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Small subsets of first countable spaces

by

Eric K. van Douwen (Athens, Ohio) and **Michael L. Wage** (New Haven, Conn.)

Abstract. The existence of two types of first countable spaces is shown to be equivalent to a certain structure on the rationals. This structure, whose intuitive content is that discrete subsets of the rationals are small, is consistent with the usual axioms for set theory.

Introduction. In this paper we present two consistent examples of first countable spaces both of which require careful handling of certain sets which are small in an intuitive sense. We use two combinatorial principles, called $P(c)$ and $BF(c)$, which will be explained in Section 2. Both are strictly weaker than Martin's Axiom, hence strictly weaker than the Continuum Hypothesis, and $BF(c)$ is strictly weaker than $P(c)$. However, it is consistent with ZFC that $P(c)$ and $BF(c)$ be false.

We first recall some definitions. A space X is *collectionwise Hausdorff*, abbreviated CWH, if for each closed discrete subset D of X there is an open family $\{U_x|\ x \in D\}$ in X such that $x \in U_x$, for all $x \in D$, and $U_x \cap U_y = \emptyset$, for all $x \neq y \in D$. A space is *σ-discrete* if it is the union of countably many closed discrete subsets. A space is *pseudonormal* (or has *property* D) if any two disjoint closed subsets, one of which is countable (and discrete) have disjoint neighborhoods. (This is not the usual definition of property D, [M, p. 69], but is equivalent to it in first countable regular spaces.)

Our first example answers Mike Reed's question of whether every CWH σ-discrete Moore space is normal (hence metrizable) in the negative. This question is quite natural, since in a CWH space closed discrete subsets are "small", so a CWH σ-discrete space is σ-"small".

1.1. Example 1. $[P(c)]$ There is a CWH σ-discrete Moore space which is not pseudonormal.

The fact that there exists a nonnormal CWH Moore space was known already, see [W]. The example in [W] does not require any additional set theoretic axioms. Interest in collectionwise Hausdorffness in Moore spaces stems from Fleissner's Theorem that $V = L$ (which implies CH, hence $P(c)$) implies that first countable normal spaces are CWH (in fact this is true for normal spaces with character $\leqslant c$), [F].

The existence of Example 1 will be deduced from the existence of Example 2, which answers Mike Reed's question of whether property D implies pseudonormality in Moore spaces in the negative.

1.2. EXAMPLE 2. [P(c)] There exists a separable Moore space with property D which is not pseudonormal.

(That propery D and pseudonormality are not equivalent in regular spaces is shown in [vD$_2$]. The example in [vD$_2$] does not require any additional set theoretic axioms but is not first countable.) Example 2 shows that a countable closed discrete subset is much smaller than just a countable closed subset. It should be contrasted with the following Lemma from [vD$_2$], which will be used in the construction.

1.3. LEMMA ([vD$_2$]). *The following conditions are equivalent*

(1) BF(c),

(2) *any first countable regular space with cardinality less than c is pseudonormal, and*

(3) *any separable Moore space with cardinality less than c has property* D.

Example 2 is more than just an example in the fine structure of weak separation axioms in Moore spaces. As stated above, the existence of Example 1 follows from the existence of Example 2. If separability is dropped, then the converse is also true, and it does not simplify the construction if one looks just for a first countable regular space, instead of for a Moore space, see Lemma 3.1. This lemma also shows that each space one may construct for Example 2 gives rise to a space looking like ours.

Another point of interest is that the existence of Example 2 (or Example 1 without separability) is equivalent to the existence of a certain structure on Q, the space of rationals. The intuitive content of this structure is that closed discrete subsets of Q are very small. It is unknown if the existence of this structure is independent of ZFC.

2. Preliminaries. As usual, a cardinal is an initial ordinal, and an ordinal is the set of smaller ordinals; ω is ω_0, and c is 2^{ω}. A family of sets is *strongly centered* if every finite subfamily has infinite (rather than nonempty) intersection. A set A is called an *almost intersection* of a family $\mathscr{F}$ if $A\setminus F$ is finite for each $F\in\mathscr{F}$. If $\varkappa$ is a cardinal, then P$(\varkappa)$ is the following assertion.

