Some examples in the dimension theory of Tychonoff spaces

by

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Abstract. In this paper we give some examples related to the covering dimension \( \dim \) in the class of Tychonoff spaces. In particular, we construct a Tychonoff space \( X \) with \( \dim X > 0 \) which is the union of two functionally closed subspaces \( X_L \) and \( X_R \) such that \( \dim X_L = 0 \) and a noncompact weakly paracompact space of local dimension zero which is not \( N \)-compact. The common idea of our constructions bases on the well-known theorem of M. Bockstein on products of real lines. These results were summarized in [12], where the reader is also referred for the remarks about the main idea of our constructions.

1. Terminology and notation. Our terminology follows [4]. All our spaces are assumed to be Tychonoff. By the dimension we mean the covering dimension \( \dim \) defined as in [4] or [6] (\(^1\)). A space \( X \) is strongly zero-dimensional if \( \dim X = 0 \); \( X \) is zero-dimensional if it has a base consisting of open-end-closed sets. The local dimension of the space \( X \) is at most \( n \) (abbreviated \( \text{loc.dim} X \leq n \)) if each point \( x \) of \( X \) has an open neighbourhood \( U \) such that \( \dim U \leq n \) (see [8], Chapter 2, § 11). The symbol \( \mathbb{R} \) denotes the real line, \( I \) — the real interval \([0, 1]\) and \( C = I \) — the Cantor discontinuum. By a countable set we mean a set of cardinality \( \aleph_0 \), \( c \) denotes the power of continuum. If \( X \) is a topological space and \( F \) a subspace of \( X \) then the symbol \( X_F \) denotes the set \( X \) equipped with the following topology: the set \( U \) is open in \( X_F \) iff it is of the form \( V \cup K \), where \( V \) is open in \( X \) and \( K \subseteq X \setminus F \) (see [4], Example 5.1.2); notice that if \( X \) is a Tychonoff space then \( X_F \) is also a Tychonoff space. A set \( F \subseteq X \) is functionally closed in the space \( X \) if it is of the form \( F = f^{-1}(0) \) for some continuous function \( f: X \to I \); we say that \( F \) is \( G_F \)-closed in the space \( X \) if the complement \( X \setminus F \) of \( F \) is the union of \( G_F \)-sets in \( X \). A space \( X \) is scattered if it has no subset dense in itself. A space \( X \) is \( N \)-compact if it is homeomorphic with a closed subspace of a product of natural numbers. To avoid the confusion between the various topologies we will sometimes denote by \( K^X \) the closure of the set \( K \) in the space \( X \).

\(^{1}\) We say that \( \dim X \leq n \) if every finite functionally open covering of \( X \) can be refined by a functionally open covering whose order is at most \( n \).
2. Auxiliary lemmas. In Section 3 we shall need the following.

**Lemma 1.** Let $Y$ be a dense subspace of the Cartesian product $M$ of metrizable separable zerodimensional spaces. Then for every subspace $F$ of $Y$ the space $X = Y_x$ is strongly zerodimensional.

**Proof.** First remark that by the theorem of K. A. Ross and A. H. Stone (see [4], P. 2712) each regular open (7) subset of $M$ depends on countably many coordinates, hence it is functionally open in $M$ and is strongly zerodimensional as the Cartesian product of zero-dimensional metrizable separable spaces (see [7], Theorem 3). Let us take two arbitrarily functionally closed and disjoint subsets $K_0$ and $K_1$ of the space $X$ and let $U_0$ and $U_1$ be open subsets of $X$ such that $K_i \subseteq U_i$ for $i = 0, 1$ and $U_0^c \cap U_1^c = \emptyset$. Then there exist open subsets $V_0$ and $V_1$ of the space $Y$ such that $K_i \cap F = V_i \cap U_i$ for $i = 0, 1$. Because $Y$ is dense in $M$, the sets $V_i^Y$ are functionally closed in $M$ and by the initial remark the set $U = M \setminus (V_0^Y \cap V_1^Y)$ is strongly zerodimensional and the sets $V_0^M \cap U$ and $V_1^M \cap U$ are functionally closed disjoint subsets of $U$. Thus there exist disjoint open subsets $W_0$ and $W_1$ of $U$ such that $V_0^M \cap U \subseteq W_i$ for $i = 0, 1$ and $W_0 \cup W_1 = U$. Observe that $U = F$. Indeed, we have

$$F \cap V_0^M \cap V_1^M = F \cap V_0^M \cap V_1^M = F \cap V_0^M \cap V_1^M \subseteq F \cap U_0^c \cap U_1^c = \emptyset$$

(whose second equality follows from the fact that the points of $F$ have the same neighbourhoods in $X$ and in $Y$). Let us put $W_0' = (W_0 \setminus K_0) \cap (X \setminus F)$ and $W_1' = (W_1 \setminus K_0) \cap (X \setminus F)$. The sets $W_0'$ and $W_1'$ are open disjoint subsets of $X$ containing $K_0$ and $K_1$ respectively such that $W_0' \cap W_1' = \emptyset$. Indeed we have

$$W_0' = (W_0 \setminus K_0) \cap (X \setminus F) = (W_0 \setminus K_0) \cap (X \setminus F) = (W_0 \setminus K_0) \cap (X \setminus F) = K_0$$

$$W_1' = (W_1 \setminus K_0) \cap (X \setminus F) = (W_1 \setminus K_0) \cap (X \setminus F) = K_1$$

and

$$W_0' \cap W_1' = (W_0 \setminus K_0) \cap (X \setminus F) = (W_0 \setminus K_0) \cap (X \setminus F) = \emptyset$$

and

$$W_0' = (W_0 \setminus K_0) \cap (X \setminus F) = (W_0 \setminus K_0) \cap (X \setminus F) = \emptyset$$

and

$$W_1' = (W_1 \setminus K_0) \cap (X \setminus F) = (W_1 \setminus K_0) \cap (X \setminus F) = \emptyset$$

Because the points of the set $X \setminus F$ are isolated, the sets $W_0'$ and $W_1'$

$$X \setminus W_0' = W_1' \cup (X \setminus F \setminus W_0')$$

are disjoint, open-end-closed subsets of $X$ containing $K_0$ and $K_1$ respectively. This finishes the proof that $\dim X = 0$.

