught sentences.

Some open questions on the absoluteness of the Morley derivative are listed in Section 6.

2. Rank. Notation and definitions are taken from [12]. $\mathscr{K}(T)$ is the category of all substructures of all models of T. $f\colon \mathscr{A} \to \mathscr{B}$ is a monomorphism of $\mathscr{A}$ into $\mathscr{B}$. $S\mathscr{A}$ is the Stone space of 1-types over $\mathscr{A}$. A typical basic open subset of $S\mathscr{A}$ is

$$\{p \mid F(\underline{a}, x) \in p\},$$

where $F(\underline{a}, x)$ is a formula in the language of T with parameters $\underline{a}$ in A, the universe of $\mathscr{A}$.

$Sf\colon S\mathscr{B} \to S\mathscr{A}$ is the continuous onto map induced by f. If $q \in S\mathscr{B}$, then

$$F(\underline{a}, x) \in (Sf)q \leftrightarrow F(f\underline{a}, x) \in q.$$

S is a contravariant, limit preserving functor from $\mathscr{K}(T)$ into the category of Stone spaces.

DS is the Morley derivative of S. $p \in DS\mathscr{A}$ iff $p \in S\mathscr{A}$ and there exists an $f\colon \mathscr{A} \to \mathscr{B}$ in $\mathscr{K}(T)$ such that $(Sf)^{-1}(p)$ contains a limit point of $S\mathscr{B}$. DSf is the restriction of Sf to $DS\mathscr{B}$, and maps $DS\mathscr{B}$ onto $DS\mathscr{A}$.

$D^\beta S$, the βth Morley derivative of S, is defined by transfinite recursion. $D^0 S = S$, $D^{\beta+1} S = D(D^\beta S)$, and

$$D^\lambda S\mathscr{A} = \bigcap \{D^\beta S\mathscr{A} \mid \beta < \lambda\}$$

when λ is a limit ordinal. $D^\beta S$ is a contravariant, limit preserving functor from $\mathscr{K}(T)$ into the category of Stone spaces. If there is a β (necessarily unique) such that

$$p \in D^\beta S\mathscr{A} - D^{\beta+1} S\mathscr{A},$$

then p is said to be a *Morley ranked point of* $S\mathscr{A}$ of Morley β.

$dS\mathscr{A}$, the Cantor–Bendixson derivative of $S\mathscr{A}$, is the set of all limit points of $S\mathscr{A}$. $d^\beta S\mathscr{A}$ is defined by transfinite recursion. $d^0 S\mathscr{A} = S\mathscr{A}$, $d^{\beta+1} S\mathscr{A} = dd^\beta S\mathscr{A}$, and

$$d^\lambda S\mathscr{A} = \bigcap \{d^\beta S\mathscr{A} \mid \beta < \lambda\}.$$

If there is a β such that $p \in d^\beta S\mathscr{A} - d^{\beta+1} S\mathscr{A}$, then p is said to be a *Cantor–Bendixson ranked point of* $S\mathscr{A}$ of Cantor–Bendixson rank β. The latter state of affairs is indicated by: $r_{CB}(p) = \beta < \infty$.

Proposition 2.1. $r_{CB}(p) \leqslant r_M(p)$.

Proof. It suffices to show $d^\beta S\mathscr{A} \subset D^\beta S\mathscr{A}$ for all β by transfinite induction. The latter is straightforward, because the identity monomorphism $1_A\colon \mathscr{A} \to \mathscr{A}$ belongs to $\mathscr{K}(T)$. ∎

Suppose $\mathscr{U}, \mathscr{V}, \mathscr{W} \in \mathscr{K}(T)$. $\mathscr{V}$ is finitely generated if there exists a finite $Y \subset V$ such that $\mathscr{V}$ is the least substructure of $\mathscr{V}$ whose universe contains Y. Suppose $p \in D^\beta S\mathscr{V}$ and $i\colon \mathscr{V} \subset \mathscr{W}$ is an inclusion. p is said to *split* in $D^\beta S\mathscr{W}$ if $(D^\beta Si)^{-1}(p)$ has at least two members.

$\mathscr{U}$ is an α-universal domain for T if for all $\beta < \alpha$, all finitely generated $\mathscr{V} \subset \mathscr{U}$, and all isolated $p \in D^\beta S\mathscr{V}$, the following holds: if p splits in $D^\beta S\mathscr{W}$ for some $\mathscr{W} \in \mathscr{K}(T)$, then p splits in $D^\beta S\mathscr{W}_0$ for some finitely generated $\mathscr{W}_0 \subset \mathscr{U}$.

$\mathscr{U}$ is a universal domain for T if $\mathscr{U}$ is an α-universal domain for T for all α.