P$(\varkappa)$: If $\mathscr{F}$ is a strongly centered collection of subsets of some countable set and $|\mathscr{F}|<\varkappa$, then $\mathscr{F}$ has an infinite almost intersection.

It is easy to show that P(ω_1) is true, hence CH implies P(c). More generally, Martin's Axiom implies P(c), [MS, p. 54], but not conversely, [KT]. Also, if ZFC is consistent, then so is ZFC$+\neg$P$(\omega_2)+\neg$CH.

If A and B are sets, ${}^AB$ is the set of functions from A to B; we will use the fact that $f\subset A\times B$ for $f\in{}^AB$. If $f,g\in{}^A\omega$, then $f<^*g$ means that $f(a)<g(a)$ for all but finitely many $a\in A$. If $\varkappa$ is a cardinal, then BF$(\varkappa)$ is the following assertion.

BF$(\varkappa)$. If K is a countable set, and if $F\subset{}^K\omega$ has cardinality less than $\varkappa$, then there is a $g\in{}^K\omega$ such that $f<^*g$ for all $f\in F$.

F. Rothberger, who studied P$(\varkappa)$ and BF$(\varkappa)$ extensively, [R$_1$], [R$_2$], [R$_3$], showed that P$(\varkappa)$ implies BF$(\varkappa)$, [R$_3$, Thm. 3^a], in particular P(c) implies BF(c). Solomon showed that BF(c) does not imply P(c), [So]. Hechler proved that if ZFC is consistent, then so is ZFC$+\neg$BF$(\omega_2)+\neg$CH, [H].

A space X is *developable* if there is a sequence $\langle\mathscr{G}_n\rangle_{n\in\omega}$ of open covers such that $\{\mathrm{St}(x,\mathscr{G}_n)|\ n\in\omega\}$ is a local base at x in X for all $x\in X$. (Recall that $\mathrm{St}(x,\mathscr{G})=\cup\{G\in\mathscr{G}|\ x\in G\}$.) A Moore space is a regular developable space. The following lemma will be used repeatedly, the easy proof is omitted.

2.1. LEMMA. *If X is a first countable T_1-space, containing disjoint closed subsets F and G such that F is discrete, G is countable, all points of $X\setminus(F\cup G)$ are isolated, and $X\setminus(F\cup G)$ is an F_σ-subset of X, then X is developable.*

3. The examples.

3.1. Construction of Example 2. It is shown in [vD$_1$] that there is a collection $\mathscr{U}$ of open subsets of Q, the rationals, such that

(1) for each closed discrete subset D of Q there is a $U\in\mathscr{U}$ with $D\subset U$,

(2) if $\mathscr{F}$ is any finite subcollection of $\mathscr{U}$ then $Q\setminus\cup\mathscr{F}$ is unbounded;

this does not require P(c). We can enumerate $\mathscr{U}$ as $\{U_\alpha|\ \alpha\in c\}$. Then

$$\{Q\setminus U|\ U\in\mathscr{U}\}\cup\{Q\setminus(-n,n)|\ n\geqslant 1\}$$

is strongly centered by (2). So P(c) implies that there is a family $\{G_\alpha|\ \alpha\in c\}$ of infinite subsets of Q, such that

(3) $U_\alpha\cap G_\beta$ is finite whenever $\alpha\leqslant\beta$,

(4) $G_\beta\cap(-n,n)$ is finite, for all $\beta<c$ and $n\geqslant 1$.

Let $\{V_\alpha|\ \alpha\in c\}$ enumerate all neighborhoods of $Q\times\{\omega\}$ in the usual product $Q\times(\omega+1)$. For each α there is a v_α: $Q\to\omega$ such that

$$\{\langle a,k\rangle\in Q\times\omega|\ k\geqslant v_\alpha(a)\}\subset V_\alpha.$$

Since P(c) implies BF(c), we can find a subset $\{f_\alpha|\ \alpha\in c\}$ of ${}^Q\omega$ such that

(5) $f_\alpha<^*f_\beta$ whenever $\alpha<\beta$, and

(6) $f_\alpha\subset V_\alpha$ for all $\alpha\in c$.

