Proof. Since γ is countable, we may extend to a conservative strict scale $\langle a_\alpha |\ \alpha<\gamma+1\rangle$. Choose $a_{\gamma+\omega}$ so that $a_{\gamma+\omega}(n)\geqslant b(n)$ for all $n\in\omega$ and so that $c(n) = a_{\gamma+\omega}(n)-a_\gamma(n)$ defines an increasing function $c\in{}^\omega\omega$. Now if we define

$$a_{\gamma+n}(k) = a_\gamma(k)+n \quad \text{for} \quad k, n\in\omega$$

then it is easily verified that $\langle a_\alpha |\ a<\gamma+\omega+1\rangle$ is a conservative strict scale.

3.2. THEOREM. *If* $\{b_\alpha |\ \alpha<\omega_1\}\subseteq{}^\omega\omega$ *then there is an* f: $[\omega_1]^2\to\omega$ *such that*

1. *There is a scale which is governed by* f.

2. *Every scale which is governed by* f *majorizes* $\{b_\alpha |\ \alpha<\omega_1\}$.

Proof. By Corollary 2.6 it suffices to construct a conservative strict scale $\langle a_\alpha |\ \alpha<\omega_1\rangle$ which majorizes $\{b_\alpha |\ \alpha<\omega_1\}$. This is easily accomplished using Lemma 3.1 to recursively choose the sequence $\langle a_\alpha |\ \alpha<\omega_1\rangle$ so that

$$(\forall\beta<\omega_1)(\exists_\alpha<\omega_1)[b_\beta\leqslant a_\alpha] \text{ s.b.e.p.}$$

Thus if there is an unbounded (major) scale $\langle a_\alpha |\ \alpha<\omega_1\rangle$ then there is an f: $[\omega_1]^2\to\omega$ such that

1. There is a scale which is governed by f.

2. Every scale governed by f is unbounded (major).

References

[1] E. Borel, *Leçons sur la théorie des functions*, Gauthier-Villars et fils, 2-ed., Paris 1914.
[2] E. Ellentuck, *Godel's square axioms for the continuum*, unpublished manuscript.
[3] F. Hausdorff, *Summen von* $\aleph_1$ *Mengen*, Fund. Math. 26 (1936), pp. 241–55.
[4] S. Hechler, *Short complete nested sequences in* βN–N *and small maximal almost disjoint families*, Gen. Topology and Appl. 2 (1972), pp. 1–11.
[5] D. Martin and R. Solovay, *Internal Cohen extensions*, Ann. Math. Logic 2 (1970), pp. 143–178.
[6] F. Rothberger, *On some problems of Hausdorff and Sierpiński*, Fund. Math. 35 (1948), pp. 29–46.
[7] — *On the property C and a problem of Hausdorff*, Cand. J. Math. 4 (1952), pp. 111–119.
[8] — *Sur un ensemble toujours de première catégorie qui est dépourvu de la propriété* λ, Fund. Math. 32 (1939), pp. 294–300.
[9] — *Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété C*, Proc. of Camb. Philo. Soc. 37 (1941), pp. 109–126.
[10] D. Scott and R. Solovay, *Boolean-valued models for set theory*, unpublished manuscript.
[11] R. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. of Math. 9 (1971), pp. 201–245.
[12] F. Tall, *Hausdorff's gaps and limits and a new iterated extension axiom*, unpublished manuscript.

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Non-finitizability of a weak second-order theory

by

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Abstract. The weak second-order theory R_2, based on the axioms for ordered fields and the continuity scheme, and Tarski's weak second-order geometry $\mathcal{E}'_2$ are shown to be not finitely axiomatizable.

Introduction. Weak second-order theories are understood here in the sense of Tarski [11] (using finite sequences). Mostowski pointed out that the weak second-order theories of familiar mathematical structures are either finitely axiomatizable or not recursively axiomatizable (with respect to the notion of weak second-order consequence), in fact, the theory of real numbers does not even have an analytic axiom system (see [6]) while the theories of natural numbers, integers, rational numbers, and complex numbers turn out to be finitely axiomatizable.

Then, Vaught proved the existence of weak second-order theories which are recursively but not finitely axiomatizable (see appendix of [8]). The axiom systems in his example, however, are just constructed to get this result by a diagonal argument, namely, they are of the form

$$c = \bar{k}\to\neg\alpha_k \quad (k\in N)$$

where c is an individual constant, $\bar{k}$ is the numeral for the number k, and α_k is the kth sentence in a recursive enumeration of all sentences or of all first-order sentences.

So, it remained an open problem to find "mathematically motivated" weak second-order theories of the same kind. Already when writing [8], the author had two candidates for such theories — which are recursively axiomatizable by definition — and he discussed them with colleagues.

The aim of this paper is to show that one of these candidates (for the other one see 7.1) is indeed not finitely axiomatizable. It is the theory R_2 based on the axioms for ordered fields and the weak second-order continuity scheme.

With this, one also gets a negative answer to the question raised by Tarski in [12], p. 25 — if a corresponding weak second-order geometry $\mathcal{E}'_2$ is finitely axiomatizable (6.1).

The proof of our result makes use of the observation that all models of R_2 are Archimedian ordered fields. Then, a translation from R_2 into the system A_ω of second-order arithmetic of [2] is constructed which transforms the weak second-order continuity scheme into the comprehension scheme of A_ω. With this, the result is reduced to the following result of Mostowski:

(*) A_ω is not finitely axiomatizable.

The reduction is done in two steps (in Sections 3 and 4) using an intermediate theory R_N, which is formally first-order but contains a predicate N for being natural, for which standard interpretation is required.

I wish to express my thanks to Professors Mostowski and Dana Scott. Scott helped me by remarks in a discussion, which encouraged me to resume older attempts in this direction and to write to Mostowski asking if (*) holds. Mostowski sent me some letters, in which he gave a sketch of a proof of (*) and other helpful informations. Moreover, I could discuss this paper in his seminar. A proof of (*) has been published in the paper [5] of Zbierski.

1. Basic notions. We assume that two kinds of denumerably many variables are given, namely the *individual variables* (as in first-order theories) (usually denoted here by x, y, z, ..., sometimes with indices) and the *sequence variables* (variables for finite sequences) (X, Y, Z, ...). For defining *terms* and *formulas* of a *weak second-order language L* (determined by its "primitive" relation and operation symbols), we use all the rules familiar from first-order logic (¹) and, in addition to them, the following ones.

1. *Sequence terms* (terms for finite sequences) are introduced by the rules:

a) if X is a sequence variable and x an individual variable, then X and Ix are sequence terms.

b) if T_1, T_2 are sequence terms, then also $(T_1 \circ T_2)$ (parantheses will be omitted according to the usual conventions).

2. Equations $T_1 \doteq T_2$ between sequence terms T_1, T_2 are subsumed among the atomic formulas.

3. Quantification is allowed also for sequence variables.

A *structure* $\mathfrak{A}$ for L is understood as in first-order logic, i.e., it has only one universe $U_{\mathfrak{A}}$ (for interpreting the individual variables), while sequence variables are always interpreted as finite sequences of individuals (elements of $U_{\mathfrak{A}}$), I is interpreted as the operation of forming one-termed sequences from individuals ($\langle x\rangle$ from x), and $\circ$ is interpreted as the operation $^\frown$ of concatenation of finite sequences (if $X = \langle x_1, \ldots, x_k\rangle$ and $Y = \langle y_1, \ldots, y_l\rangle$, then $X^\frown Y = \langle x_1, \ldots x_k, y_1, \ldots, y_l\rangle$).

With this, it is assumed to be clear what it means that a *valuation* h over $\mathfrak{A}$ (i.e. a mapping which assigns to each individual variable an individual of $\mathfrak{A}$ and to each sequence variable a finite sequence of individuals of $\mathfrak{A}$) *satisfies* a formula α in $\mathfrak{A}$ (abbreviated h $\mathrm{Sat}_{\mathfrak{A}}\alpha$), also that α is *valid* in $\mathfrak{A}$ (i.e. satisfied by each valuation, abbreviated $\models_{\mathfrak{A}}\alpha$), that $\mathfrak{A}$ is a *model* of a set Σ of formulas, and that α is a (weak second-order or WII-) *consequence* from Σ (abbreviated $\alpha \in \mathrm{Cn}_{\mathrm{WII}}(\Sigma)$). (For more details compare, e.g., [8], however, we do not need here the case that a relation or operation of $\mathfrak{A}$ may have finite sequences as arguments.)

(¹) See, e.g., [9] as well as for other things, which are not defined here and can be naturally transferred to weak second-order logic when used so. We use $\doteq$ as equality sign of L. Fml_L denotes the set of formulas of L.

Intuitively speaking, the notion of weak second-order consequence differs from first-order consequence in that we have "standard interpretation" (not an arbitrary second universe) for the sequence variables and that we consider I and $\circ$ as logical constants (their meaning does not depend on the given model).

