On concentrated sets

by

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Abstract. With the help of the continuum hypothesis, a hierarchy is constructed of sets which are not concentrated on any countable set yet which have the property C of Sierpiński.

1. Introduction. A set E in a topological space X is concentrated on a countable set D if whenever G is open in X, and $G \supset D$, then $(E \setminus G)$ is countable. Besicovitch [1] (see also [4], p. 74, Theorem 38) showed that if the continuum hypothesis is true, there exist uncountable concentrated sets in each uncountable complete separable metric space.

Here we construct and study sets which are not concentrated on a countable set yet which have the property C of Sierpiński (see e.g. [5], or 3.1 for the definition). We show that there is a hierarchy of such sets, similar to the hierarchies of scattered and Borel sets.

We define the operation of concentration (see 3.3), and show in Section 3 that the class of sets with property C is closed under concentration and the operation of taking countable unions. Starting from the countable sets and applying these two operations we construct a class of sets which is also closed under these operations and which is contained in the class of sets with property C.

I thank Dr. A. Ostaszewski for drawing to my attention papers by E. Michael [3] and R. Telgársky [7]. I am also grateful to Prof. C. A. Rogers for reading an early draft, and to R. Telgársky for his comments.

2. Definitions and preliminary results. We assume the continuum hypothesis throughout this paper, denoting it by CH. Without loss of generality, all sets will be subsets of the real line, R.

DEFINITION 2.1. We shall say that countable sets are *of type* 0, and denote the class of countable sets by $\mathscr{B}_0$. A set E is 1-*concentrated* on a set A if G open and $G \supset A$ imply that $E \setminus G$ is countable. If A is countable, E is of *type* 1. The class of sets of type 1 we denote by $\mathscr{B}_1$.

Thus a set concentrated in the ordinary sense is of type 1, and belongs to $\mathscr{B}_1$.

Suppose we have defined ξ-concentration and sets of type ξ for ordinals $\xi<\alpha$. Then a set E is *α-concentrated* on a set A if G open and $G\supset A$ imply that $E\setminus G$ is of type ξ for some $\xi<\alpha$.

Suppose α is either a non-limit ordinal, or a limit ordinal not cofinal with ω_0, the first infinite ordinal. A set E is *of type* α if E is α-concentrated on a set A of type ξ for some $\xi<\alpha$.

Suppose α is a limit ordinal cofinal with ω_0. Then E is *of type* α if $E=\bigcup_n E_n$, where E_n is of type $\xi(n)$ for $\xi(n)<\alpha$.

We denote the class of sets of type α by $\mathscr{B}_\alpha$.

We justify these definitions in Section 4.

The following lemmas are easily proved:

LEMMA 2.2. *If E is of type α, and $F\subset E$, then F is of type α.*

LEMMA 2.3. *Let α,β be ordinals with $\alpha<\beta$. If E is of type α, then E is of type β.*

THEOREM 2.4. (i) *The countable union of sets each of type α is again of type α.*

(ii) *If $\alpha\geqslant 1$ is a non-limit ordinal or a limit ordinal not cofinal with ω_0, the countable union of sets E_n of type $\xi(n)$ for $\xi(n)<\alpha$ is of type ξ for some $\xi<\alpha$.*

Proof. (i) is true for $\alpha=0$ and (ii) for $\alpha=1$. Suppose (i) and (ii) are true for all $\alpha<\eta$. If η is a limit ordinal cofinal with ω_0, (i) is clearly true for $\alpha=\eta$ by definition. Suppose η is either a non-limit ordinal or a limit ordinal not cofinal with ω_0. First we prove (ii).