The underlying set of our example is $T=c\,\dot\cup\,(Q\times(\omega+1))$. We topologize T as follows. $Q\times(\omega+1)$ is an open subspace of T, retopologized by making all points of $Q\times\omega$ isolated, and giving all points of $Q\times\{\omega\}$ their usual product neighborhoods. A basic neighborhood of $\alpha\in c$ contains α and all but finitely many points of $f_\alpha[G_\alpha]$. (This explains why we made points of $Q\times W$ isolated.) Clearly T is separable and first countable, and Lemma 2.1 shows that T is developable. T is regular at all points of $Q\times\{\omega\}$, by (4), and is regular at all points of c by (4) and the fact that $\{f_\alpha[G_\alpha]|\ \alpha\in c\}$ is an almost disjoint family, which follows from (5). So T is regular. Consequently T is a separable Moore space.

Each neighborhood of $Q\times\{\omega\}$ in T contains some V_α. It follows from (6) that T is not pseudonormal, since $Q\times\{\omega\}$ and c are disjoint closed subsets of T, one of which is countable.

We conclude by showing that T has property D. Let A be a countable closed discrete subset of T, and let V be any neighborhood of A. We want to construct a neighborhood W of A in T such that $W^- \subset V$. Since all points of $Q\times\omega$ are isolated, it suffices to consider the following two cases.

Case 1. $A\subset Q\times\{\omega\}$. There is an α such that $A\subset U_\alpha\times(\omega+1)$. The open subspace $S_1 = \alpha\cup Q\times(\omega+1)$ of T has cardinality less than c, hence is pseudonormal by Lemma 1.3. So we can find an open W_1 in T with

$$A\subset W_1\subset V\cap U_\alpha\times(\omega+1) \quad\text{and}\quad S_1\cap W_1^-\subset V.$$

But (3) implies that $\beta\notin(U_\alpha\times(\omega+1))^-$ whenever $\beta\geqslant\alpha$, so $W_1^-\subset V$.

Case 2. $A\subset c$. Fix α such that the subspace

$$S_2 = \bigcup\{\{\gamma\}\cup\{\langle k, f_\gamma(k)\rangle|\ k\in Q, f_\gamma(k)<f_\alpha(k)\}|\ \gamma<\alpha\}$$

of T is a neighborhood of A and has cardinality less than c. It is pseudonormal by Lemma 1.3, so we can find an open subset W_2 of T with

$$A\subset W_2\subset V\cap S_2 \quad\text{and}\quad S_2\cap W_2^-\subset V.$$

But (5) implies that $\beta\notin S_2^-$ whenever $\beta\geqslant\alpha$, so $W_1^-\subset V$.

The following lemma shows that the existence of Example 1 follows from the existence of Example 2, and that the existence of Example 1 is intimately related to the structure of Q. We organize the proof in such a way that the reader can skip this relationship.

3.2. Lemma. *The following conditions are equivalent.*

(1) *There is a Moore space with property* D *which is not pseudonormal.*

(1′) *There is a first countable regular space with property* D *which is not pseudonormal.*

(2) *There is a σ-discrete* CWH *Moore space which is not pseudonormal.*

(2′) *There is a first countable regular σ-discrete* CWH *space which is not pseudonormal.*

(3) *There exists a family $\mathcal{U}$ of open subsets of Q and a collection $\underline{V}$ consisting of countably infinite families of open subsets of Q, such that*

(a) *for each closed discrete subset D of Q there is a $U\in\mathcal{U}$ with $D\subset U$,*

(b) *if $\mathcal{W}$ is any open cover of Q, then there is a $\mathcal{V}\in\underline{V}$ which refines $\mathcal{W}$,*

(c) *for each $U\in\mathcal{U}$ and $\mathcal{V}\in\underline{V}$ there are at most finitely many $V\in\mathcal{V}$ with $V\subset U$.*

That (2) implies (1) was observed by Mike Reed, after hearing the converse implication. In the proof we use a Moore-ificator inspired by the example in [W], and use the following machine, inspired by examples in [vD$_2$], four times.