We shall not need here the equivalence of the notion of weak second-order consequence with a notion of formal provability (with infinitary proofs) as introduced by Lopez-Escobar in [4].

A set T of formulas is a *weak second-order theory* (WII-*theory*) iff $T = \mathrm{Cn}_{\mathrm{WII}}(T)$. T is called *finitely axiomatizable* (for short: *finitizable*) or *recursively axiomatizable* iff there is a finite or a recursive set Σ, respectively, which is an *axiom system* for T (i.e. $T = \mathrm{Cn}_{\mathrm{WII}}(\Sigma)$).

2. The WII-theory R_2, the Main Theorem. Let $L_2 = L(S, P, <)$ be the weak second-order language with two ternary relation symbols S, P and one binary relation symbol $<$. We shall read Sxyz as "the sum of x and y is z", Pxyz as "the product of x and y is z", and x<y as "x is less than y". It is known how other familiar notions as $+, -, \cdot, 0, 1, \leqslant, >, \geqslant$ can be expressed in this language. We shall freely use such symbols when writing up (abbreviations for) special formulas of L_2.

Let Σ_{OF} be a finite axiom system for ordered fields, expressed in L_2 (by first-order formulas). The WII-*continuity scheme* is the set (Ct) consisting of the following (infinitely many) formulas $\mathrm{Ct}_{\alpha,\beta}$:

(Ct) $$\forall x \forall y [\alpha(x) \wedge \beta(y) \to x<y] \wedge \exists x \alpha(x) \wedge \exists y \beta(y) \wedge \forall x [\alpha(x) \vee \beta(x)] \to \exists z \forall x \forall y [\alpha(x) \wedge \beta(y) \to x \leqslant z \leqslant y],$$

where $\alpha(x)$, $\beta(y)$ are arbitrary formulas of L_2 such that the variables y, z are not free in $\alpha(x)$ and x, z not free in $\beta(y)$. $\mathrm{Ct}_{\alpha,\beta}$ expresses the well known axiom of the Dedekind cut for the lower class and upper class defined by $\alpha(x)$ and $\beta(y)$, respectively. The definitions for these classes may contain "parameters", i.e. $\alpha(x)$ and $\beta(y)$ may contain free variables u, v, ..., X, Y, ... other than x, y. If one wants to have sentences only as axioms, one can consider the corresponding sentences $\overline{\mathrm{Ct}}_{\alpha,\beta} = \forall u \forall v \ldots \forall X \forall Y \ldots \mathrm{Ct}_{\alpha,\beta}$ instead of $\mathrm{Ct}_{\alpha,\beta}$.

Let R_2 be the WII-theory based on the axiom system $\Sigma_{\mathrm{OF}} \cup (\mathrm{Ct})$ in the language L_2.

We want to prove the

2.1. MAIN THEOREM. R_2 *is not finitely axiomatizable.*

This will be done by reducing the Theorem to the non-finitizability of some other theory T (in the second step $T = A_\omega$). The general idea of this reduction is the following one.

A "translation" is constructed by which already a finitizable subtheory R_2^- of R_2 is "equivalent" with a finitizable subtheory T^- of T (and R_2 with T). This ensures that finitely many (additional) axioms will be translated into finitely many additional axioms on the other side, and the finitizability of R_2 would imply that of T. To get a subtheory, the axioms of R_2^- have to be formulas $\alpha \in R_2$ ("theorems of R_2").

We put $R_2^- = Cn_{WII}(\Gamma)$, where Γ consists of the axioms of Σ_{OF} and the axioms A1, A2, to be fixed below.

3. Reduction to the theory R_N. In L_2, we can express that x is a "natural element" (i.e. a natural multiple of the unit element) of the field considered, in fact, this holds iff there is a "natural sequence for x", i.e. a finite sequence consisting of the consecutive natural elements from 0 to x. We may use then the natural elements as natural numbers.

For later use, we list here some more notions and formulas of L_2 which obviously express these notions.

3.1. "X is an initial segment of Y":

$$\mathrm{Is}(X, Y) := \exists Z\, X \circ Z \doteq Y \text{ (}^2\text{)}.$$

3.2. "X is a final segment of Y":

$$\mathrm{Fs}(X, Y) := \exists Z\, Z \circ X \doteq Y .$$

3.3. "X is a segment of Y":

$$\mathrm{Sg}(X, Y) := \exists U \exists V\, U \circ X \circ V \doteq Y .$$

3.4. "X and Y have the same length ("equal lengths"):

$$\begin{aligned}\mathrm{El}(X, Y) := \exists t \exists Z \{ &\neg \mathrm{Sg}(\mathrm{I}t, X) \wedge \neg \mathrm{Sg}(\mathrm{I}t, Y) \wedge \\ &\wedge \mathrm{Is}(\mathrm{I}t \circ X \circ \mathrm{I}t, Z) \wedge \mathrm{Fs}(\mathrm{I}t \circ Y \circ \mathrm{I}t, Z) \wedge \\ &\wedge \forall W \forall W' [\mathrm{Sg}(\mathrm{I}t \circ W \circ \mathrm{I}t \circ W' \circ \mathrm{I}t, Z) \wedge \\ &\wedge \neg \mathrm{Sg}(\mathrm{I}t, W) \wedge \neg \mathrm{Sg}(\mathrm{I}t, W') \\ &\to \exists x \exists y \exists U \exists V (W \doteq U \circ \mathrm{I}x \circ V \wedge W' \doteq U \circ \mathrm{I}y \circ V)]\} .\end{aligned}$$

In fact, $\mathrm{El}(X, Y)$ says that there is a finite sequence $\mathfrak{Z} = \langle W_1, \ldots, W_n \rangle$ of finite sequences W_ν beginning with X and ending with Y such that always $W_{\nu+1}$ arises from W_ν by changing one element only; $\mathfrak{Z}$ is described by a sequence

$$Z = \langle t \rangle^\frown W_1^\frown \langle t \rangle^\frown \ldots {}^\frown \langle t \rangle^\frown W_n^\frown \langle t \rangle$$

of individuals. (For structures with possibly finite universe, which we do not need here, one can express the same notion by using more complicated "separating sequences" C instead of one-termed sequences $\langle t \rangle$, see, e.g., [8], p. 75 or [7], p. 107.)

(²) We use $:=$ for equality by definition.

3.5. "X is a natural sequence for x":

$$\mathrm{Ns}(x, X) := \mathrm{Is}(\mathrm{I}0, X) \wedge \mathrm{Fs}(\mathrm{I}x, X) \wedge \forall y \forall z [\mathrm{Sg}(\mathrm{I}y \circ \mathrm{I}z, X) \to z \doteq y+1] .$$

3.6. "x is natural":

$$\mathrm{N}x := \exists X\, \mathrm{Ns}(x, X) .$$

3.7. "X has length k":

$$\mathrm{Lh}(k, X) := \{k \doteq 0 \wedge X \circ X \doteq X\} \vee \{k \geqslant 1 \wedge \exists Y [\mathrm{Ns}(k-1, Y) \wedge \mathrm{El}(X, Y)]\} .$$

3.8. "x is the kth member of X":

$$\mathrm{Mb}(x, k, X) := \exists U \exists V [X \doteq U \circ \mathrm{I}x \circ V \wedge \mathrm{Lh}(k, U \circ \mathrm{I}x)] .$$

In R_2^-, obviously, $\mathrm{Lh}(k, X)$ implies $\mathrm{N}k$, and $\mathrm{Mb}(x, k, X)$ implies $\mathrm{N}k \wedge k \geqslant 1$.

Using 3.6 (for which 3.4 is not needed) we can express the Archimedian axiom

A1: $$\forall x \exists n [\mathrm{N}n \wedge x < n] .$$

Moreover, A1 is a theorem of R_2, since the proof well-known from foundations of analysis for the Archimedian axiom uses a Dedekind cut which can be brought to the form $\mathrm{Ct}_{\alpha,\beta}$.

Putting A1 to the axioms of R_2^-, we get that each model of R_2^- (the more each model of R_2) is an Archimedian ordered field and, hence, isomorphic to a subfield of the ordered field $\boldsymbol{R}$ of real numbers (cf., e.g., [13], p. 245). So, we shall consider, in the sequel, only subfields of $\boldsymbol{R}$ as models of R_2^-, which is certainly sufficient. With this, natural elements are the same as natural numbers, and each model $\mathfrak{A}$ is determined by its universe $U_{\mathfrak{A}}$, which is a set of real numbers.