Let E_n be of type $\xi(n)$ for $\xi(n)<\eta$, and $E=\bigcup_n E_n$. Let $\xi=\sup_n\xi(n)$. Then $\xi<\eta$ and by Lemma 2.3 each E_n is of type ξ, so by the inductive hypothesis E is of type ξ. To prove (i), let E_n be of type η for each n, and $E=\bigcup_n E_n$. Then each E_n is η-concentrated on a set A_n of type $\xi(n)$ for $\xi(n)<\eta$. By (ii) for $\alpha=\eta$, $A=\bigcup_n A_n$ is of type ξ for some $\xi<\eta$. If G is open and $G\supset A$, then $G\supset A_n$ for each n, so $E_n\setminus G$ is of type $\nu(n)$ for some $\nu(n)<\eta$. Again by (ii) for $\alpha=\eta$, $E\setminus G=\bigcup_n(E_n\setminus G)$ is of type ν for some $\nu<\eta$. Thus E is η-concentrated on A, so E is of type η.

3. Property C.

DEFINITION 3.1. A set E has *property C* if for each sequence of positive numbers $\{a_n\}$ there is a sequence of sets $\{A_n\}$ such that $E\subset\bigcup_n A_n$ and $d(A_n)\leqslant a_n$, where $d(A_n)$ is the diameter of A_n. The class of sets with property C we shall denote by $\mathscr{C}$.

The next lemma is easy to prove:

LEMMA 3.2. *The countable union of sets each with property C is a set with property C.*

DEFINITION 3.3. Suppose $\mathscr{A}$ is a class of sets. We define $\mathscr{A}_c$ to be the class of sets E such that: there exists an $A\in\mathscr{A}$ such that if G is open and $G\supset A$, $E\setminus G\in\mathscr{A}$.

(For example, $\mathscr{B}_{0C}=\mathscr{B}_1$). We say *$\mathscr{A}_C$ is obtained from $\mathscr{A}$ by concentration*. If $\mathscr{A}_C=\mathscr{A}$, we say *$\mathscr{A}$ is closed under concentration*.

THEOREM 3.4. *$\mathscr{C}_C=\mathscr{C}$, i.e., $\mathscr{C}$ is closed under concentration.*

Proof. Let E be a set such that there exists a set A with property C such that if G is open and $G\supset A$ then $E\setminus G$ has property C. We must show that E has property C. Let a sequence $\{a_n\}$ of positive numbers be given. As A has property C, we have $A\subset\bigcup_n A_{2n}$ where $d(A_{2n})\leqslant a_{2n}$ for each n and where we may assume that the sets A_{2n} are open. Then $(E\setminus\bigcup_n A_{2n})$ also has property C, so there are sets $\{A_{2n-1}\}$ with $d(A_{2n-1})\leqslant a_{2n-1}$ and $(E\setminus\bigcup_n A_{2n})\subset\bigcup_n A_{2n-1}$. Thus $E\subset\bigcup_n A_n$, and $d(A_n)\leqslant a_n$ for each n, and the theorem is proved.

COROLLARY. *For each α, $\mathscr{B}_\alpha\subset\mathscr{C}$.*

Proof. Follows from Lemma 3.2 and Theorem 3.4 by transfinite induction.

DEFINITION 3.5. A set is called *totally imperfect* if it has no non-empty perfect subsets.

THEOREM 3.6. *A set of type α, for any α, is totally imperfect.*

Proof. By [2], p. 529, Th. 9, a set with property C is totally imperfect. The result follows from Theorem 3.4, corollary.

COROLLARY. *An uncountable analytic set is not of type α for any α.*

Proof. Every uncountable analytic set contains a non-empty perfect set.

4. Sets of type α. The definition of a set of type α is different in the case where α is a limit ordinal cofinal with ω_0. We could have defined a set E to be of type α if E is α-concentrated on a set A of type ξ for some $\xi<\alpha$. However, as we shall see (Theorem 6.3), Theorem 2.4 would then no longer be true. Here we show that no generality is lost with our definition.

