## Table des matières du tome CIII, fascicule 1




Les FUNDAMENTA MATHEMATICAE publient, en langues des congrès internationaux, des travaux consacrés à la *Théorie des Ensembles*, *Topologie*, *Fondements de Mathématiques*, *Fonctions Réelles*, *Théorie Descriptive des Ensembles*, *Algèbre Abstraite*

Chaque volume paraît en 3 fascicules

Adresse de la Rédaction:
FUNDAMENTA MATHEMATICAE, Śniadeckich 8, 00-950 Warszawa (Pologne)

Adresse de l'Échange:
INSTITUT MATHÉMATIQUE ACADÉMIE POLONAISE DES SCIENCES
Śniadeckich 8, 00-950 Warszawa (Pologne)

Tous les volumes sont à obtenir par l'intermédiaire de
ARS POLONA Krakowskie Przedmieście 7, 00-068 Warszawa (Pologne)

Correspondence concerning editorial work and manuscripts should be addressed to:
FUNDAMENTA MATHEMATICAE, Śniadeckich 8, 00-950 Warszawa (Poland)

Correspondence concerning exchange should be addressed to:
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, Exchange
Śniadeckich 8, 00-950 Warszawa (Poland)

The Fundamenta Mathematicae are available at your bookseller or at
ARS POLONA, Krakowskie Przedmieście 7, 00-068 Warszawa (Poland)





ISBN 83-01-00405-3 ISSN 0016-2736

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE


# On weakly $n$-dimensional spaces

by


**B. Tomaszewski** (Warszawa)



**Abstract.** It is shown that if $X$ is a weakly $n$-dimensional space and $Y$ is a weakly $m$-dimensional space, then $\operatorname{ind}(X\times Y)\leqslant n+m-1$; a simple example of a weakly $n$-dimensional space is given for $n = 1, 2, \ldots$


We shall consider only metric separable spaces. We denote by $X_{(k)}$ the set $\{x \in X\colon \operatorname{ind}_x X = k\}$; the symbol $B(A, \varepsilon)$ denotes the $\varepsilon$-ball about the set $A$. A space $X$ is called *weakly $n$-dimensional* if $\operatorname{ind} X = n$ and $\operatorname{ind} X_{(n)} < n$. If $rX$ is a compactification of the space $X$ and $r\colon X\to rX$ is the homeomorphic embedding corresponding to the compactification $rX$ (see [1], p. 125), then we identify the points $x \in X$ and $r(x) \in rX$.

K. Menger in [6] asked whether there exists a weakly one-dimensional space $X$ such that $\operatorname{ind} X^n = n$ for $n = 1, 2, \ldots$ In this paper we answer this question, showing that there is no such space. More exactly, we prove the following

THEOREM. *If $X$ is a weakly $n$-dimensional space and $Y$ is a weakly $m$-dimensional space, then* $\operatorname{ind}(X\times Y)\leqslant n+m-1$.

In the second part of the paper we give a simple example of a weakly $n$-dimensional space for $n = 1, 2, \ldots$ which applies a construction due to K. Kuratowski [3].

In the proof of our theorem the following five easy lemmas will be applied:

LEMMA 1. *Every one-dimensional space $X$ has a one-dimensional compactification $rX$ such that $X_{(0)} \subset (rX)_{(0)}$.*

This is a particular case of a theorem proved in [2].

Straightforward proofs of Lemmas 1-4 are omitted.

A closed subset $L\subset X$ is called a *partition* between the sets $A$ and $B$ provided that there exist two open sets $U$ and $V$ in $X$ such that

$$A\subset U, \quad B\subset V, \quad U\cap V = \emptyset \quad \text{and} \quad X\setminus L = U\cup V.$$

LEMMA 2. *Let $rX$ be a compactification of the space $X$. If for any distinct points $x$ and $y$ of the space $rX$ there exists a partition $L$ between $x$ and $y$ such that* $\operatorname{ind}(L\cap X)\leqslant n-1$, *then* $\operatorname{ind} X\leqslant n$.