Lemma 2.2. *Suppose* $\mathscr{U}$ *is an* α*-universal domain for* T. *Then* $D^\beta S\mathscr{U} = d^\beta S\mathscr{U}$ *for all* $\beta < \alpha$.

Proof. By induction of β. For details, see the proof of Lemma 31.3 on page 190 of [12]. The point to remember is that $D^\beta S$ preserves limits. Let $\{\mathscr{V}_i\}$ be the direct system of all finitely generated substructures of $\mathscr{U}$. Then $D^\beta S\mathscr{U}$ is the inverse limit of the inverse system $\{D^\beta S\mathscr{V}_i\}$. ■

The existence of a countable universal domain for T is implied by Proposition 31.2 of [12]. Unfortunately there is a gap in the proof of 31.2. It will be filled in Section 5 by the type omitting argument of Section 3. The next proposition retains all that is correct in the proof of 31.2. It will be needed in Sections 3 and 4. card is an abbreviation for cardinality.

PROPOSITION 2.3. *Suppose* $\mathscr{A} \in \mathscr{K}(T)$ *is infinite and* $\mathrm{card}\,\alpha \leqslant \mathrm{card}\,\mathscr{A}$. *Then there exists a* $\mathscr{U} \supset \mathscr{A}$ *such that* $\mathrm{card}\,\mathscr{U} = \mathrm{card}\,\mathscr{A}$ *and* $\mathscr{U}$ *is an* α*-universal domain.*

Proof. A chain $\{\mathscr{U}_n \mid n<\omega\}$ is defined by recursion. $\mathscr{U}$ will be $\bigcup\{\mathscr{U}_n \mid n<\omega\}$.

1. $\mathscr{U}_0 = \mathscr{A}$.
2. For each $\beta<\alpha$, each finitely generated $\mathscr{V} \subset \mathscr{U}_n$, and each isolated $p \in D^\beta S\mathscr{V}$, choose a $\mathscr{W}^{\mathscr{V}}_{\beta,p}$ finitely generated over $\mathscr{V}$ with the following property: if p splits in $D^\beta S\mathscr{W}$ for some $\mathscr{W} \in \mathscr{K}(T)$, then p splits in $D^\beta S\mathscr{W}^{\mathscr{V}}_{\beta,p}$. The existence of $\mathscr{W}^{\mathscr{V}}_{\beta,p}$ follows from the limit preserving property of $D^\beta S$ described in the proof of Lemma 2.2. Let $\mathscr{U}_{n+1}$ be a model of T which extends $\mathscr{U}_n$ and every $\mathscr{W}^{\mathscr{V}}_{\beta,p}$, and which has the least possible cardinality.

$\mathscr{U}$ is α-universal, since each finitely generated substructure of $\mathscr{U}$ is contained in some $\mathscr{U}_n$. Assume $\mathrm{card}\,\mathscr{U}_n = \mathrm{card}\,\mathscr{A}$ in order to see that $\mathrm{card}\,\mathscr{U}_{n+1} = \mathrm{card}\,\mathscr{A}$. The set of all finitely generated $\mathscr{V} \subset \mathscr{A}$ has cardinality at most that of $\mathscr{A}$. For each such $\mathscr{V}$, $S\mathscr{V}$ has a countable base for its topology, hence only countably many isolated points. ■

A set N is said to be admissible (cf. Barwise [1]) if it is transitive, closed under the operations of pairing and set union, and satisfies the axiom schemes of Δ_0 separation and collection. Every admissible set N in this paper also satisfies the axiom of infinity: $\omega \in N$. Lemma 2.4 sums up the strong absoluteness properties of the Cantor–Bendixson derivative alluded to in Section 1. It is not known if the Morley derivative is equally absolute. 2.4 is inspired by an early result of Kreisel [7] to the effect that the recursive ordinals, and none less, suffice for the Cantor–Bendixson analysis of a closed, bounded Σ^1_1 set of reals.

LEMMA 2.4 (strong absoluteness of the Cantor–Bendixson derivative). *Let* N *be an admissible set and* $\mathscr{A}$ *a substructure of a model of* T. *If* $\mathscr{A} \in N$, *then* (i) *and* (ii) *are equivalent for all* p *and all* $\beta<\infty$.

(i) $p \in S\mathscr{A}$ & $r_{\mathrm{CB}}(p) = \beta$.

(ii) $p \in N$ & $\beta \in N$ & $N \models [p \in S\mathscr{A}$ & $r_{\mathrm{CB}}(p) = \beta]$.