3.9. The "translation" mentioned before will transform each sentence α of L_2 into a sentence $\mathrm{Rd}(\alpha)$ (the "reductum" of α) which is equivalent in R_2^- with α but does not contain sequence variables in other connection than in subformulas of the form Nx. Formally, we introduce a new unary relation symbol N and require that it has "standard interpretation" only (i.e. $\mathrm{N}_{\mathfrak{A}}x$ holds iff x is natural). Then, N has the same meaning as in 3.6, and the following things are independent on how N is introduced. Then, R_2 will be translated into a theory R_N, which has a first-order language $L_N = L^1(S, P, <, N)$ but a notion Cn_N of consequence different from first-order logic (using models with standard interpretation for N only). Similarly, or R_2^- and a finitizable subtheory R_N^- of R_N.

3.10. For the natural numbers in our models, we can use well-known techniques for encoding finite sequences of natural numbers by one number. We use here the following results (see, e.g., [9] p. 115 ff.).

A. There are (primitive recursive) functions lh from N to N and β' from N^2 to N (with $\beta'(a, i)$ abbreviated by $(a)_i$) with the following properties:

1. lh and β' can be expressed by first-order formulas with primitive symbols $+$ and $\cdot$ only (interpreted in N).

2. For each finite sequence $\langle a_0, \ldots, a_{k-1}\rangle$ of natural numbers, there is a natural number a such that $\mathrm{lh}(a) = k$ and $(a)_\varkappa = a_\varkappa$ for each $\varkappa<k$. The least such number a will be called the *sequence number* of the given sequence.

3. $\mathrm{lh}(0) = 0$ (hence, 0 is the sequence number of the empty sequence).

Result A yields a technique of replacing inductive definitions by explicit ones. Especially, we get

B. There is an explicit definition (as in A.1) for the function e_2: $N\to N$ with $\mathrm{e}_2(n) = 2^n$.

We can extend the functions lh, β', and e_2 to universes of arbitrary models of R_2^- by assigning, say, the (not needed) value 0 if at least one argument is not natural. Using A.1 and B, we can express these functions in the languages L_2 and L_N. We shall freely use corresponding terms in these languages.

3.11. *Dyadic representations.* It is well-known that each real number x has a uniquely determined dyadic representation

(a) $$x = \sum_{\nu=0}^{\infty} \frac{x_\nu}{2^\nu},$$

where

(b) x_0 is an integer,

(c) for each $\nu\geqslant 1$: $x_\nu = 0$ or $x_\nu = 1$, but

(d) there is no n such that $x_\nu = 1$ for all $\nu\geqslant n$.

We call then x_ν the *ν-th digit in the dyadic representation of x.*

The *n-th dyadic approximation*

(e) $$\bar{x}_n = \sum_{\nu=0}^{n} \frac{x_\nu}{2^\nu}$$

of x is then uniquely determined by

(f) $$\bar{x}_n\cdot 2^n \text{ is an integer}$$

and

(g) $$0\leqslant x-\bar{x}_n<\frac{1}{2^n}.$$

Thus, the following notions are expressed (in any model of R_2^-) by the given formulas.

3.12. "x is an integer":

$$\mathrm{Int(x)}:= \mathrm{Nx}\vee \mathrm{N{-}x}.$$

3.13. "x is even":

$$\mathrm{Ev(x)}:= \exists \mathrm{u}[\mathrm{Int(u)}\wedge \mathrm{x} = 2\cdot \mathrm{u}].$$

3.14. "d is the nth digit (in the dyadic representation) of x":

$$\begin{aligned}\mathrm{D(d,n,x)}:= \mathrm{Nn}\wedge\exists \mathrm{s}\{&\mathrm{Int(s)}\wedge 0\leqslant 2^n\cdot \mathrm{x}-\mathrm{s}<1\wedge\\ &\wedge[(\mathrm{n}\doteq 0\wedge \mathrm{d}\doteq \mathrm{s})\vee\\ &\vee(\mathrm{n}\geqslant 1\wedge \mathrm{Ev(s)}\wedge \mathrm{d}\doteq 0)\vee\\ &\vee(\mathrm{n}\geqslant 1\wedge \neg\mathrm{Ev(s)}\wedge \mathrm{d}\doteq 1)]\}.\end{aligned}$$

Note that these formulas are (or may be considered as) formulas of L_N.

For eliminating sequence variables, we encode finite sequences of real numbers by single real numbers.

3.15. DEFINITION. For each integer x, we introduce the *natural number $\hat{x}$ encoding* x (the *code* of x) by

$$\hat{x} = \begin{cases}2\cdot x, & \text{if } x\geqslant 0,\\ 2\cdot(-x)-1, & \text{if } x<0.\end{cases}$$

3.16. DEFINITION. The real number y *encodes*, or, is the *code* of the *k-termed sequence* $X = \langle x_1, \ldots, x_k\rangle$ of reals (abbr. $y = \mathrm{c}(X)$) iff y_0 is the sequence number of the finite sequence $\langle \hat{x}_{1,0}, \ldots, \hat{x}_{k,0}\rangle$ of naturals, and for any $\nu\geqslant 1$: $y_{(\nu-1)\cdot k+\varkappa} = x_{\varkappa,\nu}$ ($\varkappa = 1, \ldots, k$); here, the additional indices are used for the digits of the numbers y and $x_\varkappa$ as in 3.11.

Obviously, we have a one-to-one correspondence between integers and their encodings and, also, between finite sequence of reals and their encodings. Moreover, also the last encoding is "absolute" in the sense that X has the same code y in all models of R_2^- containing y and the members of X.

The function $\hat{}$ (extended as in 3.10) can clearly be expressed in L_N. We can also express the following notions by formulas (as before).

3.17. "s is the sequence number of a k-termed sequence (of naturals)":

$$\begin{aligned}\mathrm{Sn(k,s)}:= \mathrm{Ns}\wedge \mathrm{lh(s)}\doteq \mathrm{k}\wedge\forall \mathrm{t}\{&\mathrm{Nt}\wedge \mathrm{lh(t)}\doteq \mathrm{k}\wedge\\ &\wedge\forall \mathrm{i}[\mathrm{Ni}\wedge \mathrm{i}<\mathrm{k}\to(\mathrm{s})_\mathrm{i}\doteq(\mathrm{t})_\mathrm{i}]\to \mathrm{s}\leqslant \mathrm{t}\}.\end{aligned}$$

3.18. "y encodes the k-termed sequence X (of reals)":

$$\begin{aligned}\mathrm{C(y,k,X)}:= \mathrm{Lh(k,X)}\wedge\exists \mathrm{y}_0\{&\mathrm{D(y_0,0,y)}\wedge \mathrm{Sn(k,y_0)}\wedge\\ &\wedge[\mathrm{k}\doteq 0\to \mathrm{y}\doteq 0]\wedge\\ &\wedge\forall \mathrm{i}\forall \mathrm{x}\forall \mathrm{u}\forall \mathrm{n}\forall \mathrm{d}[\mathrm{Ni}\wedge \mathrm{i}<\mathrm{k}\wedge \mathrm{Mb(x,i+1,X)}\wedge \mathrm{Nn}\wedge \mathrm{n}\geqslant 1\\ &\quad\to\langle \mathrm{D(u,0,x)}\leftrightarrow\hat{\mathrm{u}}\doteq(\mathrm{y}_0)_\mathrm{i}\rangle\wedge\\ &\quad\wedge\langle \mathrm{D(d,n,x)}\leftrightarrow \mathrm{D(d,(n-1)\cdot k+i+1,y)}\rangle]\}.\end{aligned}$$

3.19. "y encodes the finite sequence X":

$$C(y, X) := \exists k C(y, k, X) .$$

Since we want to encode finite sequences by single reals in an arbitrary model of R_2^-, we want to have

A2′: $$\forall X \exists y C(y, X)$$

as a theorem of R_2^-. We prefer to get this from an axiom A2 which will be formulated in L_N (and, hence, can be used for R_N^- also).

Already in L_N, we can express the following (as before).

3.20. "y encodes a k-termed (finite) sequence":

$$F(k, y) := \exists y_0\{D(y_0, 0, y) \wedge Sn(k, y_0) \wedge [k \doteq 0 \rightarrow y \doteq 0]\} \wedge \wedge \forall i\{Ni \wedge i<k \rightarrow \exists x \forall n \forall d[Nn \wedge n \geqslant 1 \rightarrow \cdot D(d, n, x) \leftrightarrow D(d, (n-1)\cdot k+i+1, y)]\} .$$

3.21. "y encodes a finite sequence":

$$F(y) := \exists k F(k, y) .$$

3.22. "y encodes the one-termed sequence $\langle x \rangle$":

$$yIx := \exists y_0 \exists x_0[D(y_0, 0, y) \wedge D(x_0, 0, x) \wedge Sn(1, y_0) \wedge (y_0)_0 \doteq x_0] \wedge Int(y-x) .$$

In this case, y and x have the same digits except the 0th one. Thus, one-termed sequences trivially have encodings.