LEMMA 3. *If $Z$ is a compact space and $F \subset Z$ is a closed subset of $Z$, then for every $\varepsilon \geqslant 0$ the union of all components of the space $Z$ having diameter not smaller than $\varepsilon$ and intersecting the set $F$ is a compact set.*

LEMMA 4. *If a closed subset $F$ of a compact space $Z$ is equal to the union of a family of components of the space $Z$, then for every $\varepsilon > 0$ there exists an open-and-closed set $U \subset Z$ such that $F \subset U \subset B(F, \varepsilon)$.*

LEMMA 5. *If a subset $A$ of a compact space $Z$ is equal to the union of a family of components of the space $Z$ and $\bar{A} \setminus A$ is a compact set, then for every $\varepsilon > 0$ there exist sets $U_1, U_2, \ldots \subset Z$ such that*

(a) *$U_i$ are open-and-closed and pairwise disjoint.*

(b) $A \subset \bigcup_{i=1}^{\infty} U_i \subset B(A, \varepsilon)$.

(c) $\mathrm{Fr}(\bigcup_{i=1}^{\infty} U_i) \subset \bar{A} \setminus A$.

Proof. Let us denote by $\mathscr{F}$ the family of all components of the space $Z$ intersecting the set $\bar{A} \setminus A$ and let us consider their union $F = \bigcup \mathscr{F}$. The set $F$ is compact by virtue of Lemma 3; moreover $F \cap A = \emptyset$. Applying Lemma 4, let us take for $i = 1, 2, \ldots$ an open-and-closed set $V_i$ such that $F \subset V_i \subset B(F, 1/i)$. The set $F_i = A \setminus V_i$ is compact for every $i$; since $\bigcap_{i=1}^{\infty} V_i = F$, the inclusion $A \subset \bigcup_{i=1}^{\infty} F_i$ holds. Each of the sets $V_i$, being open-and-closed, is the union of a family of components of the space $Z$. Hence, there exists an open-and-closed set $U_i'$ such that $F_i \subset U_i' \subset B(F_i, \varepsilon/i)$. Let us define $U_i = U_i' \setminus (U_1' \cup \ldots \cup U_{i-1}')$ for $i = 1, 2, \ldots$ The sets $U_i$ are open-and-closed, pairwise disjoint and, moreover,

$$A = \bigcup_{i=1}^{\infty} F_i \subset \bigcup_{i=1}^{\infty} U_i' = \bigcup_{i=1}^{\infty} U_i \subset B(A, \varepsilon).$$

In order to conclude the proof of our lemma it remains to show that condition (c) is satisfied.

Let us assume that $p \in \mathrm{Fr}(\bigcup_{i=1}^{\infty} U_i)$ and $p = \lim p_i$, where $p_i \in U_{k_i}$. Since the sets $U_i$ are open-and-closed, it follows that $\lim k_i = \infty$. For an arbitrary number $\eta > 0$ there exists a natural number $j$ such that $k_i > \varepsilon/\eta$ for $i > j$. Since

$$\varrho(p_i, A) < \varrho(p_i, F_{k_i}) < \eta \quad \text{for} \quad i > j,$$

we have the inequality $\varrho(p, A) \leqslant \eta$. As the number $\eta$ is arbitrary, we conclude that $p \in \bar{A}$. Obviously, the relation $p \in A$ cannot hold.

The theorem will be proved by induction. We shall start with the case of two weakly one-dimensional spaces $X$ and $Y$. Let $rX$ and $rY$ be compactifications of $X$ and $Y$, respectively, such that $X_{(0)} \subset (rX)_{(0)}$ and $Y_{(0)} \subset (rY)_{(0)}$; such compactifications exist by virtue of Lemma 1.



For arbitrary distinct points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ of the space $Z = rX \times rY$ we shall construct a partition $L$ in the space $Z$ between the points $z_1$ and $z_2$ such that $\mathrm{ind}\, L \cap (X \times Y) \leqslant 0$. By virtue of Lemma 2 this will yield our theorem in the case where $n = m = 1$. We can assume that $x_1 \neq x_2$.