The following lemma is a special case of Lemma 1.

**Lemma 2.** Every dense subspace of the Cartesian product of metrizable separable zerodimensional spaces is strongly zerodimensional.

This lemma can be proved also using a theorem of A. Arhangelskii [1], which states that every continuous function defined on a dense subspace of the Cartesian product of metrizable separable spaces can be factored by a countable subproduct (compare the proof of Theorem 3 of [7]).

3. A space of positive covering dimension which is the union of two functionally closed strongly zerodimensional subspaces.

3.1. In this section we shall construct a space $X$ having the following properties:

(a) $\dim X > 0$;

(b) $X = X_1 \cup X_2$, where $X_i$ are functionally closed strongly zerodimensional subspaces of $X$;

(c) $X_i = G_i \cup F$, where $G_i$ is a functionally open discrete subspace of $X$ and $F$ is a discrete subspace of $X$ (notice that the conditions (b) and (c) imply that $\text{locdim} X = 0$).

Next, we shall slightly strengthen our construction in order to obtain the following two examples.

**Example 3.1.** A space $X$ having the properties (a)-(c) and the property (d) $X$ is separable.

**Example 3.2.** A space $X$ having the properties (a)-(c) and the property (e) $X$ is weakly paracompact.

Let us mention that if we want to obtain a space having only the properties mentioned in the title, then the construction is simpler than that given below (see Subsection 3.5).

3.2. We pass to the construction of a space $X$ satisfying the conditions (a)-(c).

Let $Q_i$ be the set of all left ends and $Q_2$ — the set of all right ends of the contiguous intervals of the Cantor discontinuum $C$, let $P = C \setminus (Q_1 \cup Q_2)$ and $C_i = P \cup Q_i$ for $i = 1, 2$. Let $F : C \to F$ be a function such that $F(x) = f(y)$ if $x = y$ or $x$ and $y$ are the ends of the same contiguous interval. Let $S$ be a set of cardinality $c$.

Fix $x_0 \in S$ and let $\{A_i \mid i \in T\}$ be the family of all countable subsets of $S \setminus \{x_0\}$. Let $M_i = (C_i \cup \{2\} \cup \{3\})^x$ for $i = 1, 2$. For $A \subseteq S$ and $i = 1, 2$ the symbol $p_i^A : M_i \to (C_i \cup \{2\} \cup \{3\})^x$ denotes the projection; for $x \in S$ we put $p_i^x = p_i^A$. For each pair $(i, p)$, where $i \in T$ and $p \in C$ let us choose an index

$$w_{ip} \in S \setminus \{A_i \cup \{x_0\}\}$$
so that if \( (t, p) \neq (t', p') \) then \( w_{tp} \neq w_{t'p'} \). For \( i = 1, 2, t \in T \) and \( p \in C_i \) let us define a point \( x_{tp} \in M_i \) as follows:

\[
x_{tp}(s) = \begin{cases} 
  p & \text{if } s \in A_i, \\
  0 & \text{if } s = s_p, \\
  2 & \text{if } s = w_{tp}, \\
  3 & \text{otherwise.}
\end{cases}
\]

Put \( F_i = \{x_{tp}: t \in T, p \in C_i \} \). Take an arbitrary dense subset \( E_i \) of \( M_i \) such that \( E_i \cap (\rho_i)^{-1}(0) = \emptyset \). Let \( Z_i = E_i \cup F_i \) be the subspace of \( M_i \).

Now let us define the equivalence relation \( \mathcal{R} \) on the discrete sum \( Z = Z_1 \oplus Z_2 \) by the formula

\[
y \mathcal{R} z \iff [(y = z) \lor (y = x_{tp} \wedge z = x_{tp} \wedge f(s_i) = f(s_j))] .
\]

Let \( Y = Z/\mathcal{R} \) be the quotient space, \( \pi: Z \to Y \) — the natural quotient mapping, \( Y_1 = \pi(Z_1), Y_2 = \pi(Z_2), G_1 = \pi(E_1) \) and \( F = \pi(F_1) = \pi_2(F_2) \). Finally, let us put \( X = Y_2 \) and \( X_i = (Y_i)_2 \).

First let us establish some properties of the spaces \( Z \) and \( Y \).

1. The set \( F_i \) is a discrete functionally closed subspace of \( Z_i \).

Indeed, the set \( F_i \) is of the form \( F_i = (\rho_i)^{-1}(0) \cap Z_i \) (thus it is functionally closed) and for each \( t \in T \) and \( p \in C_i \) the set

\[
U_{tp} = \{x \in Z_i: |x(w_{tp}) - 2| < 1\}
\]

is an open neighbourhood of the point \( x_{tp} \) such that \( U_{tp} \cap F_i = \{x_{tp}\} \).