THEOREM 4.1. *Let α be a limit ordinal cofinal with ω_0. Let E be a set which is $(\alpha+1)$-concentrated on a set A of type η for some $\eta<\alpha$. Then E is of type α.*

Proof. If G is open and $G\supset A$, then $E\setminus G$ is of type α, so

$$E\setminus G=\bigcup_n E^{(n)},$$

where $E^{(n)}$ is of type ξ for $\xi<\alpha$.

Using CH, well order the open sets containing A as $\{R_\varkappa\}_{\varkappa<\omega_1}$, where ω_1 is the first uncountable ordinal and where $R_1=R$. Then $E\setminus R_2=C_1$ is of type α, and $C_1\subset R_1$. For any $\lambda<\omega_1$,

$$(E\setminus R_\lambda)\cap\bigcap_{\varkappa<\lambda}R_\varkappa=C_\lambda$$

is of type α, and $C_\lambda\subset\bigcap_{\varkappa<\lambda}R_\varkappa$. Then

$$E\subset A\cup\bigcup_{2\leqslant\lambda<\omega_1}E\cap\Big(\big(\bigcap_{\varkappa<\lambda}R_\varkappa\big)\setminus R_\lambda\Big)=A\cup\bigcup_{1\leqslant\lambda<\omega_1}C_\lambda.$$

As α is cofinal with ω_0 we can choose a sequence $\{\nu(n)\}$ of non-limit ordinals with $\sup_n \nu(n) = \alpha$, and express each C_λ as

$$C_\lambda = \bigcup_n C_\lambda^{(n)}$$

where $C_\lambda^{(n)}$ is of type $\nu(n)$.

Let $E_n = A \cup \bigcup_{1\leqslant\lambda<\omega_1} C_\lambda^{(n)}$ for each n. Then

$$E \subset A \cup \bigcup_{1\leqslant\lambda<\omega_1} C_\lambda = A \cup \bigcup_{1<\lambda<\omega_1} \bigcup_n C_\lambda^{(n)} = \bigcup_n \Big(A \cup \bigcup_{1\leqslant\lambda<\omega_1} C_\lambda^{(n)}\Big) = \bigcup_n E_n .$$

By Lemma 2.2 it will suffice to show that each E_n is of type $\xi(n)$ for $\xi(n)<\alpha$.

Let $G \supset A$ be open. Then $G = R_{\lambda^*}$ for some $\lambda^* < \omega_1$. We have $C_\lambda^{(n)} \subset C_\lambda \subset R_{\lambda^*}$ for each n and $\lambda > \lambda^*$. Thus

$$E_n \setminus R_{\lambda^*} \subset \bigcup_{1\leqslant\lambda\leqslant\lambda^*} C_\lambda^{(n)},$$

which is a countable union of sets each of type $\nu(n)$. By Theorem 2.4 (i), $E_n\setminus R_{\lambda^*}$ is of type $\nu(n)$. As G was any open set containing A, E_n is $(\nu(n)+1)$-concentrated on A, the set A being of type $\eta<\alpha$. We can choose $\xi(n)$ such that

$$\max(\nu(n)+1, \eta) < \xi(n) < \alpha$$

and $\xi(n)$ is a non-limit ordinal, for each n. Then E_n is $\xi(n)$-concentrated on A, and A is of type $\eta<\xi(n)$, so E_n is of type $\xi(n)$. This completes the proof.

Corollary. *If α is a limit ordinal cofinal with ω_0, and E is α-concentrated on a set of type $\xi<\alpha$, then E is of type α.*

It follows from this result and Theorem 2.4 that our classes $\mathscr{B}_\alpha$, as α varies, contain all sets which are obtainable from the countable sets under the operations of concentration (as defined in 3.3) and taking countable unions.

Theorem 4.2. *Let ω_2 be the first ordinal whose power is greater than the power of ω_1. Then $\mathscr{B}_{\omega_2} = \bigcup_{\alpha<\omega_2} \mathscr{B}_\alpha$.*

Proof. Suppose $E \in \mathscr{B}_{\omega_2}$. Then E is ω_2-concentrated on a set A of type ξ for some $\xi<\omega_2$.

