Some examples in the dimension theory of Tychonoff spaces

by

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Abstract. In this paper we give some examples related to the covering dimension dim in the class of Tychonoff spaces. In particular, we construct a Tychonoff space X with $\dim X > 0$ which is the union of two functionally closed subspaces X_1 and X_2 such that $\dim X_i = 0$ and a realcompact weakly paracompact space of local dimension zero which is not N-compact. The common idea of our constructions bases on the well-known theorem of M. Bockstein on products of real lines. These results were summarized in [12], where the reader is also referred for the remarks about the main idea of our constructions.

1. Terminology and notation. Our terminology follows [4]. All our spaces are assumed to be Tychonoff. By the dimension we mean the covering dimension dim defined as in [4] or [6] (¹). A space X is strongly zerodimensional if $\dim X = 0$; X is zerodimensional if it has a base consisting of open-and-closed sets. The local dimension of the space X is at most n (abbreviated $\operatorname{loc}\dim X \leqslant n$) if each point x of X has an open neighbourhood U such that $\dim \overline{U} \leqslant n$ (see [8], Chapter 2, § 11). The symbol R denotes the real line, I — the real interval $[0, 1]$ and $C \subset I$ — the Cantor discontinuum. By a countable set we mean a set of cardinality $\aleph_0$, $\mathfrak{c}$ denotes the power of continuum. If X is a topological space and F a subspace of X then the symbol X_F denotes the set X equipped with the following topology: the set U is open in X_F iff it is of the form $V \cup K$, where V is open in X and $K \subset X \setminus F$ (see [4], Example 5.1.2); notice that if X is a Tychonoff space then X_F is also a Tychonoff space. A set $F \subset X$ is functionally closed in the space X if it is of the form $F = f^{-1}(0)$ for some continuous function $f\colon X \to I$; we say that F is G_δ-closed in the space X if the complement $X \setminus F$ of F is the union of G_δ-sets in X. A space X is scattered if it has no subset dense in itself. A space X is N-compact if it is homeomorphic with a closed subspace of a product of natural numbers. To avoid the confusion between the various topologies we will sometimes denote by $\overline{K}^X$ the closure of the set K in the space X. If $M = X^S$ then for $A \subset S$ the symbol $p_A\colon X^S \to X^A$ denotes the projection and $p_s = p_{\{s\}}$; if $x \in M$ then $x(s) = p_s(x)$ is the sth coordinate of x.

(¹) We say that $\dim X \leqslant n$ if every finite functionally open covering of X can be refined by a functionally open covering whose order is at most n.

2. Auxiliary lemmas. In Section 3 we shall need the following

LEMMA 1. *Let Y be a dense subspace of the Cartesian product M of metrizable separable zerodimensional spaces. Then for every subspace F of Y the space $X = Y_F$ is strongly zerodimensional.*

Proof. First remark that by the theorem of K. A. Ross and A. H. Stone (see [4], P. 2.7.12) each regularly open (²) subset of M depends on countably many coordinates, hence it is functionally open in M and is strongly zerodimensional as the Cartesian product of zerodimensional metrizable separable spaces [see [7], Theorem 3). Let us take two arbitrary functionally closed and disjoint subsets K_0 and K_1 of the space X and let U_0 and U_1 be open subsets of X such that $K_i \subset U_i$ for $i = 0, 1$ and $\overline{U}_0^X \cap \overline{U}_1^X = \emptyset$. Then there exist open subsets V_0 and V_1 of the space Y such that $K_i \cap F \subset V_i \subset U_i$ for $i = 0, 1$. Because Y is dense in M, the sets $\overline{V}_i^M$ are regularly closed in M and by the initial remark the set $U = M \setminus (\overline{V}_0^M \cap \overline{V}_1^M)$ is strongly zerodimensional and the sets $\overline{V}_0^M \cap U$ and $\overline{V}_1^M \cap U$ are functionally closed disjoint subsets of U. Thus there exist disjoint open subsets W_0 and W_1 of U such that $\overline{V}_i^M \cap U \subset W_i$ for $i = 0, 1$ and $W_0 \cup W_1 = U$. Observe that $U \supset F$. Indeed, we have

$$F \cap \overline{V}_0^M \cap \overline{V}_1^M = F \cap \overline{V}_0^Y \cap \overline{V}_1^Y = F \cap \overline{V}_0^X \cap \overline{V}_1^X \subset F \cap \overline{U}_0^X \cap \overline{U}_1^X = \emptyset$$