In addition all values of the function g, *defined by*

$$g(\beta) = \{p \mid p \in S\mathscr{A} \ \&\ r_{\mathrm{CB}}(p) = \beta\}\,,$$

belong to N, *and* g *is* Δ_1 *over* N.

Proof. Since $\mathscr{A}$ belongs to N, its complete first order theory also belongs to N. That theory is given by $T \cup \mathrm{Di}(\mathscr{A})$, where $\mathrm{Di}(\mathscr{A})$ is the diagram of $\mathscr{A}$, since it was assumed in Section 1 that T is substructure complete. A typical $p \in S\mathscr{A}$ is a consistent set of formulas of the form $F(x)$ with parameters in A. "Consistent" means consistent with respect to $T \cup \mathrm{Di}(\mathscr{A})$, and has the same meaning inside N that it has outside N. $F(x)$ has a Cantor–Bendixson rank $\leqslant\infty$ defined by

$$r_{\mathrm{CB}}(F(x)) = \sup\{r_{\mathrm{CB}}(p) \mid F(x) \in p\}\,.$$

Part I of the proof is an induction on β intended to show the equivalence of (i) and (ii) for all $\beta \in N$. The induction is performed simultaneously with a recursion on β designed to show that the relation

$$r_{\mathrm{CB}}(F(x))<\beta \tag{1}$$

is Δ_1 over N. Part II of the proof will show that $r_{\mathrm{CB}}(p)<\infty$ implies $r_{\mathrm{CB}}(p) \in N$.

Part I. Fix $\beta \in N$ and suppose (i) holds. Let $G(x)$ isolate p in $d^\beta S\mathscr{A}$. Thus $G(x) \in p$, and for all q,

$$G(x) \in q \ \to\ q = p \vee r_{\mathrm{CB}}(q)<\beta\,. \tag{2}$$

Then p is generated by the set $Z(G)$ defined by

$$\{G(x)\} \cup \{\sim H(x) \mid r_{\mathrm{CB}}(H(x))<\beta\}\,; \tag{3}$$

that is p is the unique 1-type containing all of $Z(G)$. By induction (1) is Δ_1 over N, so $Z(G)$, and hence p, are members of N. Also by induction, $r_{\mathrm{CB}}(p) \geqslant \beta$ and (2) are true in N. Consequently (ii) holds.

Now suppose (ii) holds. Then p is generated in N by a set $Z(G)$ as in (3). Since the concept of first order logical consequence is Δ_0, $Z(G)$ generates p outside N. By induction $Z(G)$ insures that $r_{\mathrm{CB}}(p) = \beta$.

To see that $r_{\mathrm{CB}}(F(x)) \leqslant \beta$ is Δ_1, observe that $r_{\mathrm{CB}}(F(x)) = \beta$ if and only if there is at least one p, but not infinitely many, such that $F(x) \in p$ and p is generated by some $Z(G)$ as in (3). It is a Δ_0 matter to decide if $Z(G)$ is consistent and complete (that is generates a 1-type), and if two $Z(G)$'s generate the same 1-type.

Part II. Let α be the least ordinal not in N. Suppose $p \in S\mathscr{A}$ and $r_{\mathrm{CB}}(p) \leqslant \alpha$. Then there is a $G(x)$ such that

$$\{G(x)\} \cup \{\sim H(x) \mid r_{\mathrm{CB}}(H(x))<r_{\mathrm{CB}}(p)\} \tag{4}$$

generates p. Since (1) is Δ_1; it follows that (4) belongs to N, and is equal to (3) for some $\beta<\alpha$. But then $r_{\mathrm{CB}}(p)<\alpha$.

Each p of Cantor–Bendixson rank β is generated by some $Z(G)$ as in (3). The Δ_1-ness of (1) over N implies that $g(\beta) \in N$, and that g is Δ_1 over N. ■

COROLLARY 2.5. *If* $\mathscr{A}$ *is countable,* $p \in S\mathscr{A}$ *and* $r_{\mathrm{CB}}(p)<\infty$, *then* p *is hyperarithmetic in* $\mathscr{A}$ *and* $r_{\mathrm{CB}}(p)<\omega_1^{\mathscr{A}}$ (cf. Kreisel [7]).

Proof. Let N be $\mathscr{A}^+$, the least admissible set with $\mathscr{A}$ as a member, and then apply 2.4. ■

3. Bounding Morley rank.

THEOREM 3.1 (weak absoluteness of the Morley derivative, cf. Lemma 5.2). *Let N be an admissible set, and $\mathscr{A}$ a substructure of a model of T. Suppose $\mathscr{A} \in N$. Then*

$$p \in S\mathscr{A} \;\&\; r_M(p) = \beta \;\rightarrow\; p \in N \;\&\; \beta \in N\,.$$

Proof. By 2.1, $r_{CB}(p) \leqslant \beta$, and so by 2.4, $p \in N$. Let ZFC_n be a finite fragment of set theory strong enough to carry out all routine arguments of model theory. The following collection $\mathscr{F}$ of axioms is Σ_1 over N, and talks about an end extension of N in which ZFC_n is true.