Since $\mathrm{ind}\, X_{(1)} = 0$, in the space $rX$ there exists a partition $M$ between the points $x_1$ and $x_2$ such that $M \cap X_{(1)} = \emptyset$. Obviously the set $P = M \times rX$ is a partition in the space $Z$ between the points $z_1$ and $z_2$. Let $Z \setminus P = G_1 \cup G_2$, where $G_1$ and $G_2$ are disjoint open sets such that $z_1 \in G_1$ and $z_2 \in G_2$. Since $M \cap X_{(1)} = \emptyset$, we have $P \cap (X_{(1)} \times rY) = \emptyset$. We can also assume that

(1) $$\varrho(x_1, M) > 2 \quad \text{and} \quad \varrho(x_2, M) > 2.$$

Let $B_m$ be the union of the family of all components of the space $rX$ having a diameter less than $1/m$ and intersecting the set $M$. Obviously $B_m = E \setminus F_m$, where $E$ is the union of all components of the space $rX$ intersecting $M$ and $F_m$ is the union of all components of the space $rX$ having a diameter not less than $1/m$. By virtue of Lemma 3 the sets $E$ and $F_m$ are compact. Hence the set $\bar{B}_m \setminus B_m = \bar{B}_m \cap F_m$ is compact. Applying Lemma 5, take open-and-closed pairwise disjoint sets $U_{im}$ such that

(2) $$B_m \subset \bigcup_{i=1}^{\infty} U_{im} \subset B(B_m, 1/m)$$

and

(3) $$\mathrm{Fr}(\bigcup_{i=1}^{\infty} U_{im}) \subset F_m.$$

Since $B_m \subset B(M, 1/m)$, it follows from (2) that

(4) $$\bigcup_{i=1}^{\infty} U_{im} \subset B(M, 2/m).$$

Let us denote by $A_m$ the union of the family of all components of the space $rY$ having a diameter not smaller than $1/m$. By virtue of Lemma 3 the set $A_m$ is compact. Applying Lemma 4, take for $i = 1, 2, \ldots$ an open-and-closed set $V_{im}$ such that

$$A_m \subset V_{im} \subset B(A_m, 1/i).$$

Let us consider the set

$$W = \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{\infty} (U_{im} \times V_{im}).$$

We shall prove the inclusion

(5) $$[(M \cap X) \times Y_{(1)}] \subset W.$$

Let us assume that $(x, y) \in [(M \cap X) \times Y_{(1)}]$ and let us take a natural number $m$ such that $y \in A_m$. Since $M \subset rX \setminus X_{(1)}$, we have $M \cap X \subset X_{(0)} \subset (rX)_{(0)}$ and this implies that $(M \cap X) \cap \bigcup_{n=1}^{\infty} F_n = \emptyset$. Since $M \cap X \subset E$, we conclude that $(M \cap X)$

$\subset E\setminus F_m = B_m$. From the inclusion $B_m \subset \bigcup_{i=1}^{\infty} U_{im}$ follows the existence of a natural number $i$ such that $x \in U_{im}$. Since $y \in A_m \subset V_{im}$, we have $(x, y) \in U_{im} \times V_{im} \subset W$.

Now we shall prove that

$$\operatorname{Fr} W \subset [(rX)_{(1)} \times (rY)_{(1)}] \cup P. \tag{6}$$

Let $p = (r, s) \in \operatorname{Fr} W$. Let us choose a sequence of points $p_1, p_2, \ldots$, where $p_k = (r_k, s_k) \in W$, converging to the point $p$. Let $p_k \in U_{i_k m_k} \times V_{i_k m_k}$. Since the sets $U_{im} \times V_{im}$ are open-and-closed, we can assume that $(i_k, m_k) \neq (i_j, m_j)$ for $k \neq j$. Moreover, we can assume (passing to a subsequence, if necessary) that either

(a) $m_1 < m_2 < \ldots$ or

(b) $m = m_1 = m_2 = \ldots$ and $i_1 < i_2 < \ldots$

First we shall consider the case (a). Let us take an arbitrary natural number $n$ and such a number $K$ that $m_k > n$ for $k > K$. Applying (4) we have $r_k \in U_{i_k m_k} \subset B(M, 2/m_k) \subset B(M, 2/n)$. Since the number $n$ is arbitrary, it follows from the compactness of the set $M$ that $\lim r_k \in M$, so that $p = (r, s) \in P$.