(the second equality follows from the fact that the points of F have the same neighbourhoods in X and in Y). Let us put $W_0' = (W_0 \setminus K_1) \cup (K_0 \setminus F)$ and $W_1' = (W_1 \setminus K_0) \cup (K_1 \setminus F)$. The sets W_0' and W_1' are open disjoint subsets of X containing K_0 and K_1 respectively such that $W_0' \cup W_1' \supset F$. Indeed we have

$$W_i' \supset (W_i \cap K_i) \cup (K_i \setminus F) \supset (\overline{V}_i^M \cap U \cap K_i) \cup (K_i / F)$$
$$\supset (V_i \cap F \cap K_i) \cup (K_i \setminus F) = (K_i \cap F) \cup (K_i \setminus F) = K_i,$$

$$W_0' \cap W_1' = [(W_0 \setminus K_1) \cap (W_1 \setminus K_0)] \cup [(W_0 \setminus K_1) \cap (K_1 \setminus F)] \cup$$
$$\cup [(K_0 \setminus F) \cap (W_1 \setminus K_0)] \cup [(K_0 \setminus F) \cap (K_1 \setminus F)] = \emptyset$$

and

$$W_0' \cup W_1' \supset (W_0 \setminus K_1) \cup (W_1 \setminus K_0)$$
$$\supset [U \setminus (K_0 \cup K_1)] \cup [(\overline{V}_0^M \cap U) \setminus K_1] \cup [(\overline{V}_1^M \cap U) \setminus K_0)]$$
$$\supset [F \setminus (K_0 \cup K_1)] \cup (K_0 \cap F) \cup (K_1 \cap F) = F.$$

Because the points of the set $X \setminus F$ are isolated, the sets W_0' and

$$X \setminus W_0' = W_1' \cup (X \setminus F \setminus W_0')$$

(²) We say that the set U is regularly open in the space M if $U = \operatorname{Int} \overline{U}$.

are disjoint, open-and-closed subsets of X containing K_0 and K_1 respectively. This finishes the proof that $\dim X = 0$.

The following lemma is a special case of Lemma 1.

LEMMA 2. *Every dense subspace of the Cartesian product of metrizable separable zerodimensional spaces is strongly zerodimensional.*

This lemma can be proved also using a theorem of A. Arhangel'skiĭ [1], which states that every continuous function defined on a dense subspace of the Cartesian product of metrizable separable spaces can be factorized by a countable subproduct (compare the proof of Theorem 3 of [7]).

3. A space of positive covering dimension which is the union of two functionally closed strongly zerodimensional subspaces.

3.1. In this section we shall construct a space X having the following properties:

(a) $\dim X > 0$,

(b) $X = X_1 \cup X_2$, where X_i are functionally closed strongly zerodimensional subspaces of X,

(c) $X_i = G_i \cup F$, where G_i is a functionally open discrete subspace of X and F is a discrete subspace of X

(notice that the conditions (b) and (c) imply that $\operatorname{loc}\dim X = 0$).

Next, we shall slightly strengthen our construction in order to obtain the following two examples.

EXAMPLE 3.A. A space X having the properties (a)-(c) and the property

(d) X is separable.

EXAMPLE 3.B. A space X having the properties (a)-(c) and the property

(e) X is weakly paracompact.

Let us mention that if we want to obtain a space having only the properties mentioned in the title, then the construction is simpler than that given below (see Subsection 3.5).