(i) ZFC_n.

(ii) The diagram of N. For each $a \in N$ there is an individual constant $\underline{a}$ and a sentence

$$(y)[y \in \underline{a} \leftrightarrow \bigvee_{b \in a} (y = \underline{b})]\,.$$

(iii) $\infty > r_M(\underline{p}) \geqslant \underline{\delta}$. One such axiom for each ordinal $\delta \in N$.

(iv) $\underline{\mathscr{A}} \subset \mathscr{U}$ and $\mathscr{U}$ is a countable δ-universal domain. One such axiom for each $\delta \in N$.

Assume $r_M(p) \notin N$ in the hope of a contradiction. For the moment assume N is countable. Then $\mathscr{F}$ is countable, and by 2.3, $\mathscr{F}$ has a model, namely V, the real world. It follows that $\mathscr{F}$ has a model $\mathscr{N}$ that omits α, the least ordinal not in N. (For details concerning the construction of such an $\mathscr{N}$, see Keisler [6, p. 58]). (iii) implies $\mathscr{N}$ is a proper end extension, because $r_M(\underline{p})$ must have a value in $\mathscr{N}$, and that value cannot belong to N.

For each $z \in \mathscr{N}$, let $\mathrm{tc}(z)$ be the transitive closure of z with respect to $\in_{\mathscr{N}}$, the $\in$ relation of $\mathscr{N}$. If the restriction of $\in_{\mathscr{N}}$ to $\mathrm{tc}(z)$ is well-founded, then z is said to be a *standard set of* $\mathscr{N}$. Let $\mathrm{St}(\mathscr{N})$ be the set of all standard sets of $\mathscr{N}$. $\mathrm{St}(\mathscr{N})$ is admissible, contains N, and its ordinals are just those less than α (cf. [1], p. 60). $\mathscr{N}$ can be regarded as an end extension of $\mathrm{St}(\mathscr{N})$.

The countability of $\mathscr{U}$ in $\mathscr{N}$ makes it possible to construe $\mathscr{U}$ as a member of $\mathrm{St}(\mathscr{N})$. The universe of $\mathscr{U}$ becomes ω, and the relations of $\mathscr{U}$ become sets of n-tuples of finite ordinals.

From now on superscripts will be used to indicate in which universe ranks are being computed. Since $\mathscr{A}$ can be injected into $\mathscr{U}$, there is a $q \in S\mathscr{U} \cap \mathscr{N}$ such that q is a pre-image of p and $r_M^{\mathscr{N}}(q) = r_M^{\mathscr{N}}(p)$. Since q is an actual pre-image of p in the real world and p has a Morley rank, q must also have a Morley rank. It follows from 2.1 and 2.4 that $r_{CB}^{\mathrm{St}}(q) < \alpha$. A straightforward induction shows that any assignment of Cantor–Bendixson rank in $\mathrm{St}(\mathscr{N})$ persists in $\mathscr{N}$. In short an isolated point of $d^{\gamma}\mathscr{S}\mathscr{U}$ in $\mathrm{St}(\mathscr{N})$ remains isolated in $\mathscr{N}$. Thus $r_{CB}^{\mathscr{N}}(q) < \alpha$; hence $r_M^{\mathscr{N}}(q) < \alpha$ by 2.2. But this last contradicts axiom (iii).

Now suppose N is uncountable. It suffices to show β is countable. There exist a countable $\mathscr{A}_0 \subset \mathscr{A}$ and a $p \in S\mathscr{A}_0$ such that $r_M(p_0) = \beta$. $\mathscr{A}_0^+$, the least admissible set having $\mathscr{A}_0$ as a member, is countable, hence susceptible to the above argument for countable N. ∎

COROLLARY 3.2. *Suppose $\mathscr{A}$ is a countable substructure of T and p is a 1-type over $\mathscr{A}$. If p has a Morley rank, then p is hyperarithmetic in $\mathscr{A}$ and its Morley rank is an ordinal recursive in $\mathscr{A}$.*

4. Bounding d_T and α_T. d_T, Blum's density number, was defined in Section 1.

THEOREM 4.1. *If T is quasi-totally transcendental, then d_T is recursive in T.*

Proof. Similar to that of 3.1. Let $\mathscr{A}$ be a substructure of a model of T. Define $d_T^{\mathscr{A}}$ to be the least β such that the points of $S\mathscr{A}$ of Morley rank less than β are dense in β. As in the proof of 3.1, the following set $\mathscr{G}$ of axioms is Σ_1 over T^+, the least admissible set with T as a member.

