Note on category in Cartesian products of metrizable spaces

by

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Abstract. We characterize the pairs X, Y of metrizable spaces such that if $A \subset X$ is of second category in X and $B \subset Y$ is of second category in Y then $A \times B$ is of second category in $X \times Y$.

Throughout this note *all spaces are assumed to be metrizable.*

A classical theorem of Kuratowski and Ulam [9] asserts that *if X and Y are spaces such that at least one of them is separable and if $A \subset X$ is of second category in X and $B \subset Y$ is of second category in Y, then $A \times B$ is of second category in $X \times Y$*; the assumption of separability can be replaced here by local separability, as was pointed out by Sikorski [13] (cf. also [10]).

Recently, Krom [5] showed that under continuum hypothesis there exist metrizable spaces X and Y of second category whose product $X \times Y$ is of first category, and quite recently Fleissner [4] constructed such spaces X and Y without any additional axioms of set theory (see Added in proof (1)).

In this note we characterize the pairs of spaces X, Y which satisfy the assertion of the theorem of Kuratowski and Ulam.

The inspiring source for our reasonings has been the idea of Fleissner [4]; however, our approach is different and, in particular, it yields a relatively simple construction of two spaces of second category whose product is of first category.

Our terminology follows [3] and [6]. The weight of a space X is denoted by $w(X)$, ω_1 stands for the set of all countable ordinals and N is the set of natural numbers. We say that a space X is *non-separable* at a point $x \in X$ provided that every neighbourhood of x is non-separable. For a space X the letter ϱ denote a metric compatible with the topology of X and $B(M, \varepsilon) = \{x\colon \varrho(x, M) < \varepsilon\}$, where $M \subset X$ and $\varepsilon > 0$. The product of countably many copies of the discrete space of cardinality $\mathfrak{m}$ is denoted by $B(\mathfrak{m})$ (cf. [3]).

LEMMA 1. *Let spaces X and Y be the unions of increasing sequences $X_1 \subset \ldots \subset X_\xi \subset \ldots$ and $Y_1 \subset \ldots \subset Y_\xi \subset \ldots$ of type ω_1 of closed and boundary subsets of X and Y, respectively. Let us put*

$$G_\xi = X_\xi \setminus \bigcup_{\alpha<\xi} X_\alpha \quad \text{and} \quad H_\xi = Y_\xi \setminus \bigcup_{\alpha<\xi} Y_\alpha .$$

Then the set

$$E = \bigcup \{G_\xi \times H_\eta:\ \xi \neq \eta,\ \xi, \eta < \omega_1\}$$

is an F_σ-set of first category in the product $X \times Y$.

Proof. Let us put for each $\xi < \omega_1$ and $n \in N$

$$U_\xi^n = B(X_\xi, 1/n), \quad W_\xi^n = B(Y_\xi, 1/n),$$
$$X_\xi^n = X \setminus U_\xi^n, \quad Y_\xi^n = Y \setminus W_\xi^n,$$
$$E_\xi^n = (X_\xi \times Y_\xi^n) \cup (X_\xi^n \times Y_\xi), \quad E_n = \bigcup \{E_\xi^n:\ \xi < \omega_1\}.$$

We have then $E = \bigcup_n E_n$.

Obviously, $E_\xi^n \subset E$ for each ξ and n. Conversely, let $(x, y) \in E$, i.e., $x \in G_\xi$ and $y \in H_\eta$ for $\xi \neq \eta$. Assume for example that $\xi < \eta$. Then $x \in X_\xi$ and $y \notin Y_\xi$, and therefore $y \notin W_\xi^n$ for some n; this gives $(x, y) \in X_\xi \times Y_\xi^n \subset E_n$.

It is enough now to show that each E_n is an F_σ-set of first category in $X \times Y$. To this end let us consider the open covering $\{U_\xi^n \times W_\xi^n:\ \xi < \omega_1\}$ of the product $X \times Y$. For each $\xi < \omega_1$ the set $(U_\xi^n \times W_\xi^n) \cap E_n = (U_\xi^n \times W_\xi^n) \cap \bigcup \{E_\alpha^n:\ \alpha < \xi\}$ is the countable union of closed and boundary subsets of $X \times Y$, and hence it is an F_σ-set of first category in $X \times Y$. Thus E_n is locally an F_σ-set of first category in $X \times Y$ and our conclusion follows from a Montgomery's theorem and the Banach localization principle, respectively (cf. [6; § 30, X and § 10, III]).

Remark 1. For every $\mathfrak{m} > \aleph_0$ the space $X = B(\mathfrak{m})$ is the union of a sequence $X_1 \subset \ldots \subset X_\xi \subset \ldots$ of type ω_1 of closed and boundary subsets. Indeed, let $X = S^N$, where S is the discrete space of cardinality $\mathfrak{m}$ and let $S = \bigcup_{\xi < \omega_1} S_\xi$, where $S_1 \subset \ldots \subset S_\xi \subset \ldots$ and $S_{\xi+1} \setminus S_\xi \neq \emptyset$; then the sets $X_\xi = S_\xi^N$ have the required properties.

