On decompositions of hereditarily unicoherent continua

by

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Abstract. Charatonik has defined a monotone upper semi-continuous decomposition of a continuum to be admissible if the layers of its irreducible subcontinua are contained in the elements of the decomposition. He has constructed an admissible decomposition which, for many classes of continua, e.g. λ-dendroids, is unique and minimal with respect to having an hereditarily arcwise connected quotient space. This paper studies hereditarily unicoherent continua and uses collections of closed separators (closed sets which separate the space) with certain properties to obtain an equivalent description of Charatonik's decomposition. This viewpoint enables one to describe precisely when such a continuum has a non-trivial admissible decomposition. One of the difficulties in showing the equivalence is due to the lack of an adequate description of the layers of an irreducible continuum having non-void interiors. A secondary purpose of this paper is to provide such a description. A monostratic continuum is one which does not have a non-trivial admissible decomposition and an example is given at the end to show that a λ-dendroid may have no interior containing monostratic subcontinua yet not admit an admissible decomposition each of whose elements has void interior.

The study of monotone upper semi-continuous decompositions of continua with "nice" quotient spaces has been undertaken by a large number of authors. Of particular interest are three such works. FitzGerald and Swingle have described in [5] a construction that yields a unique decomposition which is the finest possible with respect to having a semi-locally connected quotient space. By a different technique in [6], McAuley, using closed separators, has constructed a decomposition equivalent to the one above. And Charatonik in [3] has given a decomposition which, for certain continua, e.g., λ-dendroids and atriodic continua, is unique and minimal with respect to having an hereditarily arcwise connected quotient space.

If I is an irreducible continuum there exists a unique minimal monotone upper semi-continuous decomposition whose quotient space is degenerate or an arc [7, p. 10]. The elements of this decomposition are called *layers* and Charatonik [3, p. 115] has defined a decomposition D of a continuum M to be *admissible* if

1. D is upper semi-continuous,

2. D is monotone,

3. for every irreducible subcontinuum I in M, every layer of I is contained in some element of D.

If D and E are decompositions of a continuum M, then $D \leqslant E$ means that every element of D is contained in some element of E, i.e., D refines E. Charatonik has proved in [3, pp. 117–118] that every continuum admits a unique admissible decomposition which is minimal with respect to $\leqslant$. This decomposition will be denoted hereafter by C.

In this paper we devote our attention to hereditarily unicoherent continua and use the notion of closed separator (defined below) to obtain an equivalent description of Charatonik's decomposition C. This will enable us to give an internal characterization of hereditarily unicoherent continua which have a non-degenerate admissible decomposition and thus answer a question of Charatonik's which he raised for λ-dendroids [4], a special class of hereditarily unicoherent continua. It is considerable easier to show the equivalence of Charatonik's decomposition and the one of this paper for λ-dendroids than for hereditarily unicoherent continua in general. The difficulty stems from the layers having non-void interiors in an irreducible continuum. λ-dendroids do not contain this type of subcontinuum since they are hereditarily decomposable and therefore the layers of an irreducible subcontinuum I have void interiors relative to I [7, Theorem 10, p. 15]. The author is unaware in the literature of a satisfactory description of the structure of the layers having non-void interiors in an irreducible continuum. This paper provides one and this is the key in establishing the equivalence of the two decompositions.

At the end an example is provided that answers in the negative a question raised by Charatonik. A continuum M is *monostratic* if the minimal admissible decomposition C is degenerate. Suppose that M is a λ-dendroid and that every monostratic subcontinuum of M has void interior. Does it follow that each element in the minimal admissible decomposition has void interior [4]? The example answers this negatively.

By a *continuum* we mean a compact metric connected space and the continuum M is *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected. If M is also hereditarily decomposable it is a *λ-dendroid.* A closed set k is a *closed separator* of M if $M-k$ is not connected. Now suppose that K is a collection of closed separators of M with the following property:

(*) If $k \in K$ and $M-k = A_1 \cup A_2$, a separation, and if $a_1 \in A_1$, $a_2 \in A_2$, $c \in k$, then for either $i = 1, j = 2$ or $i = 2, j = 1$ there exist $k' \in K$ and a continuum Q such that $\{a_i, c\} \subset Q$ and k' separates Q from a_j.