(i) ZFC_n.

(ii) T is quasi-totally transcendental.

(iii) $\mathscr{A}$ is a countable substructure of a model of T.

(iv) $d_T^{\mathscr{A}} > \delta$. One such axiom for each $\delta < \omega_1^T$.

Assume $d_T \geqslant \omega_1^T$ in hope of a contradiction. Then for each $\delta < \omega_1^T$, there is a countable $\mathscr{A}$ such that $d_T^{\mathscr{A}} > \delta$, and so each T^+-finite (²) subset of $\mathscr{G}$ is modeled by the world. It follows that $\mathscr{G}$ has a model $\mathscr{N}$ that omits ω_1^T. (ii) requires $d_T^{\mathscr{A}}$ to have a value in $\mathscr{N}$, and (iv) implies that that value is nonstandard. As in 3.1, the countability of $\mathscr{A}$ implies $\mathscr{A} \in \mathrm{St}(\mathscr{N})$, the standard part of $\mathscr{N}$. Since T is quasi-totally transcendental, the isolated points of $S\mathscr{A}$ are dense in $S\mathscr{A}$ (cf. [12], page 200), and every isolated point of $S\mathscr{A}$ has a Morley rank. Let I be the set of isolated points of $S\mathscr{A}$. By 2.4, $I \in \mathrm{St}(\mathscr{N})$.

Fix $p \in I$ to show $r_M(p) < \omega_1^T$. Suppose $r_M^{\mathscr{N}}(p) = c$; by 3.1 c is a countable ordinal (possibly nonstandard) of $\mathscr{N}$. By (i) and 2.3, there is a $\mathscr{U} \in \mathscr{N}$ such that $\mathscr{A} \subset \mathscr{U}$ and

$$\mathscr{N} \models [\mathscr{U} \text{ is a countable, } (c+1)\text{-universal domain}].$$

Let $q \in S\mathscr{U}$ be a pre-image of p such that $r_M^{\mathscr{N}}(p) = r_M^{\mathscr{N}}(q)$. By 2.2, $r_{CB}^{\mathscr{N}}(q) = r_M^{\mathscr{N}}(q)$. It follows from 2.4 that $r_{CB}^{\mathrm{St}(\mathscr{N})}(q) < \omega_1^T$. It was noted in the proof of 3.1 that any assignment of Cantor–Bendixson rank in $\mathrm{St}(\mathscr{N})$ persists in $\mathscr{N}$. Consequently $c < \omega_1^T$.

It now follows that the least upper bound (in $\mathscr{N}$) of

$$\{r_M^{\mathscr{N}}(p) \mid p \in I\} \tag{1}$$

is less than ω_1^T, since $\omega_1^T \notin \mathscr{N}$. But (iv) implies that the least upper bound of (1) is nonstandard. ∎

COROLLARY 4.2. $d_T < \aleph_1$.

α_T, the Morley rank of T, was defined in Section 1. Morley [9] showed that $\alpha_T < \aleph_1$ when T is totally transcendental.

THEOREM 4.3. *If T is totally transcendental, then α_T is recursive in T.*

(²) I.e. member of T^+.

Proof. Same as that of 4.1. Let $\mathscr{A} \in \mathscr{K}(T)$. Define $\alpha_T^{\mathscr{A}}$ to be the least β such that every $p \in S\mathscr{A}$ has Morley rank less than β. $\mathscr{G}_0$ is the result of two changes in the axiom set $\mathscr{G}$ of 4.1: "quasi-totally" is replaced by "totally", and $d_T^{\mathscr{A}}$ by $\alpha_T^{\mathscr{A}}$. The part played by I in 4.1 is now played by all of $S\mathscr{A}$. The concluding contradiction concerns the least upper bound (in $\mathscr{N}$) of $\{r_M^{\mathscr{N}}(p) |\ p \in S\mathscr{A}\}$. ∎

5. Countable universal domains. The purpose of this section is not to prove the existence of a countable universal domain $\mathscr{U}$ for T, for that fact follows easily from the Lachlan bound on Morley rank expressed by formula (2) of Section 1, but rather to estimate the least ordinal needed to construct such a $\mathscr{U}$ from T.

Let WO be the set of all countable wellorderings of integers, and for each $Y \in \mathrm{WO}$, let $|Y|$ be the ordinal represented by Y. Suppose $P(X, \beta)$ is a predicate with variable X ranging over the reals, and β ranging over the countable ordinals. $P(X, \beta)$ is said to be Σ_1^1 if $Y \in \mathrm{WO} \rightarrow P(X, |Y|)$ is Σ_1^1.