Let us now consider the case (b). The sets $U_{im}$, where $i = 1, 2, \ldots$, are pairwise disjoind and open-and-closed. Hence

$$r \in \operatorname{Fr}\Big(\bigcup_{k=1}^{\infty} U_{i_k m}\Big) \subset \operatorname{Fr}\Big(\bigcup_{i=1}^{\infty} U_{im}\Big) \subset F_m \subset (rX)_{(1)}.$$

Moreover, $s_k \in V_{i_k m} \subset B(A_m, 1/i_k)$, so that $\lim s_k = s \in A_m \subset (rY)_{(1)}$. This implies that $p = (r, s) \in (rX)_{(1)} \times (rY)_{(1)}$ and the proof of (6) is concluded.

By virtue of (1) and (4) we have

$$z_1, z_2 \in Z \setminus \overline{W}. \tag{7}$$

Let us consider open sets $G_1' = (G_1 \setminus \operatorname{Fr} W) \cup W$ and $G_2' = G_2 \setminus \overline{W}$ contained in $Z$, where $G_1$ and $G_2$ are open disjoint sets determined by the partition $P$; obviously, $G_1' \cap G_2' = \emptyset$. By virtue of (7) $z_1 \in G_1'$ and $z_2 \in G_2'$, so that the set

$$\begin{aligned} L = Z \setminus (G_1' \cup G_2') &= Z \setminus [(G_1 \setminus \operatorname{Fr} W) \cup W \cup (G_2 \setminus \overline{W})] \\ &= \operatorname{Fr} W \cup [Z \setminus (G_1 \cup W \cup G_2)] = \operatorname{Fr} W \cup [(Z \setminus (G_1 \cup G_2)) \setminus W] \\ &= \operatorname{Fr} W \cup (P \setminus W) \end{aligned}$$

is a partition between the points $z_1$ and $z_2$ in the space $Z = rX \times rY$.

In order to conclude the proof of the particular case under consideration, it suffices to show that $\operatorname{ind}(L \cap (X \times Y)) \leqslant 0$.

Let us consider the decomposition of the set $L' = L \cap (X \times Y)$ into the sets $L_1 = L' \cap P$ and $L_2 = L' \setminus P$. The set $L_1$ is closed in $L$. Hence, in order to prove that $\operatorname{ind} L' \leqslant 0$ it suffices to show that $\operatorname{ind} L_1 \leqslant 0$ and $\operatorname{ind} L_2 \leqslant 0$.

The first coordinate of every point from $L' \cap P$ belongs to $X_{(0)}$. By virtue of (5), the second coordinate belongs to $Y_{(0)}$. Hence we have $L_1 \subset X_{(0)} \times Y_{(0)}$ and this



implies that $\operatorname{ind} L_1 \leqslant 0$. Applying (6), we conclude that $L_2 \subset X_{(1)} \times Y_{(1)}$; hence — by the weak one-dimensionality of $X$ and $Y$ — also $\operatorname{ind} L_2 \leqslant 0$. Hence we have proved that $\operatorname{ind}(X \times Y) = 1$ if the spaces $X$ and $Y$ are weakly one-dimensional.