Let $\tilde{K}$ be the union of all the collections of closed separators satisfying (*). Then $\tilde{K}$ is itself a collection of closed separators satisfying (*) and is clearly the unique maximal such collection. Denote by S_x the set of all points y of M such that there does not exist $k \in \tilde{K}$ which separates x from y. By [5, p. 49], $S = \{S_x |\ x \in M\}$ is an upper semi-continuous decomposition of M into closed sets.

THEOREM 1. *If M is a hereditarily unicoherent continuum then $C = S$.*

Proof. Let f be the quotient map of M onto the quotient space (M, C) of the decomposition C and let $K = \{f^{-1}(k) |\ k$ is a closed separator of $(M, C)\}$. Now K is a collection of closed separators of M. Suppose $M - f^{-1}(k) = A \cup B$, a separation, and $a \in A$, $b \in B$, $c \in f^{-1}(k)$. Then k separates $f(a)$ from $f(b)$ in (M, C) since f is a monotone map. From [3, Theorem 1, p. 116] we know that (M, C) is arcwise connected and so there exists an arc E such that either

$$\{f(a), f(c)\} \subset E \subset (M, C) - \{f(b)\} \quad \text{or} \quad \{f(b), f(c)\} \subset E \subset (M, C) - \{f(a)\};$$

assume the latter. Let k' be a closed set that separates E from $f(a)$ in (M, C). Then $f^{-1}(k')$ separates a from $f^{-1}(E)$ in M which shows that K has property (*). This implies that $S \leqslant C$ for suppose that $S_x \in S$ and $q \cap S_x \neq 0 \neq q' \cap S_x$ where $q, q' \in C$ and $q \neq q'$. Let k be a closed subset of (M, C) that separates $f(q)$ from $f(q')$. It follows that $f^{-1}(k)$ separates q from q' so $f^{-1}(k)$ separates two points of S_x. This contradicts the definition of S_x since $f^{-1}(k)$ belongs to K and therefore to $\tilde{K}$.

To prove that $C \leqslant S$ it suffices to prove that each layer of an irreducible subcontinuum of M lies entirely within some element of S [1, p. 28]. To do this we need some lemmas. In these lemmas I is an irreducible subcontinuum of M from a to b. Note that Lemma 5 is sufficient to prove that $C \leqslant S$ if M is a λ-dendroid.

LEMMA 1. *If J is an indecomposable subcontinuum of I with non-void interior relative to I then J is contained in some element of S.*

Proof. No generality will be lost if we assume that $I - J = A \cup B$, a separation of I, where J is irreducible from $\bar{A}$ to $\bar{B}$, the closures of A and B, respectively. If J^0 (interior of J relative to I) is contained in an element of S then J is also. So assume that $x, y \in J^0$ and $S_x \neq S_y$. There exists $k \in \tilde{K}$ such that $M - k = A_x \cup A_y$, a separation, where $x \in A_x$ and $y \in A_y$. Take $z \in k \cap J$. Clearly x, y, z can be chosen from mutually distinct composants of J. By the definition of $\tilde{K}$ there exists a continuum Q containing y and z but not x. Because M is hereditarily unicoherent, $Q \cap J$ is a proper subcontinuum of J intersecting two different composants of J. This contradiction shows that $J^0 \subset S_x$ for some $x \in M$ and thus $J \subset S_x$.