2. The space \( Y_i \) is homeomorphic to \( Z_i \) and \( \dim F_i = 0 \).

To see this, let us observe that from (1) and the definition of \( \mathcal{R} \) it follows easily that \( \pi: Z_i \to Y_i \) is a homeomorphism. Since \( Z_1 \) is a dense subset of the Cartesian product of metrizable separable zerodimensional spaces, by Lemma 2 we have \( \dim Z_i = 0 \).

3. The set \( Y_i \) is functionally closed in \( Y \).

Since, by (1), the set \( \pi^{-1}(Y_i) = Z_i \cup F_i \cup F_2 \) is functionally closed in \( Z \), there exists a function \( u_2: Z \to I \) such that \( \pi^{-1}(Y_i) = u_2^{-1}(0) \). Since \( \pi \) is one-to-one on the set \( Z \setminus \pi^{-1}(Y_i) \), the function \( u_2 \) is constant on inverses of points under \( \pi \) and hence the formula \( u_2(x) = x \leftarrow \pi^{-1}(x) \) defines a continuous function \( v: Y \to I \) such that \( Y_i = v_2^{-1}(0) \).

4. The space \( Y \) is zerodimensional (hence \( Y \) is a Tychonoff space) and the set \( F \) is a functionally closed discrete subspace of \( Y \).

Since the space \( Y_i \) is zerodimensional and the set \( G_i \) is open both in \( Y_i \), and in \( Y \), the space \( Y \) is zerodimensional at each point of \( G_i \), for \( i = 1, 2 \). If \( x \in F \), then the sets \( \pi(U_1) \cup \pi(U_2) \), where \( U_i \) is an open-and-closed neighbourhood of \( \pi^{-1}(x) \) in \( Z_i \) disjoint with \( F \setminus \pi^{-1}(x) \), are open-and-closed subsets of \( Y \) disjoint with the set \( F \).
Finally, for \( j = 0, 1 \) we have

\[
\begin{align*}
 u^{-1}(0, j) & = \bigcup_{i = 1, 2} \{ x \in Z_i : x(s_0) = 0 \wedge f(x(s_1), j) = j \} = \bigcup_{i = 1, 2} \{ x \in Z_i : x(s_0) = 0 \wedge f(x(s_1), j) = f(x(s_1), 0) = 0 \} = Z_j.
\end{align*}
\]

This finishes the proof of (9).

Now let us observe that by virtue of (9) the formula \( u(x) = u(x, \pi^{-1}(x)) \) for \( x \in Y \) defines a continuous function \( v : T \to (C \cup \{ 0 \} \cup \{ 3 \} \times (T \cup \{ 0 \} \cup \{ 3 \}) \) such that \( K = \pi(L_j) \) is the unique \( (0, j) \) -equivalence class \( \pi^{-1}(0, j) \), thus we have

(10) the sets \( K_0 \) and \( K_1 \) are disjoint and functionally closed in \( Y \) and hence in \( X \).

We shall show now that

(11) there is no open-and-closed subset of \( X \) containing \( K_0 \) and disjoint with \( K_1 \).

Suppose on the contrary that such a set \( X \) exists. Then, since each point of \( F \) has the same neighbourhoods in the spaces \( Y \) and \( X \), there exist two open subsets \( U \) and \( V \) of \( Y \) such that \( K_0 \subset U \) and \( K_1 \subset V \). Let \( U \cap V = \emptyset \). Setting \( U_i = \pi^{-1}(U) \) and \( V_i = \pi^{-1}(V) \) we obtain two open and disjoint subsets of \( Z \) such that \( \{ x_{0i} : A_i \subset U_i, A_i \subset V_i \} \) and \( \{ x_{1i} : A_i \subset U_i, A_i \subset V_i \} \). Let \( U' \) and \( V' \) be open subsets of \( M_1 \) such that \( U_i \subset U' \) and \( V_i \subset V' \). Let \( U_i \) and \( V_i \) be disjoint because \( Z \) is dense in \( M_1 \). By the theorem of M. Bockstein (see [4, P. 2.7.12]) there exists a countable set \( A \) such that \( p_0(U') \cap p_0(V') = \emptyset \). We can assume that \( s_0, s_1 \in A \). Let \( t_0 \in T \) be such that \( A = t_0 \cup \{ t_0 \} \) and let \( B_i = \pi^{-1}(x_{ti}) \) for \( p \in C_i \).

Then

(12) the set \( B_i \cap p_0(U') \) is open-and-closed in \( B_i \).

Indeed, we have \( B_i \cap p_0(U') = B_i \cap p_0(U) \) and \( B_i \cap p_0(V') = B_i \cap p_0(V) \). Thus \( B_i \cap p_0(U') \cap p_0(V') = \emptyset \). Hence the sets \( B_i \cap p_0(U') \) and \( B_i \cap p_0(V') \) are open in \( B_i \) (because \( p_0(U') \) and \( p_0(V') \) are open in \( p_0(M) \)), disjoint and they cover \( B_i \).

This proves (12).

Let us observe that

(13) if \( p_i \in C_i \) for \( i = 1, 2 \) and \( f(p_1) = f(p_2) \) then \( p_0(x_{0i}) \in U_i \).

Let \( h : B_i \rightarrow C_i \) be the function \( h(p_1(x_{0i})) = p \); it is easy to see that \( h \) is a homeomorphism. Put \( W_i = h(B_i \cap p_0(U')) \). Then, by (12) and (13) we have

(14) \( W_i \) is open-and-closed subset of \( C_i \), \( 0 \in W_i \), \( 1 \notin W_i \) and

(15) if \( p_i \in C_i \) for \( i = 1, 2 \) and \( f(p_1) = f(p_2) \) then \( p_1 \in W_1 \) iff \( p_2 \in W_2 \).