3.23. "y_3 encodes the concatenation of the finite sequences encoded by y_1 and y_2":

$$\begin{aligned} Cc(y_1, y_2, y_3) := \exists k \exists m\{&F(k, y_1) \wedge F(m, y_2) \wedge F(k+m, y_3) \wedge \\ &\wedge \forall i \forall j \forall u_1 \forall u_2 \forall u_3 \forall n \forall d[Ni \wedge i<k \wedge Nj \wedge j<m \wedge \\ &\wedge \bigwedge_{\nu=1}^{3} D(u_\nu, 0, y_\nu) \wedge Nn \wedge n \geqslant 1 \rightarrow \\ &(u_3)_i \doteq (u_1)_i \wedge (u_3)_{k+j} \doteq (u_2)_j \wedge \\ &\wedge \langle D(d, (n-1)\cdot k+i+1, y_1) \leftrightarrow \\ &\quad D(d, (n-1)\cdot(k+m)+i+1, y_3)\rangle \wedge \\ &\wedge \langle D(d, (n-1)\cdot m+j+1, y_2) \leftrightarrow \\ &\quad D(d, (n-1)\cdot(k+m)+k+j+1, y_3)\rangle]\} . \end{aligned}$$

Now, we introduce

A2: $$\forall y_1 \forall y_2[F(y_1) \wedge F(y_2) \rightarrow \exists y_3 Cc(y_1, y_2, y_3)] .$$

It is intuitively clear, that A2 is a theorem of R_2, namely, by using a continuity axiom, one can prove the existence of a number y_3 with a dyadic representation which can be defined by means of the parameters y_1, y_2. A more detailed proof can be obtained from an axiom of the scheme (DR) (concerning dyadic representations) in the next section. Thus, we may take A2 as an axiom for R_2^-.

We also get A2′ from A2, since each finite sequence can be obtained by concatenating one-termed sequences.

3.24. Now, we can introduce the "translation" described in 3.9. Let ϑ be a one-to-one mapping from the set of all (individual and sequence) variables into the set of individual variables; for abbreviation, we put $x' = \vartheta(x)$, $X' = \vartheta(X)$ (both are individual variables !). By a well-known method, we can transform any $\alpha \in Fml_{L_2}$ into a logically equivalent formula $\bar{\alpha}$ containing atomic formulas only of the forms given in (1) below. We put then $Rd(\alpha) = Rd(\bar{\alpha})$, where the latter is given by the inductive definition

(1) $Rd(Sxyz) = Sx'y'z'$.
Similarly, for $Pxyz$, $x<y$, $x \doteq y$, $X \doteq Y$.
$Rd(Y \doteq Ix) = Y'Ix'$.
$Rd(X \circ Y \doteq Z) = Cc(X', Y', Z')$.

(2) $Rd(\alpha \wedge \beta) = Rd(\alpha) \wedge Rd(\beta)$.
Similarly, for the other propositional connectives.
$Rd(\forall x \alpha) = \forall x' Rd(\alpha)$
$Rd(\forall X \alpha) = \forall X'[F(X') \rightarrow Rd(\alpha)]$.
Similarly, for $\exists$.

Clearly, always $Rd(\alpha) \in Fml_{L_N}$. By an inductive proof (following (1) and (2)), we get

3.25. Theorem. *If $\mathfrak{A}$ is a model of R_2^- and h, h' are "corresponding" valuations over $\mathfrak{A}$, i.e., such that $h'(x') = h(x)$, $h'(X') = c(h(X))$ for any variable* x *or* X, *then*

$$h \, Sat_{\mathfrak{A}} \alpha \quad \textit{iff} \quad h' \, Sat_{\mathfrak{A}} Rd(\alpha) .$$

Since valuations are not needed for sentences, we get

3.26. Corollary. *If $\mathfrak{A}$ is as before and α a sentence of L_2, then*

$$\models_{\mathfrak{A}} \alpha \textit{ iff } \models_{\mathfrak{A}} Rd(\alpha) .$$

Let R_N^- be the theory with the language L_N based again on the axioms of Σ_{OF} and A1, A2 (with consequence Cn_N, see 3.9). Then, R_N^- and R_2^- have the same models. Similarly, let R_N be the theory based on Σ_{OF} and the continuity axioms $Ct_{\alpha', \beta'}$, now for formulas α', β' of L_N. As before, we get A1, A2 as theorems of R_N, hence, R_N^- is a finitizable subtheory of R_N.

3.27. Lemma. *The formulas* $Rd(Nx)$ *and* Nx' *are equivalent in* R_N^-.

Proof. By 3.25, $Rd(Nx)$ (with Nx from 3.6) is satisfied (like the atomic formula Nx') by exactly the valuations which assign a natural element to the variable x'.

Applying 3.26 to continuity axioms, we get

3.28. THEOREM. *An arbitrary continuity axiom* $\overline{\mathrm{Ct}}_{\alpha,\beta} = \forall u_1 \dots \forall u_m \forall X_1 \dots \forall X_n \mathrm{Ct}_{\alpha,\beta}$ *of* L_2 *(written as a sentence) holds in a model* $\mathfrak{A}$ *of* R_2^- *iff*

$$\gamma' = \forall u'_1 \dots \forall u'_m \forall X'_1 \dots \forall X'_n [\bigwedge_{\nu=1}^{n} F(X'_\nu) \to \mathrm{Ct}_{\mathrm{Rd}(\alpha),\mathrm{Rd}(\beta)}]$$

holds in $\mathfrak{A}$.

Here, γ' is a consequence of a continuity axiom (without the premise occurring in γ') of L_N. On the other hand, we can obtain an arbitrary continuity axiom $\overline{\mathrm{Ct}}_{\alpha',\beta'}$ of L_N, up to equivalence, as such a γ'. Namely, let α, β be like α', β' but in L_2 (with N from 3.6). Applying 3.28 to these α, β, we have $n = 0$. Moreover, if we replace in $\mathrm{Rd}(\alpha)$, $\mathrm{Rd}(\beta)$ the corresponding parts $\mathrm{Rd}(\mathrm{Nx})$ by Nx', we get back α', β' — up to a change of variables. By 3.27, this replacement gives equivalent formulas, hence, also γ' is equivalent to $\overline{\mathrm{Ct}}_{\alpha',\beta'}$.

Thus, we get that all continuity axioms of L_2 hold in a model $\mathfrak{A}$ of R_2^- (or R_N^-) iff all continuity axioms of L_N hold in $\mathfrak{A}$. This yields

3.29. THEOREM. *Also* R_2 *and* R_N *are equivalent in the sense that they have the same models.*

Our reduction is completed by

3.30. THEOREM. *If* R_N *is non-finitizable, then also* R_2.

Proof. If R_2 would have a system of finitely many axioms — written as sentences —, then the reductums of these axioms (in addition to the axioms of R_N^-) would characterize the same class of models, hence, they would form a finite axiom system for R_N.

We can replace in 3.30 "if — then" by "iff", since the converse "translation" is obvious.

4. Reduction to the theory A_ω. We use here the set-theoretical version of A_ω, which can be described als follows. A_ω has a second-order language with L_ω with variables for individuals (a, b, c, ...) and variables for sets (A, B, C, ...) and non-logical symbols $+, \cdot, 0, 1$ (for arithmetic of natural numbers); the atomic formulas are those familiar from first-order logic and $\tau \in A$ where τ is a number term (as in first-order logic) and A a set variable.

An ω-*structure* is a structure $\mathfrak{M}$ which has the set N of natural numbers as universe for the individual variables and standard interpretation for $+, \cdot, 0, 1, \in$ (i.e., these are interpreted as the usual addition, multiplication, zero, and one in N and the element relation) while the universe $S_{\mathfrak{M}}$ for the set variables is an arbitrary subset of $\mathfrak{P}(N)$, i.e., it consists of certain (but not necessarily all) sets of natural numbers. Obviously, each ω-structure $\mathfrak{M}$ is uniquely determined by $S_{\mathfrak{M}}$; thus, also $S_{\mathfrak{M}}$ is sometimes called an ω-structure.

This notion of structures leads to a corresponding notion Cn_ω of ω-*consequence*, i.e., $\alpha \in \mathrm{Cn}_\omega(\Sigma)$ iff α holds in all ω-models (ω-structures which are models) of Σ.

The theory A_ω is the set of ω-consequences of the *comprehension scheme* (Cp), which consists of the following (infinitely many) formulas Cp_φ:

$$(\mathrm{Cp}) \qquad \exists A \forall a [a \in A \leftrightarrow \varphi(a)],$$

where $\varphi(a)$ is an arbitrary formula of L_ω such that the variable A is not free in $\varphi(a)$. Again, parameters are allowed, and one can use the corresponding sentences $\overline{\mathrm{Cp}}_\varphi = \forall a_1 \dots \forall a_m \forall A_1 \dots \forall A_n \mathrm{Cp}_\varphi$ instead of the Cp_φ.

The notion of ω-consequence can be equivalently replaced by a notion of ω-provability (with infinitary proofs, using the ω-rule and some more "arithmetical" axioms, see [2] for a version of A_ω with variables for number-theoretic functions instead of sets). However, this will not be used in this paper.