PROPOSITION 5.1. *The predicate* $p \in S\mathscr{A}$ & $r_M(p) \geqslant \beta$, *restricted to countable* $\mathscr{A}$'s, *is* Σ_1^1 (³).

Proof. The predicate in question, call it $P(p, \mathscr{A}, \beta)$, is equivalent to

$$(1) \qquad (E\mathscr{B})_{\mathscr{A} \subset \mathscr{B}} (Eq)_{p \subset q \in S\mathscr{B}} [r_{CB}(q) \geqslant \beta] .$$

(1) implies $P[p, \mathscr{A}, \beta]$ by 2.1. Suppose $P(p, \mathscr{A}, \beta)$ holds. Extend $\mathscr{A}$ to $\mathscr{B}$, a countable, $(\beta+1)$-universal domain provided by 2.3. Let $q \in S\mathscr{B}$ be a pre-image of p of the same Morley rank as p. Then $r_{CB}(q) \geqslant \beta$ by 2.2.

Since the predicate $q \in S\mathscr{B}$ is arithmetic, it need only be verified that $[r_{CB}(q) \geqslant \beta]$ is Σ_1^1. The proof of 2.4 shows that the restriction of

$$(2) \qquad Y \in \mathrm{WO} \rightarrow r_{CB}(q) \geqslant |Y|$$

to any admissible set N is Π_1 over N uniformly in N. The last phase refers to a Π_1 predicate independent of N; its only parameters are integers. Thus there is a Δ_0 formula R such that for all Y and q, (2) is equivalent to

$$(3) \qquad \mathrm{HYP}(Y, q) \vDash (v) R(v, Y, q) .$$

$\mathrm{HYP}(Y, q)$ is the least admissible set with Y and q as members; each element of $\mathrm{HYP}(Y, q)$ corresponds to a real hyperarithmetic in Y, q. A universal quantifier restricted to $\mathrm{HYP}(Y, q)$ is equivalent to an unrestricted existential quantifier (cf. [11], p. 418). Hence (3) is Σ_1^1. ∎

An admissible set N is said to be *recursively inaccessible* if for each $x \in N$, there is a $y \in N$ such that $x \in y$ and y is admissible.

LEMMA 5.2. *Let N be recursively inaccessible and $\mathscr{A}$ a substructure of a model of T. Assume $\mathscr{A} \in N$, $T \in N$ and*

$$N \vDash [T \text{ is countable} \ \& \ (x)(x \text{ is wellorderable})].$$

(³) Proposition 5.1 can be derived from Lemma 4.8 of [3] without any mention of universal domains.

Then (i) *and* (ii) *are equivalent for all p and all $\beta < \infty$:*

(i) $p \in S\mathscr{A}$ & $r_M(p) = \beta$,

(ii) $p \in N$ & $\beta \in N$ & $N \vDash [p \in S\mathscr{A}$ & $r_M(p) = \beta]$.

Proof. If p, $\mathscr{A} \in N$ and $p \in S\mathscr{A}$, let $r_M^N(p)$ be the Morley rank assigned to p inside N. By 3.1 it suffices to prove

$$(1) \qquad r_M(p) = \beta \leftrightarrow r_M^N(p) = \beta$$

for all $\mathscr{A}$, p and β in N by induction on β. Since elementary arguments of model theory can be carried out in N, the value of $r_M^N(p)$ is determined, as in the real world V, by the countable structures. Thus it is enough to prove (1) for $\mathscr{A}$'s countable in N.

Fix β and assume $r_M(p) = \beta$. By induction $r_M^N(p) \geqslant \beta$. Suppose $r_M^N(p) > \beta$. Then in N there is a $\mathscr{B} \supset \mathscr{A}$ and a $q \supset p$ such that q is a limit point of $D^\beta S\mathscr{B}$. In V there is a neighborhood F of q that isolates q from all points of $S\mathscr{B}$ save those of Morley rank less than β. But then by induction F isolates q in $D^\beta S\mathscr{B}$ in N.