3.2. We pass to the construction of a space X satisfying the conditions (a)-(c).

Let Q_1 be the set of all left ends and Q_2 — the set of all right ends of the contiguous intervals of the Cantor discontinuum C, let $P = C \setminus (Q_1 \cup Q_2)$ and $C_i = P \cup Q_i$ for $i = 1, 2$. Let $f\colon C \to I$ be a function such that $f(x) = f(y)$ iff $x = y$ or x and y are the ends of the same contiguous interval. Let S be a set of cardinality $\mathfrak{c}$. Fix $s_0 \in S$ and let $\{A_t\}_{t \in T}$ be the family of all countable subsets of $S \setminus \{s_0\}$. Let $M_i = (C_i \cup \{2\} \cup \{3\})^S$ for $i = 1, 2$. For $A \subset S$ and $i = 1, 2$ the symbol $p_A^i\colon M_i \to (C_i \cup \{2\} \cup \{3\})^A$ denotes the projection; for $s \in S$ we put $p_s^i = p_{\{s\}}^i$.

For each pair (t, p), where $t \in T$ and $p \in C$ let us choose an index

$$w_{tp} \in S \setminus (A_t \cup \{s_0\})$$

so that if $(t,p)\neq(t',p')$ then $w_{tp}\neq w_{t'p'}$. For $i=1,2$, $t\in T$ and $p\in C_i$ let us define a point $x^i_{tp}\in M_i$ as follows:

$$x^i_{tp}(s)=\begin{cases}p & \text{if } s\in A_t,\\ 0 & \text{if } s=s_0,\\ 2 & \text{if } s=w_{tp},\\ 3 & \text{otherwise}.\end{cases}$$

Put $F_i=\{x^i_{tp}\colon t\in T, p\in C_i\}$. Take an arbitrary dense subset E_i of M_i such that $E_i\cap(p^i_{s_0})^{-1}(0)=\emptyset$. Let $Z_i=E_i\cup F_i$ be the subspace of M_i.

Now let us define the equivalence relation $\mathscr{R}$ on the discrete sum $Z=Z_1\oplus Z_2$ by the formula

$$y\mathscr{R}z\Leftrightarrow[(y=z)\vee(y=x^1_{tp_1}\wedge z=x^2_{tp_2}\wedge f(p_1)=f(p_2))].$$

Let $Y=Z/\mathscr{R}$ be the quotient space, $\pi\colon Z\to Y$ — the natural quotient mapping, $Y_i=\pi(Z_i)$, $\pi_i=\pi/Z_i$, $G_i=\pi(E_i)$ and $F=\pi_1(F_1)=\pi_2(F_2)$. Finally, let us put $X=Y_F$ and $X_i=(Y_i)_F$.

First let us establish some properties of the spaces Z and Y.

(1) The set F_i is a discrete functionally closed subspace of Z_i.

Indeed, the set F_i is of the form $F_i=(p^i_{s_0})^{-1}(0)\cap Z_i$ (thus it is functionally closed) and for each $t\in T$ and $p\in C_i$ the set

$$U^i_{tp}=\{x\in Z_i\colon |x(w_{tp})-2|<1\}$$

is an open neighbourhood of the point x^i_{tp} such that $U^i_{tp}\cap F_i=\{x^i_{tp}\}$.

(2) The space Y_i is homeomorphic to Z_i and $\dim Y_i=0$.

To see this, let us observe that from (1) and the definition of $\mathscr{R}$ it follows easily that $\pi_i\colon Z_i\to Y_i$ is a homeomorphism. Since Z_i is a dense subset of the Cartesian product of metrizable separable zerodimensional spaces, by Lemma 2 we have $\dim Z_i=0$.

(3) The set Y_i is functionally closed in Y.

Since, by (1), the set $\pi^{-1}(Y_i)=Z_i\cup F_1\cup F_2$ is functionally closed in Z, there exists a function $u\colon Z\to I$ such that $\pi^{-1}(Y_i)=u^{-1}(0)$. Since π is one-to-one on the set $Z\setminus\pi^{-1}(Y_i)$, the function u is constant on inverses of points under π and hence the formula $v(x)=u\pi^{-1}(x)$ defines a continuous function $v\colon Y\to I$ such that $Y_i=v^{-1}(0)$.

(4) The space Y is zerodimensional (hence Y is a Tychonoff space) and the set F is a functionally closed discrete subspace of Y.