LEMMA 6. *For every $n$-dimensional space $X$ there exists a zero-dimensional set $D \subset X$ such that for every natural number $k \leqslant n$ and each point $x \in X_{(k)} \setminus D$ the inequalities $\operatorname{ind}_x(X \setminus D) \leqslant k-1$ and $\operatorname{ind}(X_{(n)} \setminus D) \leqslant \operatorname{ind} X_{(n)} - 1$ hold.*

Proof. Let $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$ be a countable base of the space $X$ such that for every $x \in X_{(k)}$ there exist members $U_{k_1}, U_{k_2}, \ldots$ of the base $\mathcal{U}$ which constitute a base for the space $X$ at the point $x$ and satisfy the inequality $\operatorname{ind}(\operatorname{Fr} U_{k_i}) \leqslant k-1$ for $i = 1, 2, \ldots$ For every natural number $m$ such that $\operatorname{Fr} U_m \neq 0$ let us take a zero-dimensional $F_\sigma$-set $A_m \subset \operatorname{Fr} U_m$ satisfying the equality $\operatorname{ind}(\operatorname{Fr} U_m \setminus A_m) = \operatorname{ind}(\operatorname{Fr} U_m) - 1$. Let $A = \bigcup_{m=1}^{\infty} A_m$. From the sum theorem it follows that the set $A$ is zero-dimensional; moreover, $A$ is an $F_\sigma$-set.

Let $B \subset X_{(n)}$ be a zero-dimensional $F_\sigma$-set in $X_{(n)}$ such that $\operatorname{ind}(X_{(n)} \setminus B)$ $\operatorname{ind} X_{(n)} - 1$. The set $B$ also is an $F_\sigma$-set in the space $X$, because $X_{(n)}$ is an $F_\sigma$-set in $X$. Let us define $D = A \cup B$. From the sum theorem it follows that the set $D$ is zero-dimensional; it is easy to check that the set $D$ satisfies also the remaining requirements of the lemma.

Let us now assume that the theorem holds for all natural numbers $n$ and $m$ such that $n+m \leqslant k-1 \geqslant 2$. Consider a weakly $n$-dimensional space $X$ and a weakly $m$-dimensional space $Y$ such that $n+m = k$. We shall prove that $\operatorname{ind}(X \times Y) \leqslant n+m-1$.

Let $rX$ and $rY$ be arbitrary compactifications of the space $X$ and $Y$, respectively. By virtue of Lemma 2 it suffices to prove that for any two distinct points $z_1, z_2 \in rX \times rY$, where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, there exists a partition $L \subset rX \times rY$ between $z_1$ and $z_2$ such that $\operatorname{ind}(L \cap (X \times Y)) \leqslant n+m-2$. We can assume that $x_1 \neq x_2$. Let $D \subset X$ be a set satisfying the requirements of Lemma 6. Since the set $D$ is zero-dimensional, in the space $rX$ there exists a partition $L'$ between $x_1$ and $x_2$ such that $L' \cap D = \emptyset$. From Lemma 6 it follows that either

(a) $\operatorname{ind}(L' \cap X) \leqslant n-2$ or

(b) $L' \cap X$ is a weakly $(n-1)$-dimensional space.

Obviously $L = L' \times rY$ is a partition between $z_1$ and $z_2$. Moreover, $L \cap (X \times Y) = (L' \cap X) \times Y$. Thus in case (a) we have $\operatorname{ind}[L \cap (X \times Y)] \leqslant n+m-2$; from the inductive assumption it follows that the same inequality holds in case (b).

Now we shall describe, for $n = 1, 2, \ldots$, a weakly $n$-dimensional space; our construction seems simpler than the well-known construction due to Mazurkiewicz (see [7]). By $C$ we shall denote the Cantor set. The space $X$ constructed by K. Kuratowski (see [3] or [4] § 27, VI) is the graph of the function $f\colon C \to [-1, 1]$ defined by the formula

$$f(x) = \frac{(-1)^{k_1}}{2^1} + \frac{(-1)^{k_2}}{2^2} + \ldots + \frac{(-1)^{k_n}}{2^n} + \ldots, \tag{8}$$

where

(9) $$x = \frac{2}{3^{k_1}} + \frac{2}{3^{k_2}} + \ldots + \frac{2}{3^{k_n}} + \ldots, \quad k_1 < k_2 < \ldots$$