We shall construct two "translations", from L_N to L_ω and conversely, and get theorems similar to 3.25.

For the first translation, we encode each real number by a pair of a natural number and a set of naturals as follows.

4.1. DEFINITION. Let x be an arbitrary real number, and its dyadic representation given as in 3.11. We put

$$s_1(x) := \hat{x}_0,$$

$$s_2(x) := \{n \in N \mid x_{n+1} = 0\},$$

$$s(x) := \langle s_1(x), s_2(x) \rangle.$$

We call these things the *number code*, the *set code* and the *code* of x, respectively.

Intuitively speaking, $s_1(x)$ encodes the integer part and $s_2(x)$ the fractional part of x, and we have $s_2(x) = s_2(x \pm 1)$. s gives a one-to-one correspondence between real numbers and their codes. Since we required 3.11(d) for the dyadic representations, $s_2(x)$ is always an infinite set, and each infinite set of natural numbers can be obtained as $s_2(x)$ for some real number x. On the other hand, also finite sets can be obtained in this way if we drop 3.11(d).

4.2. DEFINITION. The pair $\langle a, A \rangle$ is an *encoding* of x iff it is given as $s(x)$ in 4.1 but with a dyadic representation satisfying 3.11(a)-(c) only.

Thus, a real number x is uniquely determined by each of its (one or two) encodings; it has two encodings iff its set code is cofinite (i.e., the complement of a finite set).

We have to express the primitive notions for reals by their encodings. For this, we use some calculations and definitions.

Let the sequence $\langle \bar{x}_n \rangle_{n \in N}$ of dyadic approximations of x be defined as in 3.11 (with condition (d) not necessarily holding). Then, this sequence has the limit x, and $0 \leqslant x - \bar{x}_n \leqslant \frac{1}{2^n}$ for each n. Similarly, for y, $\bar{y}_n$, z, $\bar{z}_n$. Now, let

$$d_n = \bar{z}_n - (\bar{x}_n + \bar{y}_n), \qquad e_n = \bar{z}_n - \bar{x}_n \cdot \bar{y}_n.$$

A routine calculation gives that $x+y = z$ iff

(1) $$|d_n| \leqslant \frac{3}{2^n} \quad \text{for each } n,$$

and, that $x \cdot y = z$ iff

(2) $$|e_n| \leqslant \frac{K}{2^n} \quad \text{for each } n,$$

where K is an arbitrary number with $K \geqslant |x_0|+|y_0|+3$. We may put then $K = \hat{x}_0+\hat{y}_0+3$ (since $|u| \leqslant \hat{u}$ for any integer u).

Let g be the unary function from N into N given by

$$g(n) = \begin{cases} \dfrac{3n}{2}, & \text{if } n \text{ is even,} \\ \dfrac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Then, for any integer x, we have

(3) $$g(\hat{x}) = \hat{x}+x$$

(g was just defined to get this). g (is primitive recursive and) can be expressed by a first-order formula of arithmetic (as in 3.10, A1). We use in L_ω a corresponding operation symbol g, as well as ^ and terms for the functions considered in 3.10.

4.3. Definition. Let "r represents A on (the segment determined by) n" be the notion expressed by the following formula of L_ω:

$$\mathrm{rRp_nA}: = \forall \mathrm{i}\{\mathrm{i}<\mathrm{n} \rightarrow [(\mathrm{i} \in \mathrm{A} \rightarrow (\mathrm{r})_\mathrm{i} \doteq 0) \wedge (\neg \mathrm{i} \in \mathrm{A} \rightarrow (\mathrm{r})_\mathrm{i} \doteq 1)]\} .$$

We get then that the following notions are expressed (in any ω-structure containing the sets mentioned) by the formulas given there. (For 4.4 and 4.5, this is obtained by multiplying (1) and (2) with factors and adding summands such that these conditions can be expressed by means of natural numbers instead of rationals.)

4.4. "$\langle a, A\rangle$, $\langle b, B\rangle$, and $\langle c, C\rangle$ are encodings of real numbers x, y, z with $x+y = z$":

$$\begin{aligned} \mathrm{S(a, A, b, B, c, C)}: = \forall \mathrm{n} \forall \mathrm{r} \forall \mathrm{s} \forall \mathrm{t}\{ & \mathrm{rRp_nA} \wedge \mathrm{sRp_nB} \wedge \mathrm{tRp_nC} \\ & \rightarrow \mathrm{c} \cdot 2^\mathrm{n} + \Sigma_\mathrm{A} + \Sigma_\mathrm{B} \\ & \leqslant 3 + [\mathrm{a}+\mathrm{b}] \cdot 2^\mathrm{n} + \Sigma_\mathrm{C} \\ & \leqslant 6 + \mathrm{c} \cdot 2^\mathrm{n} + \Sigma_\mathrm{A} + \Sigma_\mathrm{B}\} , \end{aligned}$$

where

$$\Sigma_\mathrm{A} = \mathrm{g(a)} \cdot 2^\mathrm{n} + \sum_{\nu=0}^{n-1} (\mathrm{r})_\nu \cdot 2^{\mathrm{n}-1-\nu}$$

and Σ_B, Σ_C similar with a, r replaced by b, s or c, t, respectively (given by explicit definitions according to 3.10).

4.5. "$\langle a, A\rangle$, $\langle b, B\rangle$, and $\langle c, C\rangle$ are encodings of real numbers x, y, z with $x \cdot y = z$":

$$\begin{aligned} \mathrm{P(a, A, b, B, c, C)}: = \forall \mathrm{n} \forall \mathrm{r} \forall \mathrm{s} \forall \mathrm{t}\{ & \mathrm{rRp_nA} \wedge \mathrm{sRp_nB} \wedge \mathrm{tRp_nC} \\ & \rightarrow [\mathrm{c}+\mathrm{a} \cdot \mathrm{b}] \cdot 2^{2\mathrm{n}} + \Sigma_\mathrm{A} \cdot \Sigma_\mathrm{B} \\ & \leqslant [\mathrm{a}+\mathrm{b}+3+\Sigma_\mathrm{C}+\mathrm{a} \cdot \Sigma_\mathrm{B}+\mathrm{b} \cdot \Sigma_\mathrm{A}] \cdot 2^\mathrm{n} \\ & \leqslant [\mathrm{a}+\mathrm{b}+3] \cdot 2^{\mathrm{n}+1} + [\mathrm{c}+\mathrm{a} \cdot \mathrm{b}] \cdot 2^{2\mathrm{n}} + \Sigma_\mathrm{A} \cdot \Sigma_\mathrm{B}\} , \end{aligned}$$

where Σ_A, Σ_B, Σ_C as before.

Remark. From 4.4, 4.5, one can see that the relations defined by the given formulas in the standard model are Π_1^0 in the sense of the Kleene-hierarchy (see, e.g., [9], Chapter 7). On the other hand, these relations are not recursive since the (non-recursive) equality of sets (for $\langle b, B\rangle$ an encoding of 0 or 1, respectively, and, say, A neither finite nor cofinite) can be obtained as a special case.

4.6. "A is an infinite set (of naturals)":

$$\mathrm{Inf(A)}: = \forall \mathrm{a} \exists \mathrm{b} [\mathrm{a}<\mathrm{b} \wedge \mathrm{b} \in \mathrm{A}] .$$

4.7. "$\langle a, A\rangle$ and $\langle b, B\rangle$ are codes of real numbers x, y with $x<y$":

$$\begin{aligned} \mathrm{L(a, A, b, B)}: = {} & \mathrm{Inf(A)} \wedge \mathrm{Inf(B)} \wedge \\ & \wedge [\mathrm{b}+\mathrm{g(a)} < \mathrm{a}+\mathrm{g(b)} \vee \\ & \vee \{\mathrm{b}+\mathrm{g(a)} \doteq \mathrm{a}+\mathrm{g(b)} \wedge \exists \mathrm{n} [\forall \mathrm{i}(\mathrm{i}<\mathrm{n} \rightarrow \cdot \mathrm{i} \in \mathrm{A} \leftrightarrow \mathrm{i} \in \mathrm{B}) \wedge \\ & \wedge \mathrm{n} \in \mathrm{A} \wedge \neg \mathrm{n} \in \mathrm{B}]\}] . \end{aligned}$$

4.8. "$\langle a, A\rangle$ is the code of a natural number":

$$\mathrm{N(a, A)}: = \exists \mathrm{b}\, \mathrm{a} \doteq 2\mathrm{b} \wedge \forall \mathrm{n}\, \mathrm{n} \in \mathrm{A} .$$

4.9. For our first translation, let ϑ_1 be a one-to-one mapping from the set of all individual variables (of L_N) into the set of pairs of an individual and a set variable (of L_ω); for abbreviation, we put $\langle x', x''\rangle = \vartheta_1(x)$. Then, the s-reductum $\mathrm{Rd_s}(\alpha)$, for formulas α of L_N, is given by the inductive definition

(1) For α being Sxyz, Pxyz, $x<y$, Nx, $x \doteq y$, the s-reductum $\mathrm{Rd_s}(\alpha)$ is $\mathrm{S}(x', x'', y', y'', z', z'')$, $\mathrm{P}(x', x'', y', y'', z', z'')$, $\mathrm{L}(x', x'', y', y'')$, $\mathrm{N}(x', x'')$, $x' \doteq y' \wedge x'' \doteq y''$, respectively.