Let \( r_0 = \inf \{ p \in C : p \notin W \} \).

Of course \( r_0 \in C \). Since the set \( W \) is closed in \( C \), \( r_0 \) must be the right end of some contiguous interval; let \( r_0 \in C \) be the left end of this interval. Then \( r_0 \in W_i \) by the definition of \( r_i \). Since \( f(r_1) = f(r_0) = r_0 \) by virtue of (15). Because \( W \) is open in \( C \), there exists \( r_i \) such that \( C \subset \{ r_0, r_i \} \subset W \). Hence, again by (15), we have \( P \subset \{ r_0, r_i \} \subset W \). But \( W_i \) is closed, \( C \subset \{ r_0, r_i \} \subset W \), contrary to the definition of \( r_0 \).

The contradiction we have just obtained proves (11) and thereby finishes the proof of (8).

3.3. Now we shall construct Example 3A. Let us assume that in the construction of the spaces \( X \) given in Subsection 3.2 we have taken a set \( E \) which is dense in \( M_1 \), disjoint with the set \( p_0^{-1}(0) \) and, in addition, countable. Then the set \( G_1 \subset G_2 \), is a dense countable subset of \( X \).

3.4. We obtain Example 3B by assuming that the set \( E \) is dense in \( M_1 \), disjoint with \( p_0^{-1}(0) \) and, in addition, every point of \( E \) has all but finite coordinates equal to 0 (compare [4, Example 5.1.1] and P. 5.5.5(9)). In order to show that the space \( X \) is then weakly paracompact, take an arbitrary open covering \( \mathfrak{U} \) of \( X \). For each \( i \in T \) let \( \mathfrak{U} = \{ U \} \) be a neighbourhood of the point \( x_{0i} \) in \( Y \) such that \( \forall x \in U \) and \( \pi^{-1}(V_{0i}) = \{ x \in M : \pi(x_{0i}) = x \} \) is a point finite open covering of \( X \). The family

\[
\mathfrak{U} = \{ V_{i+} : i \in T, p_i \in C_i \} \cup \bigcup_{i \in T} \{ V_{i+} : p_i \in C_i \}
\]

is a point finite open covering of \( X \) which refines \( \mathfrak{U} \) (is point finite, because each point \( x \in G_1 \) belongs only to the finite number of the sets \( V_{0i} \) for \( p_i \in C_i \)).

3.5. Let us notice that the space \( Y \) defined above provides also the example having properties (a) and (b), simpler than the space \( X \). The proof of this fact follows from (2), (3), (4), (10) and from the following condition, which follows immediately from (11):

(11) there is no open-and-closed subset of \( Y \) containing \( K_0 \) and disjoint with \( K_1 \).

4. A scattered space \( X \) which is not zerodimensional and a weakly paracompact space \( X' \) with \( \dim X' = \dim X \).
4.1. In this section we shall give an example of a space $X$ having the following properties:
(a) $X = E \cup F$, where $E$ is a functionally open discrete subspace of $X$ and the subspace $F$ contains only two non-isolated points $a$ and $b$,
(b) $X$ is connected between the points $a$ and $b$.
Let us notice that from (a) it follows that the space $X$ is scattered (see (17)) and zero-dimensional at each point $x \neq a, b$.
Next, we shall modify the construction of $X$ in order to obtain the following two examples:

**Example 4.4.** A space $X$ having the properties (a) and (b) and the property (c) $X$ is separable.

**Example 4.5.** A space $X$ having the properties (a) and (b) and the property (d) $X$ is weakly paracompact.

Let us add that in Subsection 5.5 we give an example of a space $X$ having the properties (a), (b) and (c) or (d) which is in addition realcompact.

Let us notice that the first example of a scattered space which is not zero-dimensional was given by R. C. Solomon [14] (this space is not weakly paracompact).

Let the space $X = X \setminus \{a \setminus \{b\}\}$, where $X$ is the space from Example 4.B, we shall obtain the following.

**Example 4.6.** A space $X'$ having the following properties:

(a') $X' = E \cup F'$, where $E$ is a functionally open discrete subspace of $X'$ and $F'$ is a discrete subspace of $X'$ (thus, $\text{locdim } X' = 0$),

(b') $\dim X' > 0$,

(d') $X'$ is weakly paracompact.

Let us observe that the spaces described in Examples 4.B and 5.B have also the properties (a'), (b') and (d'), but their construction is more complicated.

Let us notice that the space having the properties (a'), (b') and (d') can not be normal, as $\text{locdim } = \dim$ in the class of normal weakly paracompact spaces (the equality $\text{locdim } = \dim$ for paracompact spaces was proved by Dowker [3] and Nagami [9], the proof for normal weakly paracompact spaces is given in [5]).

4.2. Let us construct a space $X$ satisfying the conditions (a) and (b); our construction is similar to that given in Section 3.

Let $S$ be a set of cardinality $c$. Let us fix a point $x_0 \in S$ and let $\{A_s\}_{s \in T}$ be the family of all countable subsets of $S \setminus \{x_0\}$. For each pair $(t, p)$, where $t \in T$ and $p \in (0, 1)$, let us choose an index $w_{tp} \in S \setminus (A_t \cup \{x_0\})$ so that $w_{tp} \neq w_{t'p'}$ for $(t, p) \neq (t', p')$ and let us define a point $x_{tp} \in \mathbb{R}^2$ in the following way:

$$ x_{tp}(s) = \begin{cases} 
  p & \text{if } s \in A_t, \\
  0 & \text{if } s = x_0, \\
  2 & \text{if } s = w_{tp}, \\
  3 & \text{otherwise}. 
\end{cases} $$

Let $a, b \in \mathbb{R}^2$ be such that $a(t) = 0$ for $s \in S, b(x_0) = 0$ and $b(s) = 1$ for $s \in S \setminus \{x_0\}$.