Well order the open sets containing A, using CH, as $\{R_\varkappa\}_{1\leqslant\varkappa\leqslant\omega_1}$, with $R_1 = R$.

Now put $C_\lambda = E\setminus R_\lambda$ for each λ with $1\leqslant\lambda<\omega_1$. Then each C_λ is of type $\xi(\lambda)$ for $\xi(\lambda)<\omega_2$. Now

$$\eta = \sup_{\lambda<\omega_1} \xi(\lambda) < \omega_2,$$

and by Lemma 2.3 each C_λ is of type η. If G is open and $G\supset A$, then $G = R_{\lambda^*}$ for some $\lambda^*<\omega_1$, and $E\setminus G = C_{\lambda^*}$, so $E\setminus G$ is of type η. Choose a non-limit ordinal γ with $\max(\xi, \lambda)<\gamma<\omega_2$. Then E is γ-concentrated on A, which is of type $\xi<\gamma$, so E is of type $\gamma<\omega_2$. This proves the theorem.

As a corollary, we deduce that not all the classes $\mathscr{B}_\alpha$ are different.

Corollary. *$\mathscr{B}_\alpha = \mathscr{B}_{\omega_2}$ for $\alpha \geqslant \omega_2$.*

Proof. It follows from Theorem 4.2 that α-concentration is equivalent to ω_2-concentration for all $\alpha\geqslant\omega_2$.

Theorem 4.3. *The class $\mathscr{B}_{\omega_2}$ is*

(i) *closed under the operation of taking countable unions, and*

(ii) *closed under concentration, i.e. $(\mathscr{B}_{\omega_2})_c = \mathscr{B}_{\omega_2}$.*

Proof. (i) follows from Theorem 2.4.

(ii) Let $E \in (\mathscr{B}_{\omega_2})_c$. Then there exists a set $A \in \mathscr{B}_{\omega_2}$ such that G open and $G\supset A$ imply that $E\setminus G \in \mathscr{B}_{\omega_2}$. By Theorem 4.2, A is of type ξ for some $\xi<\omega_2$ and $E\setminus G$ is of type $\gamma<\omega_2$ for all G, so $E\in\mathscr{B}_{\omega_2}$.

5. Other classes of sets. Our inductive definition of sets of type α was motivated by a desire for the class $\mathscr{B}_\alpha$ to be as large as possible at stage α. Other intermediate classes of sets can be distinguished, and we define and study some of those which will be useful in the next section.

Definition 5.1. We shall say $E\in\mathscr{B}_n^*$, for $n\geqslant 0$, if

$$\exists D_0 | \forall G_0 \supset D_0 \exists D_1 | \forall G_1 \supset D_1 \exists \ldots \exists D_{n-1} | \forall G_{n-1} \supset D_{n-1}$$

the set $(E\setminus \bigcup_{i=1}^{n-1} G_i) = D_n$ is countable, where for each i $(0\leqslant i\leqslant n-1)$, D_i is countable and G_i is open.

Theorem 5.2. $\mathscr{B}_n \subset \mathscr{B}_{2^n-1}^*$.

Proof. The theorem is true for $n = 0$ and 1. Assume it holds for all $n<k$. Let E be a set of type k. Then there is a set A of type $k-1$ such that G open and $G\supset A$ imply that $E\setminus G$ is of type $k-1$.

By the inductive hypothesis the following statement is true:

$$\exists D_0 | \forall G_0 \supset D_0 \exists \ldots \exists D_{2^{k-1}-2} | \forall G_{2^{k-1}-2} \supset D_{2^{k-1}-2}$$

the set $A\setminus \bigcup_{i=0}^{2^{k-1}-2} G_i = D_{2^{k-1}-1}$ is countable, where for each i, D_i is countable and G_i is open.