(2) $\mathrm{Rd_s}(\alpha \wedge \beta) = \mathrm{Rd_s}(\alpha) \wedge \mathrm{Rd_s}(\beta)$.
$\mathrm{Rd_s}(\forall x \alpha) = \forall x' \forall x'' [\mathrm{Inf}(x'') \rightarrow \mathrm{Rd_s}(\alpha)]$.

Similarly, for the other propositional connectives and for $\exists$.
Clearly, always $\mathrm{Rd_s}(\alpha) \in \mathrm{Fml}_{L_\omega}$.
For the second translation, we use the following encoding.

4.10. Definition. Let A be an arbitrary set of natural numbers. The *code* $r(A)$ is the real number x determined by the following dyadic representation (with digits x_ν as in 3.11):

a) if A is infinite: $x_0 = 0$, $x_{n+1} = 0$ iff $n \in A$,

b) if A is finite: $x_0 = 1$, $x_{n+1} = 1$ iff $n \in A$.

In 4.10, obviously, if $0 \leqslant x < 1$, then $s_2(x) = A$, and, if $1 \leqslant x < 2$, then $s_2(x) = \bar{A}$ (the complement of A). Also, if A is finite, then $r(A) = r(\bar{A}) + 1$. Again, both encodings are "absolute".

Then, the following notions are expressed by the formulas of L_N given with them.

4.11. "x is the code of a set of natural numbers":

$$S(x) := 0 \leqslant x < 1 \vee \{1 \leqslant x < 2 \wedge \exists k [Nk \wedge Int(2^k \cdot x)]\}.$$

4.12. Provided that x is such a code and n natural, the statement "n is an element of the set encoded by x" is expressed by

$$E(n, x) := [0 \leqslant x < 1 \wedge D(0, n+1, x)] \vee [1 \leqslant x < 2 \wedge D(1, n+1, x)].$$

4.13. Let ϑ_2 be a one-to-one mapping from the set of all variables of L_ω (individual and set variables) into the set of variables of L_N; for abbreviation we put $a' = \vartheta_2(a)$, $A' = \vartheta_2(A)$ (both are variables for real numbers!). By a well-known method, any $\varphi \in Fml_{L_\omega}$ can be transformed into a formula $\bar{\varphi}$ logically equivalent with φ and containing atomic formulas only of the forms given in (1) below. We put then $Rd_r(\varphi) = Rd_r(\bar{\varphi})$, where the latter is given by the inductive definition

(1) For φ being $a+b \doteq c$, $a \cdot b \doteq c$, $a \doteq b$, $a \in A$, the r-reductum $Rd_r(\varphi)$ is $Sa'b'c'$, $Pa'b'c'$, $a' \doteq b'$, $E(a', A')$, respectively.

(2) $Rd_r(\varphi \wedge \psi) = Rd_r(\varphi) \wedge Rd_r(\psi)$.
$Rd_r(\forall a \varphi) = \forall a'[Na' \to Rd_r(\varphi)]$.
$Rd_r(\forall A \varphi) = \forall A'[S(A') \to Rd_r(\varphi)]$.

Similary, for the other propositional connectives and for $\exists$.

We consider now, (relational-) substructures $\mathfrak{A}$ of $\boldsymbol{R}$ (more general than in 3.25) and corresponding ω-structures $\mathfrak{M}$ of the following kind:

4.14. $U_\mathfrak{A}$ is a set of real numbers such that (a) whenever $x \in U_\mathfrak{A}$, then also $x \pm 1 \in U_\mathfrak{A}$, (b) all numbers of the form $s/2^k$, where s is an integer and k natural, are in $U_\mathfrak{A}$.

4.15. $S_\mathfrak{M}$ is a subset of $\mathfrak{P}(\boldsymbol{N})$ such that all finite sets of naturals and their complements are in $S_\mathfrak{M}$.

4.16. Definition. For each $\mathfrak{A}$ with 4.14, let $s(\mathfrak{A})$ be the ω-structure $\mathfrak{M}$ with $S_\mathfrak{M}$ consisting of (i) the sets $s_2(x)$ with $x \in U_\mathfrak{A}$, and (ii) the finite sets A with $\bar{A}$ of the form (i).

4.17. Definition. For each $\mathfrak{M}$ with 4.15, let $r(\mathfrak{M})$ be the substructure $\mathfrak{A}$ of $\boldsymbol{R}$ with $U_\mathfrak{A}$ consisting of the real numbers $x+u$ such that x is of the form $r(A)$ with $A \in S_\mathfrak{M}$ and u is an integer.

We easily get

4.18. Theorem. *For each* $\mathfrak{A}$ *with* 4.14 *and* $\mathfrak{M}$ *with* 4.15:
$s(\mathfrak{A})$ *has Property* 4.15,
$r(\mathfrak{M})$ *has Property* 4.14,
$r(s(\mathfrak{A})) = \mathfrak{A}$,
$s(r(\mathfrak{M})) = \mathfrak{M}$.

4.19. Theorem. *The* ω-*structures of type* 4.15 *are exactly the models of the theory* A_ω^- *based on the following axioms*:

B1 $\exists A \forall a \neg a \in A,$

B2 $\neg Inf(B) \to \exists A \forall a [a \in A \leftrightarrow a \in B \vee a \doteq b],$

B3 $\neg Inf(B) \to \exists A \forall a [a \in A \leftrightarrow \neg a \in B].$

These axioms are trivial consequences of certain comprehension axioms, thus, A_ω^- is a finitizable subtheory of A_ω.

By inductive proofs following 4.9 or 4.13, respectively, we get

4.20. Theorem. *If* $\mathfrak{A}$ *is of type* 4.14 *and* h, h' *are "corresponding" valuations over* $\mathfrak{A}$, $s(\mathfrak{A})$, *respectively, i.e.* $h'(x') = s_1(h(x))$, $h'(x'') = s_2(h(x))$ *for any variable* x *of* L_N, *then*

$$h\,Sat_\mathfrak{A}\,\alpha \quad \textit{iff} \quad h'\,Sat_{s(\mathfrak{A})}\,Rd_s(\alpha).$$

4.21. Theorem. *If* $\mathfrak{M}$ *is of type* 4.15 *and* h, h' *are "corresponding" valuations over* $\mathfrak{M}$, $r(\mathfrak{M})$, *respectively, i.e.* $h'(a') = h(a)$, $h'(A') = r(h(A))$ *for any variable* a *or* A *of* L_ω, *then*

$$h\,Sat_\mathfrak{M}\,\varphi \quad \textit{iff} \quad h'\,Sat_{r(\mathfrak{M})}\,Rd_r(\varphi).$$

4.22. Corollary. *If* $\mathfrak{A}$ *is of type* 4.14 *and* $\mathfrak{M} = s(\mathfrak{A})$ (*equivalently*, $\mathfrak{M}$ *of type* 4.15 *and* $\mathfrak{A} = r(\mathfrak{M})$) *and* α, φ *are sentences of* L_N, L_ω *respectively, then*

(i) $\models_\mathfrak{A} \alpha$ *iff* $\models_\mathfrak{M} Rd_s(\alpha)$,

(ii) $\models_\mathfrak{M} \varphi$ *iff* $\models_\mathfrak{A} Rd_r(\varphi)$.

Next, we have to check how the continuity axioms behave in our translation. First, we replace them by other axioms for R_N.

Under the hypothesis of $Ct_{\alpha,\beta}$ (sect. 2), the formula $\beta(x)$ defines the complement of the set defined by $\alpha(x)$, hence, $\beta(y)$ can be replaced by $\neg\alpha(y)$. Moreover, it is well-known that one can make the additional hypothesis that the upper class has no least element. Thus, $Ct_{\alpha,\beta}$ is equivalent (under Σ_{OF}) with the "special continuity axiom":

$$Cts_\alpha: \quad \forall x \forall y [\alpha(x) \wedge \neg\alpha(y) \to x < y] \wedge \exists x \alpha(x) \wedge \exists y \neg\alpha(y) \wedge \\ \wedge \neg\exists x \{\neg\alpha(x) \wedge \forall y [\neg\alpha(y) \to x \leqslant y]\} \\ \to \exists z \forall x \forall y [\alpha(x) \wedge \neg\alpha(y) \to x \leqslant z < y],$$

and we can equivalently replace the scheme (Ct) by the scheme (Cts) consisting of all such Cts_α (again, with y, z not free in $\alpha(x)$).