Put

$$ F = \{x_{tp} : t \in T, p \in (0, 1)\} \cup \{a \cup \{b\}\}. $$

Let us take an arbitrary dense subset $E$ of $\mathbb{R}^2$ such that $E \cap p_{\mathbb{R}^2}^{-1}(0) = \emptyset$. Let $Y = E \cup F$ be the subspace of $\mathbb{R}^2$ and $X = Y$. Of course $Y$ and $X$ are Tychonoff spaces.

First let us notice that

(16) each point $x_{tp}$ is isolated in $F$.

Indeed, a set

$$ U_{tp} = \{x \in Y : |x(w_{tp}) - 2| < 1\} $$

is an open neighbourhood of $x_{tp}$ in $Y$ (and hence in $X$) such that $U_{tp} \cap F = \{x_{tp}\}$.

(17) The space $X$ is scattered.

Indeed, let us take an arbitrary nonempty subspace $A$ of $X$. If $A \cap E \neq \emptyset$ then a point from $A \cap E$ is isolated. In $A \cap E = \emptyset$ and some point $x_{tp}$ belongs to $A$, then that point is isolated. If neither of the previous cases holds, then $A \subset \{a\} \cup \{b\}$ and $a$ or $b$ is isolated.

Now we shall prove that the subspace $F$ of $\mathbb{R}^2$ has the following property:

(18) If $A \subset \mathbb{R}^2$ is countable then the projection $p_A(F)$ is connected between the points $p_A(a)$ and $p_A(b)$.

Since $p_A(F) = \{0\}$, it suffices to consider the case when $A = A_t$ for some $t_0 \in T$. Let

$$ B = \{p_{A_t}(x_{t_0}) : p \in (0, 1)\} \cup \{p_{A_t}(a)\} \cup \{p_{A_t}(b)\} $$

$$ = \{x \in \mathbb{R}^2 : x(t) = 0 \text{ for } s \in A_t, \text{ where } p \in (0, 1)\} \subseteq p_{A_t}(F). $$

The set $B$ is homeomorphic to $I$, thus $p_{A_t}(F)$ is connected between $p_A(a)$ and $p_A(b)$.

Next we shall prove that

(19) $X$ is connected between the points $a$ and $b$.

Suppose on the contrary that the points $a$ and $b$ can be separated by open-and-closed subsets of $X$. Then there exist two open subsets $U$ and $V$ of the space $Y$ such that $a \in U, b \in V, U \cap V = \emptyset$ and $F \subseteq U \cup V$. Let $U'$ and $V'$ be open subsets of $\mathbb{R}^2$ such that $U = U' \cap Y$ and $V = V' \cap Y$. Because $Y$ is dense in $\mathbb{R}^2$, $U'$ is disjoint with $V'$.

Hence by the theorem of M. Bockstein (see [4], P. 2.7.12) there exists a countable set $A \subset \mathbb{R}^2$ such that $p_A(U') \cap p_A(V') = \emptyset$. Then

$$ p_A(U) \cap p_A(F) = p_A(U') \cap p_A(F), \quad p_A(V) \cap p_A(F) = p_A(V') \cap p_A(F). $$
and \( p_\Delta(U) \cap p_\Delta(V) \supseteq p_\Delta(F) \). Thus the sets \( p_\Delta(U) \cap p_\Delta(F) \) and \( p_\Delta(V) \cap p_\Delta(F) \) are open-and-closed disjoint subsets of \( p_\Delta(F) \) containing \( p_\Delta(a) \) and \( p_\Delta(b) \) respectively, which contradicts (18). This finishes the proof of (19).

4.3. In order to obtain Example 4.4 it suffices to take in the definition of \( Y \) an arbitrary dense subspace \( E \) of \( R^2 \), disjoint with the set \( p_\Delta^{-1}(0) \), which is, in addition, countable. Then \( E \) is a dense countable subset of \( X \).

4.4. We obtain Example 4.5 by taking in the definition of \( Y \) an arbitrary dense subset \( E \) of \( R^2 \), disjoint with the set \( p_\Delta^{-1}(0) \), such that every point of \( E \) has all but finite coordinates equal to 0. Let us show that \( X \) is then weakly paracompact. Let \( \mathcal{U} \) be an arbitrary open covering of \( X \). For each \( t \in T \) and \( p \in (0, 1) \) let us choose an open subset \( V_{tp} \) of \( Y \) such that \( x_{tp} \in V_{tp} \subseteq U_{tp} \), where \( U_{tp} = \{ x \in Y : |x_{tp} - 1| < 1 \} \) and \( V_{tp} \subseteq U \) for some \( U \in \mathcal{U} \). Take \( U_{ta} \in \mathcal{U} \) such that \( a \in U_{ta}, b \in U_{tp} \). Then the family

\[
\mathcal{V}^{\prime} = \{ V_{tp} : t \in T, p \in (0, 1) \} \cup \{ U_{ta} \} \cup \{ U_{tp} \} \cup \{ \{ x \} : x \notin \bigcup_{t \in \mathcal{U}, p \in (0, 1)} V_{tp} \}
\]

is a point-finite open refinement of \( \mathcal{U} \).