Now let $G_{2^{k-1}-1}$ be any open set with $G_{2^{k-1}-1}\supset D_{2^{k-1}-1}$. Then $G = \bigcup_{i=0}^{2^{k-1}-1} G_i$ is open and contains A, so $E\setminus G$ is of type $k-1$. Again by the inductive hypothesis the following statement is true:

$$\exists D_{2^{k-1}} | \forall G_{2^{k-1}} \supset D_{2^{k-1}} \exists \ldots \exists D_{2^k-2} | \forall G_{2^k-2} \supset D_{2^b-2}$$

the set $(E\setminus G)\setminus \bigcup_{i=2^{k-1}}^{2^k-2} G_i = D_{2^k-1}$ is countable, where for each i, D_i is countable and G_i is open.

But $(E\setminus G)\setminus \bigcup_{i=2^{k-1}}^{2^k-2} G_i = E\setminus \bigcup_{i=0}^{2^k-2} G_i$. Thus the following statement is true:

$$\exists D_0|\forall G_0\supset D_0\exists \ldots \exists D_{2^k-2}|\forall G_{2^k-2}\supset D_{2^k-2}$$

the set $E\setminus \bigcup_{i=0}^{2^k-2} G_i = D_{2^k-1}$ is countable, where for each i, D_i is countable and G_i is open.

This means $E\in\mathscr{B}^*_{2^k-1}$.

The next theorem is easily proved by induction:

THEOREM 5.3. *The countable union of sets belonging to $\mathscr{B}^*_n$ also belongs to $\mathscr{B}^*_n$. Also, if $E\in\mathscr{B}^*_n$ and $F\subset E$, $F\in\mathscr{B}^*_n$.*

DEFINITION 5.4 (Telgársky, [7]). Let $\mathscr{B}^{**}_0 = \mathscr{B}_0$. A set E is *n-chain-concentrated* about a countable set D if there exist $\{E_i\colon 0\leqslant i\leqslant n\}$ such that

$$E = E_0\supseteq E_1\supseteq\ldots\supseteq E_n = D$$

and E_i is 1-concentrated on E_{i+1} for each $0\leqslant i\leqslant n-1$.

The class of n-chain-concentrated sets is now denoted by $\mathscr{B}^{**}_n$.

THEOREM 5.5. *The countable union of sets belonging to $\mathscr{B}^{**}_n$ also belongs to $\mathscr{B}^{**}_n$.*

The proof by induction is straightforward.

THEOREM 5.6. *Let i, j, and n be natural numbers with $1+i+j = n$. If $E\in\mathscr{B}^{**}_n$, then there is a set $A\subset E$ with $A\in\mathscr{B}^{**}_j$ such that G open and $G\supset A$ imply that $E\setminus G\in\mathscr{B}^{**}_i$.*

Proof. The theorem is true for $n = 1$, when $i = j = 0$. Suppose it is also true for $n<m$. Let $E\in\mathscr{B}^{**}_m$ and $1+i+j = m$. Then there exist $\{E_i\colon 0\leqslant i\leqslant m\}$ such that

$$E = E_0\supseteq E_1\supseteq\ldots\supseteq E_m = D\,,$$

where D is countable and E_i is 1-concentrated on E_{i+1} for $0\leqslant i\leqslant m-1$.

By definition the set $E_{m-j}\in\mathscr{B}^{**}_j$. Suppose G is open and $G\supset E_{m-j}$. Then $E_{m-j-1}\setminus G$ is countable. Now

$$E\setminus G\supseteq E_1\setminus G\supseteq\ldots\supseteq E_{m-j-1}\setminus G\,.$$

Let G' be any open set containing $E_{k+1}\setminus G$ for some k with $0\leqslant k\leqslant m-j-1$. Then $G'\cup G$ is open and $E_{k+1}\subset G'\cup G$, so $E_k\setminus(G'\cup G)$ is countable, that is, $(E_k\setminus G)\setminus G'$ is countable. Thus $E_k\setminus G$ is 1-concentrated on $E_{k+1}\setminus G$ for $0\leqslant k\leqslant m-j-2$, and so $E\setminus G\in\mathscr{B}^{**}_{m-j-1} = \mathscr{B}^{**}_i$. Thus we may take $A = E_{m-j}$.