Let (DR) be the set consisting of the following formulas $\mathrm{DR}_{u,\gamma}$:

$$\text{(DR)}\quad \mathrm{Int}(u)\wedge\forall m\{\mathrm{N}m\to\exists n[\mathrm{N}n\wedge m<n\wedge\gamma(n)]\}$$
$$\to\exists z\{\mathrm{D}(u,0,z)\wedge\forall n[\mathrm{N}n\to\cdot\mathrm{D}(0,n+1,z)\leftrightarrow\gamma(n)]\},$$

where $\gamma(n)$ is an arbitrary formula of L_N (possibly with "parameters" as before) such that z is not free in $\gamma(n)$. $\mathrm{DR}_{u,\gamma}$ expresses that there is a real number z with the dyadic representation given by the integer part u and the formula γ, which may be considered as defining a set of natural numbers. $\overline{\mathrm{Cts}}_\alpha$ and $\overline{\mathrm{DR}}_{u,\gamma}$ denote the sentences obtained by universal quantification as before.

Cts_α and $\mathrm{DR}_{u,\gamma}$ have the forms

$$\mathrm{Cts}_\alpha=\mathrm{HC}_\alpha\to\exists z\mathrm{CC}_\alpha(z),$$
$$\mathrm{DR}_{u,\gamma}=\mathrm{HD}_{u,\gamma}\to\exists z\mathrm{CD}_{u,\gamma}(z),$$

where HC_α denotes the hypothesis and CC_α the claim for z in Cts_α, similarly for $\mathrm{DR}_{u,\gamma}$.

The equivalence of both schemes can be obtained, intuitively speaking, by observing that, in a Dedekind cut, the dyadic representation of the separating element can be defined in terms of the lower class, and conversely. To make this precise, we put

4.23. $\alpha_{u,\gamma}(x):=\forall t\forall k\{\mathrm{N}k\wedge\mathrm{D}(u,0,t)\wedge\forall i[\mathrm{N}i\wedge i<k\to\cdot\mathrm{D}(0,i+1,t)\leftrightarrow\gamma(i)]\wedge$
$\wedge\forall i[\mathrm{N}i\wedge k\leqslant i\to\mathrm{D}(0,i+1,t)]\to x<t+1/2^k\}$

(here, the hypothesis stated for t and k means that the dyadic representation of t coincides with the given one up to the k-th digit and has zeros afterwards),

4.24. $\gamma_\alpha(n):=\exists t[\mathrm{Int}(2^n\cdot t)\wedge\alpha(t)\wedge\neg\alpha(t+1/2^{n+1})]$
(t denoting the (n+1)-st dyadic approximation of a separating element intended, where $2^{n+1}\cdot t$ is even).

Then, we have as a theorem of R_N^-:

4.25. $\mathrm{N}n\to\cdot\neg\gamma_\alpha(n)\leftrightarrow\exists t[\mathrm{Int}(2^{n+1}\cdot t)\wedge\alpha(t)\wedge\neg\alpha(t+1/2^{n+1})\wedge\neg\mathrm{Int}(2^n\cdot t)]$.

Moreover, we get

4.26. LEMMA. *If* $\gamma(n)=\gamma_\alpha(n)$ *and* $\mathfrak{A}$ *a subfield of* $\boldsymbol{R}$, *then*

$$\models_{\mathfrak{A}}\mathrm{HC}_\alpha\wedge\mathrm{Int}(u)\wedge\alpha(u)\wedge\neg\alpha(u+1)\to\mathrm{HD}_{u,\gamma}\wedge[\mathrm{CC}_\alpha(z)\leftrightarrow\mathrm{CD}_{u,\gamma}(z)],$$

and, hence

$$\models_{\mathfrak{A}}\forall u\mathrm{DR}_{u,\gamma}\to\mathrm{Cts}_\alpha.$$

4.27. LEMMA. *If* $\alpha(x)=\alpha_{u,\gamma}(x)$ *and* $\mathfrak{A}$ *is a subfield of* $\boldsymbol{R}$, *then*

$$\models_{\mathfrak{A}}\mathrm{HD}_{u,\gamma}\to\mathrm{HC}_\alpha\wedge[\mathrm{CC}_\alpha(z)\leftrightarrow\mathrm{CD}_{u,\gamma}(z)],$$

and, hence,

$$\models_{\mathfrak{A}}\mathrm{Cts}_\alpha\to\forall u\mathrm{DR}_{u,\gamma}.$$

Thus, in a subfield $\mathfrak{A}$ of $\boldsymbol{R}$, all axioms of (Cts) hold iff all axioms of (DR) hold, hence

4.28. THEOREM. $\Sigma_{\mathrm{OF}}\cup\{\mathrm{A1}\}\cup(\mathrm{DR})$ *is an axiom system for* R_N.

4.29. Remark. The same argument goes through for R_2 if we allow formulas from L_2 in the schemata (Ct), (Cts), and (DR).

Now, consider an arbitrary axiom $\mathrm{DR}_{u,\gamma}$ from (DR). For its claim, we get

$$\mathrm{Rd_s}(\exists z\mathrm{CD}_{u,\gamma})=\exists z'\exists A\{\mathrm{Inf}(A)\wedge\mathrm{Rd_s}(\mathrm{D}(u,0,z))\wedge\forall n'\forall n''$$
$$[\exists a\, n'\doteq 2a\wedge n''\doteq N\to\cdot\psi(n',n'',z',A)\leftrightarrow\varphi(n',n'')]\},$$

where, for abbreviation, N is used as a constant (which can be eliminated), $A=z''$, $\psi(n',n'',z',A)=\mathrm{Rd_s}(\mathrm{D}(0,n+1,z))$, $\varphi(n',n'')=\mathrm{Rd_s}(\gamma(n))$. Applying 4.20, we get that (i) the formulas $\mathrm{N}n\wedge\gamma(n)$ and $\varphi(2a,N)$ "define" the same (infinite) set of natural numbers (when corresponding models and valuations for the other free variables are given), and (ii) the formulas $\psi(2a,N,z',A)$ and $a\in A$ are equivalent in A_ω^-. With this (and a similar argument for the hypothesis $\mathrm{HD}_{u,\gamma}$), the reductum $\mathrm{Rd_s}(\overline{\mathrm{DR}}_{u,\gamma})$ turns out to be a consequence of a certain comprehension axiom $\overline{\mathrm{Cp}}_\varphi$.

On the other hand, consider an arbitrary comprehension axiom Cp_φ. For its reductum, we have

$$\mathrm{Rd_r}(\mathrm{Cp}_\varphi)=\exists x\{\mathrm{S}(x)\wedge\forall n[\mathrm{N}n\to\cdot\mathrm{E}(n,x)\leftrightarrow\gamma(n)]\},$$

where $x=A'$, $n=a'$, $\gamma(n)=\mathrm{Rd_r}(\varphi(a))$. Distinguishing the cases that $\gamma(n)$ fulfills the hypothesis of $\mathrm{DR}_{u,\gamma}$ or not, we get $\mathrm{Rd_r}(\overline{\mathrm{Cp}}_\varphi)$ from $\overline{\mathrm{DR}}_{0,\gamma}\wedge\overline{\mathrm{DR}}_{1,\neg\gamma}$, i.e., as a consequence of (two axioms of) (DR).

Thus, by 4.22, all axioms of (DR) hold in $\mathfrak{A}$ iff all axioms of (Cp) hold in the corresponding model, in other words,

4.30. THEOREM. R_N *and* A_ω *are equivalent in the sense that, for any* $\mathfrak{A}$, $\mathfrak{M}$ *as in* 4.22, $\mathfrak{A}$ *is a model of* R_N *iff* $\mathfrak{M}$ *is a model of* A_ω.

Our reduction is completed by

4.31. THEOREM. *If* A_ω *is non-finitizable, then* R_N *is non-finitizable.*

Proof. If R_N would have a system of finitely many axioms (sentences), their reductums (in addition to the axioms of A_ω^-) would also characterize, by 4.22, the class of models corresponding to models of R_N, hence, they would form a finite axiom system for A_ω.

Of course, we also get the converse of 4.31 (noting that all models of R_N^- are of type 4.14).

With 3.29, 4.31 and the result (*) from [5] (see Introduction), the Main Theorem 2.1 is proved.

5. Models. Since R is a model and the downward Löwenheim–Skolem–Tarski Theorem holds for weak second-order languages (see [11]), R_2 has also countable models.

By 3.28 and 4.30, the models of R_2 are (up to isomorphism) those of the form $r(\mathfrak{M})$, where $\mathfrak{M}$ is a model of A_ω. From the corresponding results for A_ω (see [2] and [1, I]), we get

5.1. THEOREM. (i) *The intersection of all models of* R_2 (*considered as subfields of* R) *is the field* $\mathfrak{F}_H$ *of the hyperarithmetic real numbers* (*i.e. of the real numbers* x *such that* $s_2(x)$ *is a hyperarithmetic set*). (ii) $\mathfrak{F}_H$ *itself is not a model of* R_2.