Let us denote by $D$ the set of all points $x \in C$ which have a finite expansion of form (9); the set $D$ is countable. One easily sees that $D$ is the set of points of discontinuity of the function $f$. We define

$$X_1 = \{(x, f(x)): x \in D\} \quad \text{and} \quad X_0 = \{(x, f(x)): x \in C \setminus D\};$$

obviously $X = X_0 \cup X_1$. K. Kuratowski showed that $X_0 = X_{(0)}$ and $X_1 = X_{(1)}$, but his result will not be needed here. The only fact we shall use is that the function $f$ has the following property:

(10) for every $x \in D$ there exist sequences $\bar{x}_k$ and $\underline{x}_k$ converging to $x$ such that $\bar{x}_k, \underline{x}_k \in D$, $\bar{x}_k, \underline{x}_k \neq x$ for $k = 1, 2, \ldots$ and

$$\bar{f}(\bar{x}_k) = \bar{f}(x), \quad \underline{f}(\bar{x}_k) = f(x),$$
$$\bar{f}(\underline{x}_k) = f(x), \quad \underline{f}(\underline{x}_k) = \underline{f}(x),$$

where $\bar{f}(x)$ and $\underline{f}(x)$ denote, respectively, the upper and the lower limit of the function $f$ at the point $x$.

The space $X^n$ is the graph of the function $k_n = f \times f \times \ldots \times f$: $C^n \to [-1, 1]^n$. Let $x = (x_1, x_2, \ldots, x_n) \in D^n$ and $\varepsilon_i = \pm 1$ for $i = 1, 2, \ldots, n$. Let $A_x^{\varepsilon_1 \ldots \varepsilon_n}$ be the boundary of the $n$-dimensional cube

$$K_x^{\varepsilon_1 \ldots \varepsilon_n} = \{(x, y_1, y_2, \ldots, y_n) \in \{x\} \times [-1, 1]^n: \bar{f}(x_j) \geqslant y_j \geqslant f(x_j),$$
$$\text{if } \varepsilon_j = 1 \text{ and } f(x_j) \geqslant y_j \geqslant \underline{f}(x_j), \text{ if } \varepsilon_j = -1\}.$$

The set $K_x = \bigcup_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} K_x^{\varepsilon_1 \ldots \varepsilon_n}$ is an $n$-dimensional cube; by $A_x$ we shall denote the boundary of this cube. We shall prove that the space

$$Y = X_0^n \cup \bigcup_{x \in D^n} \bigcup_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} A_x^{\varepsilon_1, \ldots, \varepsilon_n}$$

s a weakly $n$-dimensional space. The symbol 0 shall denote the point $(0, 0, \ldots, 0) \in D^n$. To prove that $\operatorname{ind} Y \geqslant n$ it suffices to show that the identity mapping $\mathrm{id}_{A_0}$: $A_0 \to A_0$ is not continuously extendable over the whole space $Y$.

Let us assume that $g$: $Y \to A_0$ is a continuous extension of the mapping $\mathrm{id}_{A_0}$. Let $B \subset D^n$ be the set of all points $x \in D^n$ such that the mapping $g|A_x$: $A_x \to A_0$ is not homotopic to a constant; this means that the mapping $g|A_x$ cannot be continuously extended over the set $K_x$. If $x \in B$, then there exists a sequence $\varepsilon_1, \ldots, \varepsilon_n$ of numbers equal to $\pm 1$ such that the mapping $g|A_x^{\varepsilon_1 \ldots \varepsilon_n}$ is not continuously extendable over $K_x^{\varepsilon_1 \ldots \varepsilon_n}$; otherwise, the combination of all such extensions would be an extension of $g|A_x$ over $K_x$.



Let $x = (x_1, x_2, \ldots, x_n) \in B$ and let $\varepsilon_1, \ldots, \varepsilon_n$ be a sequence of numbers equal to $\pm 1$ such that the mapping $g|A_x^{\varepsilon_1 \ldots \varepsilon_n}$ is not continuously extendable over $K_x^{\varepsilon_1 \ldots \varepsilon_n}$. Applying (10), let us take for $i = 1, 2, \ldots, n$ a sequence $\{x_i^k\}_{k=1}^{\infty}$ of points of the set $D$ converging to $x_i$ such that $x_i^k \neq x_i$ for every $k$ and $i = 1, 2, \ldots, n$ and

$$\bar{f}(x_i^k) = \bar{f}(x_i), \quad \underline{f}(x_i^k) = f(x_i), \quad \text{if} \quad \varepsilon_i = 1,$$
$$\bar{f}(x_i^k) = f(x_i), \quad \underline{f}(x_i^k) = \underline{f}(x_i), \quad \text{if} \quad \varepsilon_i = -1.$$