3.3. The disjointer. Let X be a first countable space containing two disjoint closed subsets F and G, with G countable, such that

(1) F and G do not have disjoint neighborhoods in X,

(2) each point of F has a neighborhood whose closure misses G,

(3) X is regular at each point of G,

(4) each closed discrete subset of G has a neighborhood whose closure misses F.

Then there is a first countable regular CWH space Y satisfying property D, also containing F and G as disjoint closed subsets, this time with F discrete, such that F and G do not have disjoint neighborhoods in Y.

Moreover, if $X\setminus(F\cup G)$ is an F_σ-subset of X (in particular if X is a Moore space) then Y is a Moore space.

Let $I = X\setminus(F\cup G)$. The underlying set of Y is $F\dot\cup G\dot\cup F\times I$. A base for Y is

$$\{\{x\}|\ x\in F\times I\}\cup\{(G\cap U)\cup(F\times(I\cap U))|\ U \text{ open in } X\}\cup$$
$$\cup\{\{x\}\cup(\{x\}\times(I\cap U))|\ U \text{ open in } X,\ x\in F\cup U\}.$$

Clearly Y is first countable. It follows from (2) and (3) that Y is regular. It is easy to see that F and G are disjoint closed subsets of Y, with F discrete, which do not have disjoint neighborhoods in Y. Let A be a closed discrete subset of Y with $A\subset F\cup G$. It follows from (4) that $A\cap F$ and $A\cap G$ have disjoint neighborhoods, U_F and U_G, respectively, in Y. Since Y is regular and $A\cap G$ is at most countable, there is a disjoint open family $\{V_x|\ x\in A\cap G\}$ in Y with $x\in V_x\subset U_G$ for $x\in A\cap G$. Put $V_x = (\{x\}\cup\{x\}\times I)\cap U_F$ for $x\in A\cap F$. Then $\{V_x|\ x\in A\}$ is a disjoint open family in Y with $x\in V_x$ for $x\in A$. Since all points of $Y\setminus(F\cup G)$ are isolated, it follows that Y is CWH. The fact that Y has property D can be similarily verified.

That Y is a Moore space if $X\setminus(F\cup G)$ is an F_σ-subset of X follows from Lemma 2.1.

3.4. Proof of the lemma. We prove (1)→(2)→(2′)→(1′)→(1) and (3)↔(1′).

(1)→(2) Let F and G be disjoint closed subsets, with G countable, of a Moore space X with property D, such that F and G do not have disjoint neighborhoods. Run X through the disjointer.

(2)→(2′) Trivial.

(2′)→(1′) (Reed). It suffices to show that each regular σ-discrete CWH space X has property D. Let A and B be disjoint closed subsets of X with A countable and discrete. Enumerate A as $\{a_n|\ n\in\omega\}$. Since X is σ-discrete, we can find for each $n\in\omega$ a closed discrete B_n in X such that $B = \cup_n B_n$. Since X is regular there is for each n a neighborhood U_n of a_n with $B\cap U_n^- = \emptyset$. Since X is CWH, and $A\cup B_n$ is closed discrete in X for each n, there is for each $n\in\omega$ a neighborhood V_n of B_n such that $V_n^-\cap A = \emptyset$. As in the proof that regular Lindelöf spaces are normal, it follows that

$$\bigcup_{n\in\omega}(U_n\setminus\bigcup_{k\leqslant n}V_k^-) \quad\text{and}\quad \bigcup_{n\in\omega}(V_n\setminus\bigcup_{k\leqslant n}U_k^-)$$

are disjoint neighborhoods of A and B, respectively.

(1′)→(1). Let F and G be disjoint closed subsets, with G countable, of a first countable regular space X with property D, such that F and G do not have disjoint neighborhoods. We may assume that X is CWH and that F is closed discrete in X, for otherwise we first run X through the disjointer. Denote $X\setminus(F\cup G)$ by I.