5.2. THEOREM. *For any model* $\mathfrak{A}$ *of* R_2:

(i) $\mathfrak{A}$ *is an Archimedian ordered field.*

(ii) $\mathfrak{A}$ *is real-closed.*

(iii) $\mathfrak{A}$ *has infinite transcendence degree over the field of rationals.*

Proof. (i) was stated in 3. (ii) holds, since the first-order continuity axioms (which characterize the real-closed fields by [10]) are contained in (Ct). By a theorem of Lindemann — see, e.g., Corollary 3 in [3], p. 186 — one can construct countably many independent transcendents which are computable and, hence, in $\mathfrak{F}_H$. Thus, (iii) holds.

6. WII-geometry. The weak second-order geometry $\mathscr{E}_2'$ introduced by Tarski in [12], p. 24 f., is based on a recursive axiom system, which includes a geometrical version of a weak second-order continuity scheme. "Translations" from $\mathscr{E}_2'$ to R_2 and conversely — similar to the "Translations" in Sections 3 and 4 — can be carried out using techniques from [12] and from this paper (especially for encoding finite sequences of points by finite sequences of coordinates). This gives

6.1. THEOREM. $\mathscr{E}_2'$ *is not finitely axiomatizable.*

6.2. THEOREM. *The models of* $\mathscr{E}_2'$ *are — up to isomorphism — the Cartesian planes over the fields which are models of* R_2.

The question (also raised in [12]) if $\mathscr{E}_2'$ is complete was settled negatively by Mostowski in [6] (namely, the theory of the Cartesian plane over R has no analytic axiom system and, hence, must be a proper extension of $\mathscr{E}_2'$).

7. A problem. Let R_1 be the weak second-order theory based on a first-order axiom system which differs from that of R_2 only in that the continuity axioms $Ct_{\alpha,\beta}$ are used for first-order formulas α, β only (i.e., formulas of L_2 without sequence variables).

On the other hand, consider, first-order Peano arithmetic, which is based on axioms including the induction scheme

$$\text{(Ind)} \qquad \alpha(0) \wedge \forall x[\alpha(x) \to \alpha(x+1)] \to \forall x \alpha(x).$$

Similarly let P_1 be the weak second-order theory based on the same (first-order) axioms, and let P_2 be based on axioms differing from the preceeding ones in that weak second-order formulas α are used in the induction scheme.

Note that R_1 and P_2 can easily be shown to be finitely axiomatizable as follows. The first-order continuity axioms are equivalent to axioms stating that (i) each positive element is a square, and (ii) each polynomial of odd degree has a zero. However, it is not difficult to express (i) and (ii) by a weak second-order formula. From an induction axiom of P_2, one can obtain as a theorem that, for each element x, there is a natural sequence for x (as in 3.5). This together with finitely many other axioms characterizes the models of P_2 up to isomorphism and, hence, is a finite axiom system for P_2.

7.1. PROBLEM. *Is* P_1 *finitely axiomatizable?*

Addendum. Jouko Väänänen asked the question if one gets a theory "equivalent" to R_N (and hence to R_2) by adding the quantifier Q_0 ("there are infinitely many") or its negation ("there are at most finitely many") to the first-order theory R of real numbers. The answer is negative. In fact, from Tarski's quantifier elimination method, one can get the

THEOREM. *The quantifier* Q_0, *if added to* R, *can be eliminated.*

Proof. It is sufficient to eliminate Q_0 in formulas of the form $Q_0 x\alpha$ where α is a first-order formula. By Tarski's method (and an obvious distribution of $Q_0 x$ to disjuncts) such a formula is equivalent to a disjunction of formulas of the form

$$(1) \qquad Q_0 x[\pi \doteq 0 \wedge \varrho_1 > 0 \wedge \ldots \wedge \varrho_n > 0]$$

where π and $\varrho_1, \ldots, \varrho_n$ are terms for polynomials in x (with other variables in the coefficients, in general). Since a non-trivial polynomial equation has only finitely many solutions, and the inequalities hold in an open set, (1) is equivalent to the first-order formula

$$(2) \qquad \exists x[\varrho_1 > 0 \wedge \ldots \wedge \varrho_n > 0] \wedge \gamma$$

where γ is a quantifier-free formula stating that all coefficients of π are equal to zero (for $n = 0$, (2) reduces to γ).

Thus, the situation is different from that for theories of natural numbers, where, by adding Q_0 analogously, one gets a theory equivalent to P_2 (see 7) since the formula $\forall x \neg Q_0 y\, y < x$ expresses that all elements x are standard elements.

References

[1] R. O. Gandy, G. Kreisel, and W. W. Tait, *Set existence I*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1960), pp. 577–582; part II ibid. 9 (1961), pp. 881–882.

[2] A. Grzegorczyk, A. Mostowski, and C. Ryll-Nardzewski, *The classical and the ω-complete arithmetic*, J. Symbolic Logic 23 (1958), pp. 188–206.

[3] W. J. Le Veque, *Topics in number-theory*, vol. II. Addison-Wesley, Reading, Mass. 1956.

[4] E. K. G. Lopez-Escobar, *A complete, infinitary axiomatization of weak second-order logic*, Fund. Math. 61 (1967), pp. 93–103.

[5] P. Zbierski, *Axiomatizability of secondorder arithmetic with ω-rule*, Fund. Math. 100 (1978), pp. 51–57.
[6] A. Mostowski, *Concerning the problem of axiomatizability of the field of real numbers in the weak second-order logic*, Essays on the foundations of mathematics, Jerusalem 1961, pp. 269–286.
[7] W. V. Quine, *Concatenation as a basis for arithmetic*, J. Symbolic Logic 11 (1946), pp. 105–114.
[8] W. Schwabhäuser, *Zur Axiomatisierbarkeit von Theorien in der schwachen Logik der zweiten Stufe*, Arch. math. Logik Grundlagenforsch 10 (1967), pp. 60–96.
[9] J. R. Shoenfield, *Mathematical Logic*, Addison-Wesley, Reading Mass. 1967.
[10] A. Tarski, *A decision method for elementary algebra and geometry*, 2nd ed., Berkeley and Los Angeles 1951.
[11] — *Some model-theoretical results concerning weak second-order logic*, Notices Amer. Math. Soc. 5 (1958), p. 673.
[12] — *What is elementary geometry*? Symposium on the axiomatic method, Amsterdam 1959, pp. 16–29.
[13] B. L. van der Waerden, *Algebra I*, 7th ed. Berlin–Heidelberg–New York 1966.

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Small subsets of first countable spaces

by

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Abstract. The existence of two types of first countable spaces is shown to be equivalent to a certain structure on the rationals. This structure, whose intuitive content is that discrete subsets of the rationals are small, is consistent with the usual axioms for set theory.

Introduction. In this paper we present two consistent examples of first countable spaces both of which require careful handling of certain sets which are small in an intuitive sense. We use two combinatorial principles, called $P(c)$ and $BF(c)$, which will be explained in Section 2. Both are strictly weaker than Martin's Axiom, hence strictly weaker than the Continuum Hypothesis, and $BF(c)$ is strictly weaker than $P(c)$. However, it is consistent with ZFC that $P(c)$ and $BF(c)$ be false.

We first recall some definitions. A space X is *collectionwise Hausdorff*, abbreviated CWH, if for each closed discrete subset D of X there is an open family $\{U_x|\ x \in D\}$ in X such that $x \in U_x$, for all $x \in D$, and $U_x \cap U_y = \emptyset$, for all $x \neq y \in D$. A space is *σ-discrete* if it is the union of countably many closed discrete subsets. A space is *pseudonormal* (or has *property* D) if any two disjoint closed subsets, one of which is countable (and discrete) have disjoint neighborhoods. (This is not the usual definition of property D, [M, p. 69], but is equivalent to it in first countable regular spaces.)

Our first example answers Mike Reed's question of whether every CWH σ-discrete Moore space is normal (hence metrizable) in the negative. This question is quite natural, since in a CWH space closed discrete subsets are "small", so a CWH σ-discrete space is σ-"small".

1.1. EXAMPLE 1. [$P(c)$] There is a CWH σ-discrete Moore space which is not pseudonormal.

The fact that there exists a nonnormal CWH Moore space was known already, see [W]. The example in [W] does not require any additional set theoretic axioms. Interest in collectionwise Hausdorffness in Moore spaces stems from Fleissner's Theorem that $V = L$ (which implies CH, hence $P(c)$) implies that first countable normal spaces are CWH (in fact this is true for normal spaces with character $\leqslant c$), [F].

The existence of Example 1 will be deduced from the existence of Example 2, which answers Mike Reed's question of whether property D implies pseudonormality in Moore spaces in the negative.