Next we take time to describe the structure of the layers of I that have non-void interior relative to I. To do this we use some ideas and notation from [5]. Let J be a layer of I that has non-void interior and assume without loss of generality that $I - J = A \cup B$, a separation of I, where J is irreducible about $(\bar{A} \cap J) \cup (\bar{B} \cap J)$. It is clear that J contains an indecomposable subcontinuum of I with non-void interior relative to I. Otherwise by [7, Theorem 10, p. 15] J has a non-trivial monotone upper semi-continuous decomposition with an arc for the quotient space which means that J cannot be a layer of I. Since I is a metric space obviously the number of such indecomposable subcontinua of J is countable; denote then by $I_1, I_2, \ldots$ Let $\mathrm{Ch}_0(I_i) = I_i$ and let $\mathrm{Cl}(H)$ be an alternate notation for the closure of a set H. Let $\mathrm{Ch}_1(I_i) = \mathrm{Cl}\{y \in J |\ y$ can be Ch_0-chained to $I_i\}$ (if F is a set function then y can be *F-chained* to $R \subset J$ if there exist $I_{i_1}, I_{i_2}, \ldots, I_{i_n}$ for some integer n such that $F(I_{i_1}), F(I_{i_2}), \ldots, F(I_{i_n})$ is a simple chain where $y \in F(I_{i_1})$ and $R \cap F(I_{i_n}) \neq 0$). Suppose for a countable ordinal α that $\mathrm{Ch}_\beta(I_i)$ has been defined for each i and for each $\beta < \alpha$. If α is a limit ordinal define $\mathrm{Ch}_\alpha(I_i) = \mathrm{Cl}\left(\bigcup_{\beta < \alpha} \mathrm{Ch}_\beta(I_i)\right)$. If α is a non-limit ordinal define $\mathrm{Ch}_\alpha(I_i) = \mathrm{Cl}\{y \in J \mid y$ can be $\mathrm{Ch}_{\alpha-1}$-chained to $I_i\}$. If Ω is the first uncountable ordinal then clearly $\mathrm{Ch}_\alpha(I_i)\subset \mathrm{Ch}_{\alpha+1}(I_i)$ for all $\alpha<\Omega$ and it follows that there exists a countable ordinal γ such that $\mathrm{Ch}_\gamma(I_i)=\mathrm{Ch}_{\gamma+1}(I_i)$ for all i. If $i\neq j$ then $\mathrm{Ch}_\gamma(I_i)=\mathrm{Ch}_\gamma(I_j)$ or $\mathrm{Ch}_\gamma(I_i)\cap \mathrm{Ch}_\gamma(I_j)=0$. For suppose that $\mathrm{Ch}_\gamma(I_i)\cap\mathrm{Ch}_\gamma(I_j)\neq 0$ and $x\in \mathrm{Ch}_\gamma(I_j)$. Then $x \in \mathrm{Ch}_{\gamma+1}(I_i)$ since $\mathrm{Ch}_\gamma(I_j)$, $\mathrm{Ch}_\gamma(I_i)$ is a simple chain from x to I_i. But $\mathrm{Ch}_{\gamma+1}(I_i)=\mathrm{Ch}_\gamma(I_i)$ so $x\in\mathrm{Ch}_\gamma(I_i)$ and $\mathrm{Ch}_\gamma(I_j)\subset\mathrm{Ch}_\gamma(I_i)$. Similarly $\mathrm{Ch}_\gamma(I_i)\subset\mathrm{Ch}_\gamma(I_j)$ and thus $\mathrm{Ch}_\gamma(I_i)=\mathrm{Ch}_\gamma(I_j)$. From the definition each $\mathrm{Ch}_\gamma(I_i)$ is closed and from the construction each $\mathrm{Ch}_\gamma(I_i)$ is connected.