We now run X through a Moore-ificator. Retopologize the subset

$$Y = (F \cup G)\times\{\omega\} \cup I\times\omega$$

of $X\times(\omega+1)$ as follows. A basic neighborhood of $\langle x, \{\omega\}\rangle \in Y$, with $x \in F$, has the form

$$Y \cap (U\times(\omega+1))$$

where U is a neighborhood of x in X. Points of $G\times\{\omega\}$ get their usual subspace neighborhoods, and points of $I\times\omega$ are made isolated. One easily checks that Y is a first countable regular space. To prove that Y is Moore we show that $(F\cup G)\times\{\omega\}$ is a G_δ-subset, and then apply Lemma 2.1. For each $x \in F$ choose a neighborhood base $\{U(x, n)|\ n \in \omega\}$ of x in X such that $U(x, n) \supset U(x, n+1)$ for all n, and $U(x, 1) \cap U(y, 1) = \emptyset$ if $x, y \in F$ are distinct. Then

$$\bigcap_{n\in\omega}\Big(\bigcup_{x\in F}\big(U(x, n)\times(\omega+1)\big) \cap Y\Big) = F\times\{\omega\},$$

and

$$\bigcap_{n\in\omega}\big(G\times\{\omega\} \cup I\times(\omega\setminus n)\big) = G\times\{\omega\}$$

hence $(F\cup G)\times\{\omega\}$ is a G_δ-subset of Y.

Let π denote the "projection" from Y onto X. π is continuous, and if D is a closed discrete subset of Y with $D\subset(F\cup G)\times\{\omega\}$, then $\pi[D]$ is a closed discrete subset of X, and $\pi(x) \neq \pi(y)$ for distinct $x, y \in D$. Since X is CWH and all points of $Y\setminus(F\cup G)\times\{\omega\}$ are isolated, it follows that Y is CWH.

Let V be a neighborhood of $F\times\{\omega\}$ in Y. There is a neighborhood U of F in X such that $Y \cap \big(U\times(\omega+1)\big) \subset V$. There is an $x \in G$ such that $x \in U^-$ (in X). Then $\langle x, \omega\rangle \in V^-$ (in Y). Hence Y is not pseudonormal.

This completes the proof of the first part. We now first prove (3)→(1′) to make the function of $\mathscr{U}$ and $\underline{V}$ clear.

(3)→(1′) Let $\mathscr{T}$ be the collection of all nonempty open subsets of Q. Define a space X as follows. The underlying set of X is $Q \dot\cup \underline{V} \dot\cup \mathscr{T}$. Points of $\mathscr{T}$ are isolated. If U is a neighborhood of $x \in Q$ in Q, then

$$A(U) = U \cup \{T\in\mathscr{T}|\ T\subset U\}$$

is defined to be a neighborhood of x in X. If $\mathscr{V} \in \underline{V}$, and if $\mathscr{F}\subset\mathscr{V}$ is finite, then

$$\{\mathscr{V}\} \cup (\mathscr{V}\setminus\mathscr{F})$$

is a neighborhood of $\mathscr{V}$ in X. Notice that X is first countable. We want to apply the disjointer to X with $F = \underline{V}$ and $G = Q$, so (1)-(4) of Section 3.3 must be verified.

If H is a neighborhood of Q in X, then there is an open cover $\mathscr{W}$ of the space Q such that

$$\bigcup\{A(W)|\ W\in\mathscr{W}\} \subset H.$$

Clearly $\mathscr{V} \in H^-$ for each $\mathscr{V} \in \underline{V}$ which refines $\mathscr{W}$. It follows from (b) that Q and $\underline{V}$ do not have disjoint neighborhoods in X, although they are disjoint closed subsets of X.