THEOREM 5.7. $\mathscr{B}^{**}_{2^n-1}\subset\mathscr{B}_n$.

Proof. The result is true for $n = 1$. Assume it is true for $n<k$, and let $E\in\mathscr{B}^{**}_{2^k-1}$. Take $i = j = 2^{k-1}-1$. Then $i+j+1 = 2^k-1$, so by Theorem 5.6 there exists a set $A\in\mathscr{B}^{**}_{2^{k-1}-1}$ with $A\subset E$ such that G open and $G\supset A$ imply that $E\setminus G\in\mathscr{B}^{**}_{2^{k-1}-1}$. But by the inductive hypothesis both A and $E\setminus G$ are of type $k-1$, so E is of type k.

6. Existence theorems. It may be that for each $\alpha<\omega_2$ there are sets of type α not of type ξ for any $\xi<\alpha$. However, we can only prove this for $\alpha\leqslant\omega_0$. To do this, we must first show the existence of certain sets in the classes $\mathscr{B}^{**}_n$.

THEOREM 6.1. *Let H be an uncountable Borel set. For each natural number n, there exists a set in H belonging to $\mathscr{B}^{**}_n$ but not to $\mathscr{B}^*_{n-1}$.*

Proof. If we interpret $\mathscr{B}^*_{-1}$ to be the class consisting of the empty set, then the theorem is clearly true for $n = 0$. Suppose we have proved the theorem for $n<k$. We construct a set in H belonging to $\mathscr{B}^{**}_k$ but not to $\mathscr{B}^*_{k-1}$.

As every uncountable Borel set contains a non-empty perfect subset, we may assume that H is closed. Let D_1 be a countable set dense in H. Well order the open sets containing D_1 as $\{R_\varkappa\}_{0\leqslant\varkappa<\omega_1}$, and the set of all countable sets as $\{D_\varkappa\}_{1\leqslant\varkappa<\omega_1}$, using CH.

Let λ be an ordinal with $1<\lambda<\omega_1$. The set $\bigcap_{\varkappa<\lambda} R_\varkappa\cap H$ is a Borel set, which by the Baire category theorem applied to H is uncountable, being the countable intersection of sets open and dense in H. By the corollary to Theorem 3.6, it follows that $(\bigcap_{\varkappa<\lambda} R_\varkappa\cap H)\notin\mathscr{B}_1$. So there exists an open set G_λ with $D_\lambda\subset G_\lambda$ such that

$$H_\lambda = (\bigcap_{\varkappa<\lambda} R_\varkappa\cap H)\setminus G_\lambda$$

is uncountable. Now H_λ is a Borel set, so by the inductive hypothesis there is a set C_λ belonging to $\mathscr{B}^{**}_{k-1}$ but not to $\mathscr{B}^*_{k-2}$ with $C_\lambda\subset H_\lambda$.

Let $E = D_1\cup \bigcup_{1<\lambda<\omega_1} C_\lambda$. Then $E\subset H$.

Now each $C_\lambda\in\mathscr{B}^{**}_{k-1}$. So, for each λ there are $\{C^i_\lambda\colon 0\leqslant i\leqslant k-1\}$, with

$$C_\lambda = C^0_\lambda\supseteq C^1_\lambda\supseteq\ldots\supseteq C^{k-1}_\lambda\,,$$