Let $x_k = (x_1^k, x_2^k, \ldots, x_n^k)$; obviously, $\lim x_k = x$. The set $A_x^{\varepsilon_1 \ldots \varepsilon_n} \cup \bigcup_{k=1}^{\infty} A_{x_k}$ is compact, so that the mappings $g_k$: $A_x^{\varepsilon_1 \ldots \varepsilon_n} \to A_0$ defined by the formula $g_k(x, y) = g(x_k, y)$ are uniformly convergent to the mapping $g|A_x^{\varepsilon_1 \ldots \varepsilon_n}$. Hence for almost all $k$ the mappings $g_k$ are not homotopic to a constant (see [5] § 54, II Theorem 4a). We obtain:

(11) for every $x = (x_1, x_2, \ldots, x_n) \in B$ there exists a sequence $\{x_k\}$ of points in the set $B$, where $x_k = (x_1^k, x_2^k, \ldots, x_n^k)$, such that $\lim x_k = x$ and $x_i^k \neq x_i$ for every $k$ and $i = 1, 2, \ldots, n$.

As $0 \in B$, the set $B$ is non-empty. Let us arrange all elements of $D$ into a sequence $a_1, a_2, \ldots$ From (11) it follows that there exists a point $y_1 \in B$ and its open neighbourhood $U_1 \subset C^n$ whose closure does not contain any point having at least one coordinate equal to $a_1$. From the set $U_1$ we choose a point $y_2 \in B$ and its neighbourhood $U_2 \subset U_1$ whose closure does not contain any point having at least one coordinate equal to $a_2$, and so on. In such a way we construct a sequence of points $y_1, y_2, \ldots$ in the set $B$ and a sequence of their neighbourhoods $U_1 \supset U_2 \supset \ldots$ Moreover, we can assume that $\delta(U_i) < 1/i$. The sequence $\{y_i\}$ converges to a point $y \in C^n$; from the construction if follows that $y \in (C \setminus D)^n$. None of the mappings $g|A_{y_i}$ is homotopic to a constant; hence the image of each mapping is the whole set $A_0$. From this it follows that there exist a number $\eta > 0$ and points $y_i', y_i'' \in A_{y_i}$ such that

$$\eta < \varrho((g|A_{y_i})(y_i'), (g|A_{y_i})(y_i'')) = \varrho(g(y_i'), g(y_i'')).$$

But the sequences $\{y_i'\}$ and $\{y_i''\}$ converge to the same point $(y, k_n(y)) \in X_0^n$, because the function $f$ is continuous at all points $y^i$, where $y = (y^1, y^2, \ldots, y^n)$. This contradicts the assumption of the continuity of function $g$. Hence $\operatorname{ind} Y \geqslant n$.

Let us note that for every $x \in X_0^n$ the equality $\operatorname{ind}_x Y = 0$ holds.

Let $U$ be an arbitrary open neighbourhood of the point $x = (t, k_n(t))$ in the space $Y$. Since the mapping $k_n$ is continuous at the point $t$, there exists an open-and-closed set $V \subset C^n$ such that $t \in V$ and $(t', k_n(t')) \in U$ for $t' \in V$. The set $\alpha^{-1}(V)$, where $\alpha$: $Y \to C^n$ is the projection of $Y \subset C^n \times [-1, 1]^n$ onto $C^n$, is open-and-closed as the inverse image of open-and-closed set $V$. Moreover, it is contained in $U$. Hence

$$Y_{(n)} \subset \bigcup_{x \in D^n} \bigcup_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} A_x^{\varepsilon_1 \ldots \varepsilon_n}.$$

The last set is $(n-1)$-dimensional as a countable union of compact, $(n-1)$-dimensional sets, so that $Y$ is weakly $n$-dimensional.