Fix $\mathscr{V} \in \underline{V}$, and let $H = \{\mathscr{V}\} \cup \mathscr{V}$. We want to show that the closure of H in X misses Q. Fix $q \in Q$. Since $\{q\}$ is closed discrete in Q, there is a $U \in \mathscr{U}$ with $q \in U$. By (c) the set $\mathscr{F} = \{V\in\mathscr{V}|\ V\subset U\}$ is finite. We can find a finite subset of G of $U\setminus\{q\}$ such that no member of $\mathscr{F}$ is contained in $U\setminus G$. Then $A(U\setminus G)$ is a neighborhood of q which misses H.

We next show that X is regular at all points of Q. Fix $q \in Q$. As above, there is a $U \in \mathscr{U}$ with $q \in U$. Let H be any neighborhood of q in X. We may assume that $H = A(W)$, where W is a neighborhood of q in Q. Since Q is zero-dimensional, there is a clopen K in Q with $q \in K \subset U \cap W$. Clearly $A(K) \subset A(W)$, and no point of $Q\setminus K$ is in the closure of $A(K)$. Let $\mathscr{V} \in \underline{V}$ be arbitrary. Then $\mathscr{F} = \{V\in\mathscr{V}|\ V\subset U\}$ is finite. But then $\{\mathscr{V}\} \cup \mathscr{V}\setminus\mathscr{F}$ is a neighborhood of $\mathscr{V}$ which misses $A(K)$. Since all points of $\mathscr{T}$ are isolated in X, it follows that $A(K)$ is closed.

To complete the proof, run X through the disjointer with $F = \underline{V}$ and $G = Q$.

(1′)→(3) Let X be a first countable regular space which has property D, but which contains disjoint closed sets F and G, with G countable, such that F and G do not have disjoint neighborhoods.

We first use Sierpiński's characterization that every countable first countable regular space without isolated points is homeomorphic to Q, [S], to see that we may assume without loss of generality that $G = Q$. Indeed, the subspace $X' = X\times\{0\} \cup G\times Q$ of $X\times Q$ is first countable and regular, is easily seen to have property D, and the disjoint closed subsets $F\times\{0\}$ and $G\times Q$ of X' do not have disjoint neighborhoods. But $G\times Q$ has no isolated points and hence is homeomorphic to Q.

Since Q is a countable subspace of the first countable regular space X, we can construct a countable open family $\mathscr{B}$ in X such that

(α) each $B \in \mathscr{B}$ intersect Q in a nonempty convex subset (with respect to the usual order on Q),

(β) for any $B, B' \in \mathscr{B}$, if $Q \cap B \subset Q \cap B'$ then $B \subset B'$,

(γ) for each $x \in Q$ the family $\{B\in\mathscr{B}|\ x\in B\}$ is a local base at x in X, and

(δ) $B^- \cap F = \emptyset$ for $B \in \mathscr{B}$.

Let $\{\mathscr{B}_\gamma|\ \gamma\in\Gamma\}$ be the collection of all subcollections of $\mathscr{B}$ which cover Q, where Γ is some index set.

Fix $\gamma \in \Gamma$. Since F and Q do not have disjoint neighborhoods in X, there is an $x_\gamma \in F$ with $x_\gamma \in (\bigcup\mathscr{B}_\gamma)^-$. Choose a sequence $\langle y_\gamma(k)\rangle_{k\in\omega}$ of points of $\bigcup\mathscr{B}_\gamma$ which converges to x_γ. For each k choose $V_\gamma(k) \in \mathscr{B}_\gamma$ which contains $y_\gamma(k)$. Unfix γ.

Let $\{\mathscr{U}_\delta|\ \delta\in\Delta\}$ be the collection of all subfamilies $\mathscr{S}$ of $\mathscr{B}$ satisfying

(ε) $F \cap (\bigcup\mathscr{S})^- = \emptyset$, and

(ζ) each maximal convex subset of $Q \cap (\bigcup\mathscr{S})$ has the form $Q \cap S$ for some $S \in \mathscr{S}$.

Define

$$\mathscr{U} = \{Q \cap (\bigcup \mathscr{U}_\delta) |\ \delta \in \Delta\}, \quad \mathscr{V}_\gamma = \{Q \cap V_\gamma(k) |\ k \in \omega\} \quad \text{if} \quad \gamma \in \Gamma,$$

$$\underline{V} = \{\mathscr{V}_\gamma |\ \gamma \in \Gamma\}.$$

We claim that $\mathscr{U}$ and $\underline{V}$ satisfy (a), (b) and (c) of (3).