Now assume $r_M^N(p) = \beta$. By induction $r_M(p) \geqslant \beta$. Then in V the following is true for each n. There exist a countable $\mathscr{B}_n$ and distinct $q_0, \ldots, q_n$ such that

$$(2) \qquad \mathscr{A} \subset \mathscr{B}_n \ \& \ p \subset q_i \in S\mathscr{B}_n \ \& \ r_M(q_i) \geqslant \beta \qquad (i < n) .$$

Since $r_{CB}^N \leqslant r_M^N(p) < \infty$, it follows from 2.4 that p is hyperarithmetic in $\mathscr{A}$. Suppose $\beta < \omega_1^{\mathscr{A}}$. Then 5.1 implies (2) is Σ_1^1 with parameter $\mathscr{A}$. By Kleene's basis theorem ([11], p. 420), $\mathscr{B}_n$ can be assumed to be recursive in $\mathcal{O}^{\mathscr{A}}$, the hyperjump of the real that codes $\mathscr{A}$. Since N is recursively inaccessible, $\mathcal{O}^{\mathscr{A}} \in N$. Let $\mathscr{B}_\infty$ be the amalgamation in N of all substructures (of models of T) coded by reals recursive in $\mathcal{O}^{\mathscr{A}}$. In V, p has infinitely many pre-images in $\mathscr{B}_\infty$ of Morley rank $\geqslant \beta$, and by induction the same is true in N. But then $r_M^N(p) > \beta$.

Finally suppose $\beta \geqslant \omega_1^{\mathscr{A}}$. Since $r_M(p) \geqslant \beta$, it follows as in the proof of 5.1 that there exist $\mathscr{B}$ and q such that

$$(3) \qquad p \subset q \in S\mathscr{B} \ \& \ r_{CB}(q) \geqslant \omega_1^{\mathscr{A}} .$$

The second half of (3) is equivalent to

$$(m)[m \in \mathcal{O}^{\mathscr{A}} \rightarrow r_{CB}(q) > |m|] .$$

($|m|$ is the ordinal less than $\omega_1^{\mathscr{A}}$ represented by m.) It follows from 5.1 that (3) is Σ_1^1 with parameter $\mathscr{A}$. As in the previous paragraph, (3) has a solution $\langle \mathscr{B}, q \rangle$ in N. By 2.4 and 2.5,

$$r_{CB}(q) = r_{CB}^N(q) = \infty .$$

But then $r_M^N(p) = \infty > \beta$. ∎

THEOREM 5.3. *Let $\mathscr{A}$ be a countable substructure of a model of T. Then $\mathscr{A}$ can be extended to a countable universal domain $\mathscr{U}$ hyperarithmetic in the hyperjump of $\mathscr{A}$.*

Proof. A compound of 2.3 and the last part of 5.2. $\mathscr{U}$ is the limit of a chain $\{\mathscr{U}_n |\ n < \omega\}$ defined by recursion on n. $\mathscr{U}_n$ is recursive in $\mathscr{U}_{n+1}$ uniformly in n. (This can be taken to mean that the even part of the code for $\mathscr{U}_{n+1}$ is the code for $\mathscr{U}_n$.) Each $\mathscr{U}_n$ has lower hyperdegree than $\mathcal{O}^{\mathscr{A}}$ (in symbols $\mathscr{U}_n <_h \mathcal{O}^{\mathscr{A}}$) to insure that the ordinals $\leqslant \omega_1^{\mathscr{A}}$ suffice for the construction.

Let $\mathscr{U}_0$ be $\mathscr{A}$.

Choose some effective enumeration E of ω^4 such that for all n, the nth member of E, denoted by (i, j, k, l), has the property that $i \leqslant n$. Let $\mathscr{V}$ be the jth finitely generated substructure of $\mathscr{U}_i$. If $k \in \mathcal{O}^{\mathscr{A}}$, let β be $|k|$, the ordinal represented by k; otherwise let β be $\omega_1^{\mathscr{A}}$.

The second half of the proof of 5.1 shows

$$\{p \mid p \in S\mathscr{V} \ \&\ r_{\mathrm{CB}}(p) < \infty\} \tag{1}$$

is Π_1^1 with parameter $\mathscr{V}$. By 2.5 each member of (1) is hyperarithmetic in $\mathscr{V}$. Consequently the members of (1) can be arranged in a sequence hyperarithmetic in $\mathcal{O}^{\mathscr{V}}$. Let p be the lth element of that sequence.

Assume $\mathscr{U}_n < \mathcal{O}^{\mathscr{A}}$. Since $\mathscr{A}$ is recursive in $\mathscr{U}_n$, it follows that

$$\omega_1^{\mathscr{U}_n} = \omega_1^{\mathscr{A}} \quad \text{and} \quad \mathcal{O}^{\mathscr{U}_n} \equiv_h \mathcal{O}^{\mathscr{A}}. \tag{2}$$

Consequently $\omega_1^{\mathscr{V}} \leqslant \omega_1^{\mathscr{A}}$ and $\mathcal{O}^{\mathscr{V}} \leqslant_h \mathcal{O}^{\mathscr{A}}$.