C^{k-1}_λ countable, and C^i_λ 1-concentrated on C^{i+1}_λ for $0\leqslant i\leqslant k-2$. Now for each i with $0\leqslant i\leqslant k-1$ put

$$E_i = D_1\cup \bigcup_{\lambda<\omega_1} C^i_\lambda\,.$$

Then clearly

$$E = E_0\supseteq E_1\supseteq\ldots\supseteq E_{k-1}\supseteq E_k = D_1\,.$$

Now suppose i is such that $0\leqslant i\leqslant k-1$. Let G be any open set containing E_{i+1}. Then $G\supset D_1$, so $G = R_{\lambda^*}$ for some $\lambda^*<\omega_1$. Thus

$$E_i\setminus R_{\lambda^*}\subset \bigcup_{1\leqslant\lambda\leqslant\lambda^*} (C^i_\lambda\setminus R_{\lambda^*})\,.$$

Now, for each λ, $R_{\lambda^*} = G\supset C^{i+1}_\lambda$, as $G\supset E_{i+1}$, so $C^i_\lambda\setminus R_{\lambda^*}$ is countable for each λ. It follows that $E_i\setminus R_{\lambda^*}$ is countable, so that E_i is 1-concentrated on E_{i+1} for $0\leqslant i\leqslant k-1$. This means $E\in\mathscr{B}^{**}_k$.

Finally, we must show that $E\notin\mathscr{B}^*_{k-1}$.

Let T_0 be any countable set. Then $T_0 = D_\eta$ for some $\eta<\omega_1$. Now G_η is an open set containing D_η such that $C_\eta\subset R\setminus G_\eta$, and C_η does not belong to $\mathscr{B}^*_{k-2}$. Set $U_0 = G_\eta$.

So, $E\setminus U_0\notin\mathscr{B}^*_{k-2}$, by Theorem 5.3. Thus

$$\forall T_1\exists U_1\supset T_1|\forall T_2\exists U_2\supset T_2\ldots|\forall T_{k-2}\exists U_{k-2}\supset T_{k-2}$$

such that the set $(E\setminus U_0)\setminus \bigcup_{i=1}^{k-2} U_i$ is uncountable, where each T_i is countable and U_i is open.

It follows (as T_0 was any countable set) that

$$\forall T_0 \exists U_0 \supset T_0 | \forall T_1 \exists U_1 \supset T_1 \ldots | \forall T_{k-2} \exists U_{k-2} \supset T_{k-2}$$

such that the set $(E\setminus U_0)\setminus \bigcup_{i=1}^{k-2} U_i = E\setminus \bigcup_{i=0}^{k-2} U_i$ is uncountable. But this implies $E \notin \mathscr{B}^*_{k-1}$.

Remark. Similar sets in product spaces are constructed by E. Michael [3]. Essentially it is shown there that the product of n copies of a set of type 1 need not belong to $\mathscr{B}^*_{n-1}$, whilst in [7] it is shown that such a set must belong to $\mathscr{B}^{**}_n$.

THEOREM 6.2. *Let H be an uncountable Borel set. For each $\alpha \leqslant \omega_0$, there exists a set in H of type α not of type ξ for any $\xi < \alpha$.*

Proof. It suffices to prove the theorem for $\alpha = n$, a natural number; the countable union (over n) of such sets is then of type ω_0 but not of type n for any n.

Let E_k be the set constructed in Theorem 6.1 which belongs to $\mathscr{B}^{**}_k$ but not to $\mathscr{B}^*_{k-1}$. Then E_{2^n-1} does not belong to $\mathscr{B}^*_{2^n-1-1}$, and so E_{2^n-1} is not of type $n-1$ by Theorem 5.2. However E_{2^n-1} belongs to $\mathscr{B}^{**}_{2^n-1}$, and so to $\mathscr{B}^{**}_{2^n-1}$, so by Theorem 5.7, E_{2^n-1} is of type n.

Finally we return to the remarks made at the beginning of Section 4 on the definition of the classes $\mathscr{B}_\alpha$, when α is a limit ordinal cofinal with ω_0. Here we shall just take $\alpha = \omega_0$.

We shall say a set E *is of type* ω_0^* if E is ω_0-concentrated on a set A of type $n < \omega_0$.