Let $Q_i=\{y\in J\mid y$ cannot be separated from $\mathrm{Ch}_\gamma(I_i)$ in J by $\mathrm{Ch}_\gamma(I_j)$ for any $j\}$. The set Q_i is obviously closed. Each $\mathrm{Ch}_\gamma(I_i)$ is a closed separator of J except possibly for two (one might intersect $\bar{A}$ and one might intersect $\bar{B}$). Let $\mathcal{Q}=\{Q_i\mid Q_i$ is a closed separator of $J\}$ and assume that $\mathcal{Q}\neq0$. Suppose $J-Q_i=X\cup Y$, a separation, where $x\in X$, $y\in Y$ and $c\in Q_i$. There exists j such that $\mathrm{Ch}_\gamma(I_j)$ separates x from $\mathrm{Ch}_\gamma(I_i)$ by the definition of Q_i. Let H_{ij} be an irreducible subcontinuum of J from $\mathrm{Ch}_\gamma(I_i)$ to $\mathrm{Ch}_\gamma(I_j)$. Now H_{ij} must contain I_k for some k or else by [7, p. 15] H_{ij} has a non-trivial monotone, upper semi-continuous decomposition onto an arc which implies that J is not a layer of I. It follows that $\mathrm{Ch}_\gamma(I_k)$ separates $\mathrm{Ch}_\gamma(I_i)$ from $\mathrm{Ch}_\gamma(I_j)$. Similarly there exists $\mathrm{Ch}_\gamma(I_h)$ which separates $\mathrm{Ch}_\gamma(I_j)$ from $\mathrm{Ch}_\gamma(I_k)$ and $\mathrm{Ch}_\gamma(I_m)$ which separates $\mathrm{Ch}_\gamma(I_j)$ from $\mathrm{Ch}_\gamma(I_h)$. Consequently $Q_m\subset \mathrm{Ch}_\gamma(I_j)\cup H_{jh}\cup\mathrm{Ch}_\gamma(I_h)$ and $Q_m\cap Q_i=0$. Then Q_m separates x from $Q_i\cup Y$, a continuum with y in its interior and thus $\mathcal{Q}$ satisfies Definition 8.2 of [5, p. 49]. Hence by [5, Theorem 8.3, p. 49] and [5, Theorem 2.7, p. 37], J has a non-trivial monotone upper semi-continuous decomposition whose quotient space is an arc which contradicts the fact that J is a layer. So we conclude that $\mathcal{Q}=0$. Therefore one Q_i intersects $\bar{A}$, one intersects $\bar{B}$ and from the definition of Q_i these must be the same. So $Q_1=Q_2=Q_3=\ldots$ and consequently $\mathrm{Ch}_\gamma(I_1)=\mathrm{Ch}_\gamma(I_2)=\mathrm{Ch}_\gamma(I_3)=\ldots$ It follows that $\mathrm{Ch}_\gamma(I_1)$ is an irreducible subcontinuum of J from $\bar{A}$ to $\bar{B}$.

LEMMA 2. *If I_i and $\mathrm{Ch}_\gamma(I_i)$ are defined as in the previous construction then* $\overline{\bigcup I_i}=\mathrm{Ch}_\gamma(I_1)$.

Proof. Since $\mathrm{Ch}_\gamma(I_1)$ is an irreducible continuum from $\bar{A}$ to $\bar{B}$ then for each i, $I_i\subset\mathrm{Ch}_\gamma(I_1)$. It follows that $\overline{\bigcup I_i}\subset \mathrm{Ch}_\gamma(I_1)$. Obviously $\mathrm{Ch}_0(I_j)\subset\overline{\bigcup I_i}$ for all j. Assume $\mathrm{Ch}_\beta(I_j)\subset\overline{\bigcup I_i}$ for all $\beta<\alpha$ and all j. If α is a limit ordinal it is clear from the definition that $\mathrm{Ch}_\alpha(I_j)\subset\overline{\bigcup I_i}$. If α is a non-limit ordinal, by the induction hypothesis we have that $\mathrm{Ch}_{\alpha-1}(I_j)\subset\overline{\bigcup I_i}$ for all j. Then $P=\{y\mid y$ can be $\mathrm{Ch}_{\alpha-1}$-chained to $I_j\}\subset\overline{\bigcup I_i}$ and thus the closure of P, $\mathrm{Ch}_\alpha(I_j)$, is a subset of $\overline{\bigcup I_i}$. Hence $\mathrm{Ch}_\gamma(I_1)\subset\overline{\bigcup I_i}$ and Lemma 2 is proved.

LEMMA 3. *If J^0 is the interior of J relative to I then* $\overline{J^0}=\mathrm{Ch}_\gamma(I_1)$.

Proof. Because $\mathrm{Ch}_\gamma(I_1)$ is irreducible from $\bar{A}$ to $\bar{B}$, $J^0\subset\mathrm{Ch}_\gamma(I_1)$ and thus $\overline{J^0}\subset\mathrm{Ch}_\gamma(I_1)$. On the other hand $\overline{J^0}$ is a continuum intersecting both $\bar{A}$ and $\bar{B}$ and since $\mathrm{Ch}_\gamma(I_1)$ is an irreducible continuum from $\bar{A}$ to $\bar{B}$, $\mathrm{Ch}_\gamma(I_1)\subset \overline{J^0}$. The lemma follows.