Assume $p \in D^\beta S\mathscr{V}$ and that p splits in $D^\beta S\mathscr{W}$ for some $\mathscr{W} \in \mathscr{K}(T)$. (If not, let $\mathscr{U}_{n+1}$ be $\mathscr{U}_n$.) The definition of $\mathscr{U}_{n+1}$ has two cases.

Case I. $\beta < \omega_1^{\mathscr{V}}$. Thus $\beta < \omega_1^{\mathscr{U}_n}$, and so 5.1 implies that the predicate

$$q_0 \neq q_1 \ \&\ \mathscr{U}_n \subset \mathscr{W} \ \&\ (i)_{i<2}[p \subset q_i \in D^\beta S\mathscr{W}] \tag{3}$$

is Σ_1^1 with parameter $\mathscr{U}_n$. By Gandy's basis theorem, (3) has a solution $\langle q_0, q_1, \mathscr{W}\rangle <_h \mathcal{O}^{\mathscr{U}_n}$. Let $\mathscr{U}_{n+1}$ be $\mathscr{W}$. Then $\mathscr{U}_{n+1} <_h \mathcal{O}^{\mathscr{A}}$ by (2).

Case II. $\beta \geqslant \omega_1^{\mathscr{V}}$. By 3.2, $r_{\mathrm{M}}(p) = \infty$. Hence there exist q_0, q_1 and $\mathscr{W}$ such that $q_0, q_1 \in S\mathscr{W}$ and

$$q_0 \neq q_1 \ \&\ \mathscr{U}_n \subset \mathscr{W} \ \&\ (i)_{i<2}[p \subset q_i \ \&\ r_{\mathrm{M}}(q_i) = \infty]. \tag{4}$$

(4) is Σ_1^1 with parameter $\mathscr{U}_n$, because $[r_{\mathrm{M}}(q_i) = \infty]$ is equivalent to

$$(m)[m \in \mathcal{O}^{\mathscr{W}} \to q_i \in D^{|m|} S\mathscr{W}], \tag{5}$$

and (5) is Σ_1^1 by 5.1. Now proceed as in case I.

$\mathscr{U}$ $(= \bigcup\{\mathscr{U}_n \mid n < \omega\})$ is hyperarithmetic in $\mathcal{O}^{\mathscr{A}}$, because $\mathscr{U}_n \leqslant \mathcal{O}^{\mathscr{A}}$ uniformly in n. The last assertion is a consequence of several uniformities, two of which are worth mentioning. (i) The set of indices of solvable Σ_1^1 predicates with parameter P is recursive in $\mathcal{O}^P$ uniformly in P. (ii) An index for computing $\mathcal{O}^Y$ hyperarithmetically from $\mathcal{O}^X$ can be obtained effectively from an index for computing Y hyperarithmetically from $\mathcal{O}^X$, if it is assumed that $Y <_h \mathcal{O}^X$.

To see that $\mathscr{U}$ is a universal domain for T, let $\mathscr{V}$ be a finitely generated substructure of $\mathscr{U}$. $\mathscr{V}$ is the jth finitely generated substructure of some $\mathscr{U}_i$. Fix β and p. If $\beta < \omega_1^{\mathscr{V}}$, let $k \in \mathcal{O}^{\mathscr{V}}$ be a notation for β; otherwise let k be some non-member of $\mathcal{O}^{\mathscr{V}}$. Suppose p is an isolated point of $D^\beta S\mathscr{V}$. Then $r_{\mathrm{CB}}(p) < \infty$, since $d^\gamma S\mathscr{V} \subset D^\gamma S\mathscr{V}$ for all γ. Let p be the lth element of (1). Suppose p splits in $D^\beta S\mathscr{W}$ for some $\mathscr{W} \in \mathscr{K}(T)$. If (i, j, k, l) is the nth member of E, then p splits in $D^\beta \mathscr{U}_{n+1}$, hence in some finitely generated substructure of $\mathscr{U}$. ∎

COROLLARY 5.4. *T has a countable universal domain that is hyperarithmetic in the hyperjump of T.*

6. Open questions.

1. To what extent is the assignment of Morley rank absolute? Does Lemma 2.4 remain true when the Cantor–Bendixson derivative is replaced by the Morley derivative? Can the assumption of recursive inaccessibility in Lemma 5.2 be substantially weakened?

2. Does every countable theory T have a countable universal domain $\mathscr{U}$ such that $\mathscr{U}$ is hyperarithmetic in T? If not, can it at least be arranged that $\omega_1^{\mathscr{U}} = \omega_1^T$?

Added in proof. C. Calmer (Stanford Ph. D Thesis, 1978) has answered all of the above questions. In particular, Lemma 2.4 holds for the Morley derivative.

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