4.5. Now we shall construct Example 4.6. Let \( X \) be the space constructed above having the properties (a), (b), and (d). Let \( X' = X \setminus \{ a \} \) be the subspace of the space \( X \). Of course \( X' \) has the property (a'). Let \( u : X' \rightarrow I \) be a continuous function such that \( u(a) = 0 \) and \( u(b) = 1 \). Because the space \( X \) is connected between the points \( a \) and \( b \), the sets \( u^{-1}([0, 1]) \cap X \) and \( u^{-1}((1, 2]) \cap X \) are two functionally closed disjoint subsets of \( X' \) which cannot be separated by open-and-closed sets; thus \( \dim X' > 0 \). By a slight modification of the proof of the property (d) of the space \( X \) given in 4.4 one can show that \( X' \) has the property (d').

5. A recompact locally zero-dimensional not \( N \)-compact space.

5.1. The aim of this section is to give an example of a space \( X \) having the following properties:

(a) \( X \) is recompact,

(b) \( X = E \cup F \), where \( E \) is a functionally open discrete subspace of \( X \) and \( F \) is a discrete subspace of \( X \),

(c) \( X \) is not \( N \)-compact.

Let us notice that the condition (b) implies that \( X \) is scattered and \( \text{locdim } X = 0 \), whereas (a) and (c) imply that \( \dim X > 0 \) (see [12], p. 478).

Next, we shall slightly strengthen our construction in order to obtain the following two examples:

Example 5.2. A space \( X \) having the properties (a)-(c) and the property

(d) \( X \) is separable.

Example 5.3. A space \( X \) having the properties (a)-(c) and the property

(e) \( X \) is weakly paracompact.

Let us notice that a recompact, not \( N \)-compact (and hence not strongly zero-dimensional), weakly paracompact space with \( \text{locdim } = 0 \) cannot be normal (see the final remark of Subsection 4.1).

Let us add that if we want to construct a space having only the properties mentioned in the title, then the construction is simpler (see Subsection 5.4).

The first example of a zerodimensional recompact (metrizable) not \( N \)-compact space was given by P. Nyikos [10]. For other examples of normal spaces with this phenomenon see [11] and [12] (see also [22]).

5.2. Now we shall construct a space \( X \) having the properties (a)-(c).

Let \( S \) be a set of cardinality \( c \), \( x_0 \in S \) — a fixed point of \( S \) and \( \{ A_i \}_{i \in \mathbb{R}} \) — the family of all countable subsets of \( S \setminus \{ x_0 \} \). Put \( M = (R^2)^S \). Let \( \{ C_i \}_{i \in \mathbb{R}} \) be a family of disjoint, connected, and dense subsets of the square \( I^2 \subseteq R^2 \) (it is not hard to verify that such a family exists). Let us choose a bijection \( t \rightarrow i_t \) of the set of \( t \) onto the interval \( J = \{ (x_1, x_2) : x_1 = 1, 0 \leq x_2 \leq 1 \} \). For each pair \((t, p), \) where \( t \in T \) and \( p \in C_i \) let us choose an index \( w_{tp} \in S \setminus (A_i \cup \{ x_0 \}) \) so that \( w_{tp} \neq w_{tp'} \) if \( (t, p) \neq (t', p') \) and let us define a point \( x_{tp} \in M \) as follows:

\[
x_{tp}(s) = \begin{cases} \ p & \text{if } s \in A_i, \\ (0, 0) & \text{if } s = x_0, \\ (0, 2) & \text{if } s = w_{tp}, \\ a_t & \text{otherwise.} \end{cases}
\]

Let \( F = \{ x_{tp} : t \in T, p \in C_i \} \) and \( E \) be an arbitrary dense subset of the space \( M \) such that \( E \cap p_\Delta^{-1}((0, 0)) = \emptyset \) and for every point \( x_0 \in E \) the set \( E_n(x_0) \) is \( G_\delta \)-closed in \( M \) (such a set \( E \) exists, because an arbitrary dense and countable subset of \( p_\Delta^{-1}(R^2 \setminus \{ (0, 0) \}) \) satisfies the required conditions). Let \( Y = E \cup F \) be the subspace of \( M \) and let \( X = Y_F \).

Let us observe that \( F \) is a functionally closed discrete subspace of \( Y \).

Indeed, \( F = p_\Delta^{-1}((0, 0)) \cap Y \) and for every \( t \in T \) and \( p \in C_i \), the set

\[
U_{tp} = p_\Delta^{-1}(\{(x_1, x_2) \in R^2 : \sqrt{x_1^2 + (x_2 - 2)^2} < 1\}) \cap Y
\]

is an open neighbourhood of the point \( x_{tp} \) in \( Y \) such that \( U_{tp} \cap F = \{ x_{tp} \} \).

Because the topology of \( X \) is finer than the topology of \( Y \), the set \( F \) is a discrete functionally closed subspace of \( X \). Thus, since each point of \( E \) is isolated in \( X \), the condition (b) is satisfied.

Observe that \( \text{locdim } X = 0 \).

In fact, for \( t \in T \) and \( p \in C_i \), the set \( U_{tp} \) defined above is a strongly zero-dimensional open-and-closed neighbourhood of the point \( x_{tp} \) in \( X \), whereas the points of \( E \) are isolated.
Let us prove now that (22) the space $Y$ is hereditarily realcompact.

For the purpose, it suffices to prove that for every $y \in Y$ the set $\bigcap \{y\}$ is realcompact (see [4], Exercise 3.11.B). Since, by the Mrówka's theorem ([4], P. 3.12.13), every, $G_F$-closed subset of a realcompact space is realcompact, it suffices to show that the set $\bigcap \{y\}$ is $G_F$-closed in $M$ for every $y \in Y$.