LEMMA 4. *If I is an irreducible subcontinuum of M and J is a layer of I with non-void interior relative to I, then $\overline{J^0}$ is contained in some element of the decomposition S.*

Proof. The proof involves transfinite induction. From Lemma 3 $\overline{J^0}=\mathrm{Ch}_\gamma(I_1)$ and from Lemma 1, $\mathrm{Ch}_0(I_i)=I_i\subset S_{x_i}$ for some x_i in M. Assume that for all $\beta<\alpha$ and for all i, $\mathrm{Ch}_\beta(I_i)\subset S_{x_i}$. For a limit ordinal α it follows immediately that $\mathrm{Ch}_\alpha(I_i)\subset S_{x_i}$. If α is a non-limit ordinal then $\mathrm{Ch}_{\alpha-1}(I_i)\subset S_{x_i}$ for all i. Suppose there is a simple chain $\mathrm{Ch}_{\alpha-1}(I_{i_1}), \mathrm{Ch}_{\alpha-1}(I_{i_2}), \ldots, \mathrm{Ch}_{\alpha-1}(I_{i_n})$ such that $y\in\mathrm{Ch}_{\alpha-1}(I_{i_1})$ and $\mathrm{Ch}_{\alpha-1}(I_{i_n})\cap I_i\neq0$. Since S is a decomposition $\bigcup_{j=1}^{n} \mathrm{Ch}_{\alpha-1}(I_{i_j})\subset S_{x_i}$. Hence $y\in S_{x_i}$ and $\mathrm{Ch}_\alpha(I_i)\subset S_{x_i}$ for all i. Then $\mathrm{Ch}_\gamma(I_i)\subset S_{x_i}$ so $\overline{J^0}\subset S_{x_1}$ and the proof is complete.

Although $\overline{J^0}$ is not generally equal to J in Lemma 4, the fact that J itself is contained in some element of S will follow as a corollary to the next lemma.

LEMMA 5. *If I is an irreducible subcontinuum of M and J is a layer of I with void interior relative to I, then $J\subset S_x$ for some $x\in M$.*

Proof. Suppose $J\cap S_x\neq0\neq J\cap S_y$ where $S_x\neq S_y$. If f is the quotient map of the minimal monotone upper semi-continuous decomposition of I onto $[0,1]$ then $J=f^{-1}(t)$ for some $t\in[0,1]$. Without loss of generality we can assume that $t\in(0,1)$. We have $J=J_1\cup J_2$ where $J_1=J\cap\overline{f^{-1}[0,t)}$ and $J_2=J\cap\overline{f^{-1}(t,1]}$ and we can assume that $J_1\cap S_x\neq0\neq J_1\cap S_y$. Let $p\in J_1\cap S_x$ and $q\in J_1\cap S_y$. There exists $k\in\tilde{K}$ such that k separates p from q in M. Take $t_1\in[0,t)$ such that $k\cap f^{-1}(t_1)\neq0$ and $f^{-1}(t_1)\cap\overline{f^{-1}(t_1,1]}=f^{-1}(t_1)$. Let $c\in k\cap f^{-1}(t_1)$. By the definition of $\tilde{K}$ we can assume that there exists $k'\in\tilde{K}$ and a continuum $L\subset M$ such that $\{q,c\}\subset L$ and k' separates L from p. Since $\overline{f^{-1}((t_1,t))}$ is irreducible from c to q and M is hereditarily unicoherent, $\overline{f^{-1}((t_1,t))}\subset L$. But $p\in\overline{f^{-1}((t_1,t))}-L$ and this contradiction shows that $J\subset S_x$ for some $x\in M$.

LEMMA 6. *If I is an irreducible subcontinuum of M and J is a layer of I with non-void interior relative to I, then J is contained in some element of S.*

Proof. By Lemma 4, $\overline{J^0}$ is contained in S_x for some $x\in M$. No generality is lost if we assume that $I-J=A\cup B$, a separation. Since the continuum $\bar{A}\cap J$ has void interior, with trivial modifications Lemma 5 implies that $\bar{A}\cap J\subset S_y$ for some $y\in M$. But $(\bar{A}\cap J)\cap\overline{J^0}\neq0$ so $\bar{A}\cap J\subset S_x$. Similarly $\bar{B}\cap J\subset S_x$. Thus $J\subset S_x$ since $J=(\bar{A}\cap J)\cup\overline{J^0}\cup(\bar{B}\cap J)$.