Lemma 2 (Štěpánek, Vopěnka [14]). *Every space X which is non-separable at each point is the union of an increasing sequence $X_1 \subset \ldots \subset X_\xi \subset \ldots$ of type ω_1 of closed and boundary subsets.*

Since the proof given in [14] is indirect, let us sketch a simple proof of this fact.

Let $\mathscr{K}$ be the class of all spaces X which have the required property. Then, as we have shown in Remark 1,

(1) $$B(\mathfrak{m}) \in \mathscr{K} \quad \text{for} \quad \mathfrak{m} > \aleph_0.$$

It is easy to verify that (notice, that the union of an increasing sequence of type ω_1 of closed subsets of a space Z is closed in Z)

(2) if $Y \subset Z$ and $\overline{Y} = Z$, then ($Y \in \mathscr{K}$ iff $Z \in \mathscr{K}$).

By (2) we can restrict ourselves to the case or complete spaces. We finish the proof showing that each complete space Z satisfying the assumption of Lemma 2 contains a dense subspace $Y \underset{\text{top}}{=} \bigoplus_{t \in T} B(\mathfrak{m}_t)$, where $\mathfrak{m}_t > \aleph_0$, and then using (1) and again (2).

Let $\mathscr{U} = \{U \subset Z:\ U$ is an open non-empty set such that for every open and non-empty set $V \subset U$ we have $w(V) = w(U)\}$. Let $\mathscr{V}$ be a maximal disjoint subfamily of $\mathscr{U}$; since each non-empty open set in Z contains a member of $\mathscr{U}$ we have $\overline{\bigcup \mathscr{V}} = Z$. Let $\mathscr{V} = \{V_t:\ t \in T\}$ and let $w(V_t) = \mathfrak{m}_t$; let G_t be a dense, strongly zero-dimensional, G_δ-subspace of V_t. Then each non-empty open subset of the space G_t is of weight $\mathfrak{m}_t$ and hence, by A. H. Stone's result [15; 2.3], we infer that $G_t \underset{\text{top}}{=} B(\mathfrak{m}_t)$. One can take now $Y = \bigoplus_{t \in T} G_t$.

Proposition. *Let X and Y be spaces of second category non-separable at each point. Then there exist sets $A \subset X$ and $B \subset Y$ of second category in X and Y respectively, such that the product $A \times B$ is of first category in $X \times Y$.*

Proof (cf. [6; § 10, VI]). Let X_ξ, Y_ξ, G_ξ and H_ξ be as in Lemma 1 (the existence of X_ξ and Y_ξ follows from Lemma 2). Let us put for each $L \subset \omega_1$

$$X(L) = \bigcup \{G_\xi:\ \xi \in L\}, \quad Y(L) = \bigcup \{H_\xi:\ \xi \in L\},$$

and let

$$\mathscr{I} = \{L \subset \omega_1:\ X(L) \text{ is of first category in } X\}.$$

Then $\mathscr{I}$ is a proper σ-ideal in the algebra of all subsets of ω_1 containing all one-point sets. From the Banach–Kuratowski–Ulam theorem on non-measurability of $\aleph_1$ [2], [17] we infer that $\mathscr{I}$ is not a prime ideal (see [8]), i.e., there exists a disjoint decomposition $\omega_1 = L_0 \cup L_1$ with $L_i \notin \mathscr{I}$. Each $X(L_i)$ is of second category in X. We have $Y(L_0) \cup Y(L_1) = Y$; let for example $Y(L_1)$ be of second category in Y. Since $X(L_0) \times Y(L_1) \subset E$, where E is the set defined in Lemma 1, we infer from this lemma that the sets $A = X(L_0)$ and $B = Y(L_1)$ satisfy the required conditions.

Remark 2. One can verify, using the reasonings similar to that given in [11; the proof of (iii)⇒(ii) in Theorem 1], that the sets L_i obtained in the proof are stationary, i.e., each L_i intersects each closed, cofinal set in ω_1. Thus in the case $X = Y = B(\aleph_1)$ and $X_\xi = Y_\xi$ described in Remark 1, where $S = \omega_1$ and $S_\xi = \{\alpha:\ \alpha < \xi\}$, our approach yields the spaces $M(L_i)$ defined by Fleissner [4] (cf. also [12; 3.1, 3.2]).

Remark 3. The theorem we have used in the proof yields in fact a disjoint decomposition of ω_1 into $\aleph_1$ sets which do not belong to $\mathscr{I}$; one can also consider instead of $\mathscr{I}$ the σ-ideal $\mathscr{I}_m = \{L \subset \omega_1:\ X(L)^m$ is of first category in $X^m\}$, where m is a natural number (verification that $\mathscr{I}_m$ is a σ-ideal bases on Lemma 1).