First, we shall show that (23) the set $F$ is $G_F$-closed in $M$.

Suppose that a point $y \in M$ belongs to the $G_F$-closure of $F$. Then (24) for every set $A \subseteq S$ of cardinality $\leq \kappa_0$ there exists a point $x \in F$ such that $p_A(x) = p_A(y)$

(because if $|A| \leq \kappa_0$ then the set $\bigcap_{x \in A} p_A(x)$ is a $G_F$-set in $M$). Clearly, the point $y$ belongs to the closure of the set $F$, hence $y \in (F \cup \{(0, 2)\})^{r(\kappa_0)} \times \{(0, 0)\}$. Suppose that $y(x) \in I^2$ for every $x \in S \setminus \{x_0\}$. Then, by (24), $y(x) = x(y) \in I^2$ for some $x \in F$, hence $y(x) = p \circ C_{r(\kappa_0)}$ for some $t \in T$. We shall show that there is no point $x \in F$ such that $p_{\bigcap_{x \in A}}(x) = p_{\bigcap_{x \in A}(y)}$. Indeed, if $x \neq y$, then $x(y) = (0, 2) \neq y(y) = (0, 0)$; if $x = x_{r''}$, then $x(y) = y(x_{r''})$, and if $x = y$, then $x(y) = (0, 0)$, hence $x(y) \neq y(x)$. Thus we have obtained a contradiction with the assumption that $y \in (F \cup \{(0, 2)\})^{r(\kappa_0)} \times \{(0, 0)\}$.

Hence two cases are possible:

1° There exists $x_1 \in S \setminus \{x_0\}$ such that $y(x_1) = (0, 2)$. Then $y(x_1) = x_1(y_1)$, where $x_1 \in C_{r(\kappa_0)}$, and for every $x \in F \setminus \{x_0\}$ we have $x(y_1) \neq y(x_1)$. Hence, by (24), $p_{\bigcap_{x \in A}}(x) = p_{\bigcap_{x \in A}(y_1)}$ for every $x \in S$, thus $y = x_1 \in F$.

2° There exists $x_1 \in S \setminus \{x_0\}$ such that $y(x_1) = a$, for some $t \in T$. Then $p_{\bigcap_{x \in A}}(x) = p_{\bigcap_{x \in A}(x_1)}$ for every $x \in S$, thus $y = x_1 \in F$.

Let us show now that (25) for every $y \in Y$ the set $\bigcap \{y\}$ is $G_F$-closed in $M$.

If $y \notin E$ then by the assumption on $E$ the set $E \setminus \{y\}$ is $G_F$-closed in $M$. Thus, in view of (23), the set $\bigcap \{y\}$ is $G_F$-closed in $M$. If $y = x_0 \in F$, then the set $\{y\}$ is functionally closed in $Y$ (as $\bigcap_{x \in A} p_A(x) = \bigcap_{x \in A} p_A((0, 0)) \subset \bigcap_{x \in A} p_A((0, 2)) \cap Y$, hence the set $F \setminus \{y\}$ (and thus $\bigcap \{y\}$) is $G_F$-closed in $M$. This shows (25) and thereby finishes the proof of (22).

Because $X$ can be mapped by a continuous one-to-one mapping onto $Y$, from (22) it follows that (26) $X$ is realcompact (moreover, $X$ is hereditarily realcompact) (see [4], Exercise 3.11.B).

It remains to show that (27) $X$ is $N$-compact.

Since, by (21), $X$ is zerodimensional, it suffices to verify that there exists a ultrafilter $\mathcal{U}$ in the family of open-and-closed subsets of $X$ having the countable intersection property such that $\bigcap \mathcal{U} = \emptyset$ (see [13], p. 478).

For each countable set $A \subseteq S \setminus \{x_0\}$, let us put

$$F_A = \{x_{r''}; A_r \supset A, r \neq C_{r(\kappa_0)}\}.$$  

We define

$$\mathcal{U} = \{U: U \text{ is open-and-closed in } X \text{ and there exists a countable set } A \subseteq S \text{ such that } U \supset F_A\}.$$  

(28) $\mathcal{U}$ is an ultrafilter in the family of open-and-closed subsets of $X$.

Indeed, $\mathcal{U}$ is a filter: if $U$ and $U'$ are open-and-closed in $X$, then the conditions $U \supset U'$ and $U \supset F_A$ imply that $U \supset F_A$ and the conditions $U \supset F_A$ and $U \supset F_{A'}$ imply that $U \supset F_{A \cup A'}$. In order to show that the filter $\mathcal{U}$ is maximal, let us take an arbitrary open-and-closed subset $U$ of $X$. Then there exist sets $V$ and $W$ open in $Y$ such that

$$U \cap F \subseteq V \subseteq U \text{ and } (V \setminus U) \cap F \subseteq W \subseteq Y \setminus U.$$  

Let $V'$ and $W'$ be open subsets of $M$ such that $V = V' \cap Y$ and $W = W' \cap Y$ because $Y$ is dense in $M$ and we have $V' \cap W' = \emptyset$. By the theorem of H. Bockstein (see [4], P. 2.7.12) there exists a countable set $A \subseteq S$ such that $p_A(V') \cap p_A(W') = \emptyset$. We can assume that $A = A_{\kappa(\kappa_0)}$ for some $t \in T$. Then the set

$$p_A(F_A) = p_A(F_A \cap A \supset A \supset \emptyset, p \in C_{r(\kappa_0)})$$  

is homeomorphic to the set $\bigcup \{C_t; A \supset A \supset \emptyset\}$, which is connected (because the sets $C_t$ are connected and dense in $J$). Since $p_A(V')$ and $p_A(W')$ are disjoint open subsets of $p_A(M)$ such that $p_A(V') \cup p_A(W') = p_A(V) \cup p_A(W) = p_A(F) = p_A(F_A)$ and the set $p_A(F_A)$ is connected, we have either $p_A(V') = p_A(F_A)$ or $p_A(W') = p_A(F_A)$. Thus we have $p_A(V') = F_A$ or $p_A(W') = F_A$. As $V \supset W \supset F_A$, we have $p_A(V') \cap (V \cup W) = V \supset F_A$, or $p_A(W') \cap (V \cup W) = W \supset F_A$. Thus $V \in \mathcal{U}$ or $W \in \mathcal{U}$. This finishes the proof that $\mathcal{U}$ is an ultrafilter.