Lemmas 5 and 6 prove that if J is a layer of I then $J\subset S_x$ for some $x\in M$ and the proof of Theorem 1 is complete.

The next theorems are an immediate consequence of Theorem 1.

THEOREM 2. *Let M be a hereditarily unicoherent continuum. Then M has a non-trivial admissible decomposition if and only if there exists a non-empty collection of closed separators of M with property* (*).

Or stating Theorem 2 differently we have:

THEOREM 3. *Let M be a hereditarily unicoherent continuum. Then M is monostratic if and only if no non-empty collection of closed separators of M satisfies* (*).

Charatonik has proved that if a continuum M has an admissible decomposition each of whose elements has void interior then each monostratic subcontinuum has void interior [3, p. 128]. He shows by example [3, p. 128] that the converse is false but asks if it is true for λ-dendroids. The following example shows the converse is false even for λ-dendroids.

EXAMPLE. Let Q be the Cantor middle third discontinuum in the unit interval I and form the product $Q\times I$. Let

$$I' = \{(x, 0)|\ 0\leqslant x\leqslant 1\} \quad \text{and} \quad T = \{(x, y)|\ x = |\sin(1/y)|, -1\leqslant y<0\}.$$

Denote by K the monostratic plane continuum constructed by Charatonik [2]. Replace each vertical interval i in $Q\times I$ by a copy of K, K_i, such that (1) K_i is perpendicular to the plane, (2) the cross section of K_i in the plane is i, (3) $K_i\cap I' = p_i$ where p_i is the point of the Cantor set at the foot of the interval i and is the midpoint of the interval $[a, b]$ of K_i [2, Figure 1, p. 77], (4) the point c of K_i [2, Figure 1, p. 77] coincides with the top of the vertical interval i. Let R be the set obtained from $Q\times I$ by this replacement of each vertical interval i by K_i and let $X = R\cup I'\cup T$.

Clearly X is hereditarily unicoherent and hereditarily decomposable, hence a λ-dendroid. Although X is not hereditarily stratified (X is *hereditarily stratified* if it contains no non-trivial monostratic subcontinuum) it does not contain any monostratic subcontinuum with non-void interior, e.g., $R\cup I'$ has a non-trivial admissible decomposition with an arc as a quotient space. However, each monostratic subcontinuum K_i of X must be contained in a single element of the minimal admissible decomposition C of X [3, Theorem 6, p. 124]. Also $I'\cup T$ is irreducible between the point $(\sin 1, -1)$ and each point of I' and hence I' is contained in a single element of C. But since $K_i\cap I'\neq 0$ for each i, it follows that $R\cup I'$ is contained in one element of C. Since $R\cup I'$ has non-void interior this shows that X does not have an admissible decomposition each of whose elements has void interior.

Two questions immediately arise from this example:

QUESTION 1. If X is a planar λ-dendroid and if every monostratic subcontinuum of X has void interior, then does X have an admissible decomposition each of whose elements has void interior?

QUESTION 2. If X is a hereditarily stratified λ-dendroid, then does X have an admissible decomposition each of whose elements has void interior?

Remark. The referee has pointed out that an example similar to the one above appears in a paper by T. Maćkowiak. It is Example 3 in *On some examples of monostratic λ-dendroids*, Fund. Math. 87 (1975), pp. 79–88.

References

[1] J. J. Charatonik, *On decompositions of λ-dendroids*, Fund. Math. 67 (1970), pp. 15–30.

[2] — *An example of a monostratiform λ-dendroid*, Fund. Math. 67 (1970), pp. 75–87.

[3] J. J. Charatonik, *On decompositions of continua*, Fund. Math. 79 (1973), pp. 113–130.

[4] — *Some problems concerning monotone decompositions of continua*, Colloq. Math. Soc. János Bolyai 8, Topics in Topology, Keszthely (Hungary) (1972), pp. 145–154.

[5] R. W. FitzGerald and P. M. Swingle, *Core decompositions of continua*, Fund. Math. 61 (1967), pp. 33–50.

[6] L. F. McAuley, *An atomic decomposition of continua into aposyndetic continua*, Trans. Amer. Math. Soc. 88 (1958), pp. 1–11.

[7] E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Dissertationes Math. 50, Warszawa 1966, pp. 1–74.

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