Remark 4. Let A and B be as in Proposition and let $A^* = A \cap D(A)$ and $B^* = B \cap D(B)$, where $D(M)$ is the set of all points at which M is of second category in X or Y, respectively (see [6; § 10, V]). Then A^* and B^* are Baire spaces (i.e., they are of second category at each point) whose product is a space of first category.

To formulate our main result we need the following notion introduced by A. H. Stone [16].

The *nowhere-locally separable kernel* $K(X)$ of a space X is the largest closed subspace of X which is non-separable at each of its points; the reader is referred to [16] for the properties of $K(X)$.

THEOREM. *For a pair of spaces X, Y the following conditions are equivalent:*

(a) *if $A \subset X$ is of second category in X and $B \subset Y$ is of second category in Y, then $A \times B$ is of second category in $X \times Y$;*

(b) *at least one of the nowhere-locally separable kernels $K(X)$ and $K(Y)$ is of first category in X or Y, respectively.*

Proof. Assume (b). Let, for example, the kernel $K(X)$ be of first category in X, and let $A \subset X$ and $B \subset Y$ be of second category in X or Y, respectively. By A. H. Stone's theorem [16; Theorem 4'] we have $X \setminus K(X) = \bigcup_{n \in N} F_n$, where each F_n is a locally separable and closed subspace of X; put $U_n = \operatorname{Int} F_n$. There exists n such that $U_n \cap A = C$ is of second category in X. By the theorem mentioned at the beginning of this note we infer from local separability of U_n that $C \times B$ is of second category in the space $U_n \times Y$, and hence in $X \times Y$, as U_n is open in X. Thus (b) implies (a).

Conversely, if (b) does not hold, then the negation of (a) follows immediately from the proposition applied to the spaces $\operatorname{Int} K(X)$ and $\operatorname{Int} K(Y)$.

Remark 5. Let us consider the following situation (more general than the "product problem"):

assume that $f\colon T \to X$ is an open mapping of a space T onto a space X of second category such that each fiber $f^{-1}(x)$ is a space of second category.

In this case T can be a space of first category, even if X is compact and each space $f^{-1}(x)$ is completely metrizable. To construct a correspondent example let us choose in the space $B(\mathfrak{c}^+)$ (1) a family of open sets U_{rn}, where r runs over the unit interval I and $n = 1, 2, \ldots$, such that each family $\{U_{rn}\colon r \in I\}$ is discrete and each set $U_r = \bigcup_n U_{rn}$ is dense in $B(\mathfrak{c}^+)$. This can be done as follows: let E_n be a $(1/n)$-net in $B(\mathfrak{c}^+)$; split the set $E = \bigcup_n E_n$ dense in $B(\mathfrak{c}^+)$ into $\mathfrak{c}$ disjoint dense sets E_r, $r \in I$, (cf. [8; Ch. VII, § 8, Theorem 10]) and put $U_{rn} = B(E_r \cap E_n, 1/3n)$. Now, let us take $T = \bigcup \{\{r\} \times U_r\colon r \in I\} \subset I \times B(\mathfrak{c}^+)$ and let f be the restriction to the space T of the projection of $I \times B(\mathfrak{c}^+)$ into $I = X$. The mapping f is open, as $f^{-1}(r)$ is dense in $B(\mathfrak{c}^+)$. The space T is of first category, as $T = \bigcup_n \bigcup_r \{r\} \times U_{rn}$ and each set $\{r\} \times U_{rn}$ is of first category in T.

It is worth while to notice two restrictions on the mapping f each of which guarantees in our case that T is of second category:

(a) *each fiber $f^{-1}(x)$ is separable;*

(b) *there exists a metric ϱ compatible with the topology of T such that all fibers $f^{-1}(x)$ are complete with respect to ϱ (i.e., the fibers of f are "uniformly complete").*

The statement (a) slightly improves a theorem of K. Kuratowski [7], where X is assumed to be separable; a result similar to (b) was obtained independently by E. K. van Douwen (see also [1]).

(1) $\mathfrak{c}^+$ stands for the cardinal next after the continuum $\mathfrak{c}$.

Added in proof.

(1) The first example of (metrizable) Baire spaces X and Y whose product $X \times Y$ is of first category, without additional axioms for set theory, due to P. E. Cohen, *Products of Baire spaces*, Proc. Amer. Math. Soc. 55 (1976), pp. 119–124.

(2) The result of preprint [4] are included into a joint paper of W. Fleissner and K. Kunen, *Barely Baire spaces*, Fund. Math. 101 (1978), pp. 229–240.

(3) Some examples stronger than that in Remark 5 were given by the author in a paper *On category-raising and dimension-raising open mappings with discrete fibers*, Coll. Math.

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