Check of (a). Let D be a closed discrete subset of Q. D is closed discrete in X. Since X has property D, one can use (α) and (γ) to find $\delta \in \Delta$ with $D \subset \bigcup \mathscr{U}_\delta$.

Check of (b). Let $\mathscr{W}$ be an open (in Q) cover of Q. Then there is a $\gamma \in \Gamma$ such that $\{Q \cap B |\ B \in \mathscr{B}_\gamma\}$ refines $\mathscr{W}$. But then also $\mathscr{V}_\gamma$ refines $\mathscr{W}$.

Check of (c). Let $\gamma \in \Gamma$ and $\delta \in \Delta$ be arbitrary. It follows from (ε) that there is an $n \in \omega$ such that $y_\gamma(k) \notin \bigcup \mathscr{U}_\delta$ for $k \geqslant n$. Fix $k \geqslant n$. We will show that $Q \cap V_\gamma(k) \not\subset \bigcup \mathscr{U}_\delta$. Suppose the contrary. Since $V_\gamma(k) \in \mathscr{B}$, it follows from ($\alpha$) and ($\varepsilon$) that $Q \cap V_\gamma(k) \subset Q \cap U$ for some $U \in \mathscr{U}_\delta$. But $U \in \mathscr{B}$, so $V_\gamma(k) \subset U$, by (β). This leads to the contradiction that $y_\gamma(k) \in \bigcup \mathscr{U}_\delta$. ■

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INSTITUTE FOR MEDICINE AND MATH.
OHIO UNIVERSITY
Athens, Ohio
YALE UNIVERSITY
New Haven, Conn.

Current address of M. L. Wage:
HARVARD UNIVERSITY, HST DEPARTMENT
Boston, Mass.

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Effective bounds on Morley rank

by

Gerald E. Sacks * (Cambridge, Mass.)

To the Memory of Andrzej Mostowski

Abstract. Effective bounds are obtained on the Morley rank of a 1-type, the Morley rank α_T of a totally transcendental theory T, and the Blum density number d_T of a quasi-totally transcendental theory T by means of type-omitting and absoluteness arguments over admissible sets. The above restrictions on T imply that α_T and d_T are ordinals recursive in T. Every theory T is seen to have a universal domain hyperarithmetic in the hyperjump of T.

1. Introduction. This paper might better have been titled: On the Absolute Character of the Morley Derivative. For the bounds given below on Morley rank, and on Blum's density number for quasi-totally transcendental theories, are derived from some absoluteness properties of Morley's analysis of 1-types. Let T be a countable theory of first order logic. Assume T is complete and substructure complete [1] in order to smooth the application of Morley's rank-and-degree machine to T. (The details of his machine will be reviewed in Section 2. A full account was given in [12].) Suppose $\mathscr{A}$ is a substructure of a model of T, and p is a 1-type over $\mathscr{A}$ (in symbols $\mathscr{A} \in \mathscr{K}(T)$ and $p \in S\mathscr{A}$). If p has a Morley rank, then that rank is denoted by $r_M(p)$, and the existence of that rank is indicated by the inequality: $r_M(p) < \infty$.

Let N be an admissible set as defined in Section 2. Assume T and $\mathscr{A}$ belong to N. One aspect of the absoluteness of the Morley derivative is expressed by:

(1) $$r_M(p) = \beta < \infty \rightarrow p \in N \,\&\, \beta \in N.$$

Another aspect is the fact that the relation

$$p \in S\mathscr{A} \,\&\, r_M(p) \leqslant \beta$$

is Σ_1 over N. (1) implies a bound first obtained by Lachlan [8]:

(2) $$r_M(p) < \infty \rightarrow r_M(p) < \aleph_1,$$

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(1) Substructure completeness is equivalent to admitting elimination of quantifiers.