I am grateful to Professor R. Engelking for his help and to Mr. M. Zakrzewski, who was the first reader of this text.

### References

[1] R. Engelking, *Outline of General Topology*, Amsterdam 1968.
[2] K. Kuratowski, *Sur la compactification des espaces à connexité n-dimensionelle*, Fund. Math. 30 (1938), pp. 242–246.
[3] — *Une application des images de fonctions à la construction de certains ensembles singuliers*, Mathematica 6 (1932), pp. 120–123.
[4] — *Topology*, vol. I, New York–London–Warszawa 1966.
[5] — *Topology*, vol. II, New York–London–Warszawa 1968.
[6] K. Menger, *Bemerkungen über dimensionelle Feinstruktur und Produktsatz*, Prace Mat.–Fiz. 38 (1930), pp. 77–90.
[7] S. Mazurkiewicz, *Sur les ensembles de dimension faible*, Fund. Math. 13 (1929), pp. 210–218.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

*Accepté par la Rédaction le 20. 9. 1976*

# Equivariant maps of $Z_p$-actions into polyhedra

by

**Richard J. Allen** (Northfield, Minn.)

**Abstract.** Let $X$ be an $n$-dimensional compact metric space with a free $Z_p$-action. This paper shows that for any positive number $\varepsilon$ there exists an equivariant $\varepsilon$-map from $X$ into an $n$-dimensional polyhedron $K$ with a free $Z_p$-action. Moreover, $K$ can be equivariantly embedded in $(2n+1)$-dimensional euclidean space $E$ with an orthogonal $Z_p$-action and there exists an equivariant $\varepsilon$-map arbitrarily close to a given equivariant map from $X$ into $E$.

**1. Introduction.** Let $X$ be an $n$-dimensional compact metric space with a map $a: X \to X$ of period $p$. The map $a$ then defines a $Z_p$-action on $X$ and $(X, a)$ will denote the equivariant space $(X, Z_p)$. Frequently, $(X, a)$ is called a *$Z_p$-space*. An equivariant map $f: (X, a) \to (Y, b)$ between two $Z_p$-spaces is an equivariant $\varepsilon$-map if $\operatorname{diam} f^{-1}y < \varepsilon$ for every $y \in fX$.

In the following, if $(Y, b)$ is a $Z_p$-space, then $y^* = \{y, by, \ldots, b^{p-1}y\}$ is called the *orbit of* $Y$, and $S^* = \bigcup_{j=0}^{p-1} b^j S$ is called the *orbit of* $S$, where $y$ is an element in $Y$, and $S$ is a subset of $Y$. A subset $S$ of $Y$ is called *sectional* if $S \cap y^* = \{y\}$ for each $y$ in $S$, and any one-to-one function $\chi: (Y/Z_p) \to Y$ is called a *section*.

If the action on $X$ is free, then an immediate consequence of (2.3) below is that for any positive number $\varepsilon$ there exists an equivariant $\varepsilon$-map from $X$ into an $n$-dimensional polyhedron $K$ with a free $Z_p$-action. Moreover, by (3.1) below $K$ can be equivariantly embedded in $(2n+1)$-dimensional euclidean space $R^{2n+1}$ with an orthogonal $Z_p$-action. Finally, it is shown in (3.3) that there exists an equivariant $\varepsilon$-map arbitrarily close to a given equivariant map from $X$ into $R^{2n+1}$.

A set $C$ is called a *convex body in a euclidean space* if $C$ is closed, convex and has a nonempty interior.

**2. Replacement by polyhedra.** (2.1), which is stated here and is used in proving (2.3) below, can be found in Jaworowski [7, p. 235].

Covering Lemma (2.1). *Let $(X, a)$ be a compact metric $Z_p$-space and let $A$ be an equivariant closed subspace of $X$ such that $Z_p$ acts freely outside of $A$. Suppose $C$ is an equivariant open cover of $X-A$. Then there exists a countable, locally finite,*