THEOREM 6.3. *There exists a sequence $\{E_n\}$ of sets each of type ω_0^* whose union, E, is not of type ω_0^*.*

Proof. Let Z_n be a sequence of pairwise disjoint open intervals. For each n, let D_n be a countable set dense in Z_n, and choose disjoint closed intervals $S_m^{(n)}$ $(n = 1, 2, \ldots)$ in Z_n with $d(S_m^{(n)}) \to 0$, as $m \to \infty$ (where $d(E)$ denotes the diameter of E)

$$\varrho(S_m^{(n)}, S_{m+1}^{(n)}) = \inf\{|x-y|\colon x \in S_m^{(n)}, y \in S_{m+1}^{(n)}\} \to 0 \quad \text{as} \quad m \to \infty,$$

and $x_m^{(n)} \in S_m^{(n)}$ implies $\lim_{m\to\infty} x_m^{(n)} = x^{(n)} \in D_n$.

For each n and m let $H_m^{(n)}$ be a subset of $S_m^{(n)}$ which is of type m but not of type $m-1$, satisfying: if G is open and $G \supset S_m^{(n)} \cap D_n$, then $H_m^{(n)}\setminus G$ is of type $m-1$. (This can be done, as in Theorem 6.2, by constructing $H_m^{(n)}$ so that $H_m^{(n)} \in \mathscr{B}^{**}_{2^m-1}$).

Let $E_n = \bigcup_m H_m^{(n)}$ for each n. We first show that E_n is of type ω_0^*. Let G be an open set containing D_n. Then $x^{(n)} \in G$, so there exists an $N_n(G)$ such that $S_m^{(n)} \subset G$ for $m \geqslant N_n$, and thus $H_m^{(n)} \subset G$ for $m \geqslant N_n$. Then

$$E_n\setminus G \subset \bigcup_{m \leqslant N_n} H_m^{(n)},$$

which is of type N_n. Thus E_n is of type ω_0^*.

Let $E = \bigcup_n E_n$. We have to show that E is not of type ω_0^*. Let A be any set of type k, for some $k < \omega_0$. Let $A_n = Z_n \cap A$. Now $H_m^{(n)}$ is not of type $m-1$, so for each n and $m > k+1$ we can choose an open set $G(n, m)$ such that $G(n, m) \supset A_n$ and $H_m^{(n)}\setminus G(n, m)$ is not of type $m-2$. It follows that the open set $G(n, n+2)$ contains A_n and $E_n\setminus G(n, n+2)$ is not of type n.

Let $G = \bigcup_n (G(n, n+2) \cap Z_n)$. Then G is open, $G \supset A$, and

$$E\setminus G = \bigcup_n \big(E_n\setminus (G(n, n+2) \cap Z_n)\big) = \bigcup_n \big(E_n\setminus G(n, n+2)\big),$$

which is not of type n for any $n < \omega_0$. As A was arbitrary, we deduce that E is not of type ω_0^*.

References

[1] A. S. Besicovitch, *Concentrated and rarified sets of points*, Acta Math., Stock. 62 (1934), pp. 289–300.

[2] K. Kuratowski, *Topology*, Vol. 1, New York–London–Warszawa 1966.

[3] E. A. Michael, *Paracompactness and the Lindelöf property in finite and countable Cartesian products*, Comp. Math. 23 (1971), pp. 199–214.

[4] C. A. Rogers, *Hausdorff measures*, Cambridge University Press 1970.

[5] F. Rothberger, *Eine Verschärfung der Eigenschaft C*, Fund. Math. 30 (1938), pp. 50–55.

[6] R. Telgársky, *On topological properties defined by games;* Colloquia Mathematica Societatis János Bolyai, 8. Topics in Topology, North Holland Publ. Co., Amsterdam 1974, pp. 617–624.

[7] — *Concerning product of paracompact spaces*, Fund. Math. 74 (1972), pp. 153–159.

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