(29) The ultrafilter $\mathcal{U}$ has the countable intersection property.

In fact, suppose that $U_i \in \mathcal{U}$ for $i = 1, 2, \ldots$ Then $U_i \supset F_{A_i}$, for some countable $A_i \subseteq S$ and $\bigcap_{i=1}^{\infty} U_i \supset F_{A_i}$.
It remains to show that
\[(30) \quad \bigcap W = \emptyset .\]
If \(x \notin F\) then the set \(X \setminus \{x\}\) is open-and-closed in \(X\) and contains \(F\), thus \(X \setminus \{x\} \notin W\) and \(x \notin \bigcap W\). If \(x = x_0\), then there exists a neighbourhood \(U\) of the point \(x\) such that \(x \in \cup U \subseteq X \setminus F\); thus \(X \setminus U\) is an open-and-closed subset of \(X\) which contains \(F_{\nu(x_0)}\), hence \(X \setminus U \notin W\) and \(x \notin \bigcap W\).

The proof that \(X\) has the properties (a)-(c) is completed.

5.3. We obtain Example 5.A by assuming additionally that the set \(E\) is countable.

5.4. In order to obtain Example 5.B it suffices to assume that the set \(E\) has the following properties: \(E\) is dense in \(M\), \(E \cap p_\nu^-((0,0)) = \emptyset\), for every \(x \in E\) the set \(E \setminus \{x\}\) is \(G_\nu\)-closed in \(M\) and, in addition, every point of \(E\) has all but finite coordinates equal to 0. Then the proof analogous to that given in Subsection 4.4 shows that \(X\) is weakly paracompact. Let us show that there exists a set \(E\) having all the properties mentioned above. Let \(\{B_t\}_{t \in T}\) be the family of all finite subsets of the set \(S \setminus \{x_0\}\) and \(\{C_t\}_{t \in T}\) — a family consisting of disjoint and dense subsets of \(R^\infty((0,0))\). Put
\[E = \{ \{x \in M : x(s) \in C_t, s \in B_t \cup \{x_0\} \text{ and } x(s) = (0,0) \text{ for } s \notin B_t \cup \{x_0\} \} \} .\]

First we shall show that the set \(E\) is \(G_\nu\)-closed in \(M\). For, let \(y \in M\) belongs to the \(G_\nu\)-closure of \(E\). Then \(y(s) \in C_{t_0}\) for some \(t_0 \in T\). There exists a point \(x \in E\) such that \(p_{\nu(x)}(y) = p_{\nu(x)}(x)\). Observe that if \(x\) is a point of \(E\) different from \(x\) then \(p_{\nu(x)}(x) \neq p_{\nu(x)}(x)\). Indeed, if \(x(s) \in C_{t_0}\), then \(x(s) = x(s) = (0,0)\) for \(s \notin B_{t_0}\), hence \(p_{\nu(x)}(x) = p_{\nu(x)}(x)\). and if \(x(s) \notin C_{t_0}\), then \(x(s) \neq y(s) \in C_{t_0}\). Thus \(y = x \in E\) and \(E\) is \(G_\nu\)-closed in \(M\). Now, let \(x_0\) be an arbitrary point of \(E\) and let \(t_0 \in T\) be such that \(x(s) \in C_{t_0}\). Then
\[x(s) = (p_{\nu(x)}(x_0))^{-1}p_{\nu(x)}(x) \in E\]
and so \(x_0\) is functionally closed (and hence \(G_\nu\)-open in \(E\)). Thus \(E \setminus \{x_0\}\) is \(G_\nu\)-closed in \(M\). It is easy to verify that the set \(E\) has also all the remaining properties mentioned above.

5.5. Remark. There exists a realcompact scattered space \(X^*\) which is not zero-dimensional. Moreover, \(X^*\) can be separable or weakly paracompact.

Let \(Y = Y \cup \{a\} \cup \{b\}\), where \(Y\) is the space constructed in 5.2 whereas \(a\) and \(b\) are two points of \(M\) such that \(a(s) = (0,0)\) for \(s \in S\), \(b(s) = (0,0)\) and \(b(s) = (0,1)\) for \(s \in S\). Let \(X^* = Y_{\nu(a) \cup \{b\}}\).

Then the space \(X^*\) is connected between the points \(a\) and \(b\). The proof of this property of \(X^*\) is analogous to the proof of (19), follows from the fact that for every countable set \(A \subset S\) the set \(p_\nu(F \cup \{a\} \cup \{b\})\) is connected between the points \(p_\nu(a)\) and \(p_\nu(b)\). By a slight modification of the proof of (26) one shows that \(X^*\) is realcompact and it is easy to verify that \(X^*\) is scattered (compare the proof of (17)). By an appropriate choice of \(E\) (see 5.3 and 5.4) we obtain \(X^*\) separable or weakly paracompact.

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