Since the space Y_i is zerodimensional and the set G_i is open both in Y_i and in Y, the space Y is zerodimensional at each point of G_i, for $i=1,2$. If $x\in F$, then the sets $\pi(U_1)\cup\pi(U_2)$, where U_i is an open-and-closed neighbourhood of $\pi_i^{-1}(x)$ in Z_i disjoint with $F_i\setminus\pi_i^{-1}(x)$, are open-and-closed subsets of Y disjoint with the set $F\setminus\{x\}$

and form a base of neighbourhoods of the point x in Y. Thus the space Y is zero-dimensional at each point $x\in F$ and F is a discrete subspace of Y (by (3) the set F is functionally closed in Y).

Now we shall show that the space X has the required properties. Because the space X was obtained from Y by letting the points of $Y\setminus F$ to be isolated, from (3) and (4) it follows immediately that X is a Tychonoff space,

(5) the set X_i is functionally closed in X, and

(6) the condition (c) is satisfied.

Observe that the space $X_i=(Y_i)_F$ is homeomorphic to the space $(Z_i)_{F_i}$ which is strongly zerodimensional by virtue of Lemma 1, hence we have

(7) $$\dim X_i=0.$$

It remains to show that

(8) $$\dim X>0.$$

Let us fix an arbitrary index $s_1\in S\setminus\{s_0\}$ and put

$$L_0=\bigcup_{i=1,2}\{x\in Z_i\colon x(s_0)=0, x(s_1)=0\},\quad K_0=\pi(L_0),$$

$$L_1=\bigcup_{i=1,2}\{x\in Z_i\colon x(s_0)=0, x(s_1)=1\},\quad K_1=\pi(L_1)$$

(it is easy to verify that $L_j=\bigcup_{i=1,2}\{x^i_{tj}\colon A_t\ni s_1\}$ for $j=0,1$). In order to prove that $\dim X>0$ we shall verify that K_0 and K_1 are functionally closed disjoint subsets of X which cannot be separated by open-and-closed sets.

First we shall show that

(9) there exists a continuous function $u\colon Z\to(C\cup\{2\}\cup\{3\})\times(I\cup\{2\})$ which is constant on inverses of points under π and such that $L_j=u^{-1}((0,j))$ for $j=0,1$.

For, let us define

$$u(x)=\big(x(s_0),\min\big(f(x(s_1)),2\big)\big)\quad\text{for}\quad x\in Z.$$

Clearly u is continuous. Suppose that x and y are distinct points of Z such that $\pi(x)=\pi(y)$, i.e. $x=x^1_{tp_1}$ and $y=x^2_{tp_2}$, where $t\in T$, $p_i\in C_i$ and $f(p_1)=f(p_2)$. Two cases are possible:

1° If $s_1\in A_t$, then $x^i_{tp_i}(s_1)=p_i$ and $f(p_1)=f(p_2)<2$; hence

$$u(x)=(0,f(p_1))=u(y).$$

2° If $s_1\notin A_t$, then $x^i_{tp_i}(s_1)=2$ or 3; hence $u(x)=(0,2)=u(y)$.

Finally, for $j = 0, 1$ we have

$$\begin{aligned} u^{-1}((0,j)) &= \bigcup_{i=1,2} \{x \in Z_i\colon x(s_0) = 0 \wedge \min\big(f(x(s_1)), 2\big) = j\} \\ &= \bigcup_{i=1,2} \{x \in Z_i\colon x(s_0) = 0 \wedge f\big(x(s_1)\big) = j\} \\ &= \bigcup_{i=1,2} \{x \in Z_i\colon x(s_0) = 0 \wedge x(s_1) = j\} = L_j\,. \end{aligned}$$

This finishes the proof of (9).

Now let us observe that by virtue of (9) the formula $v(x) = u(\pi^{-1}(x))$ for $x \in Y$ defines a continuous function $v\colon Y \to (C \cup \{2\} \cup \{3\}) \times (I \cup \{2\})$ such that $K_j = \pi(L_j) = \pi u^{-1}((0,j)) = v^{-1}((0,j))$. Thus we have

(10) the sets K_0 and K_1 are disjoint and functionally closed in Y and hence in X.

We shall show now that

(11) there is no open-and-closed subset of X containing K_0 and disjoint with K_1.

Suppose on the contrary that such a set exists. Then, since each point of F has the same neighbourhoods in the spaces Y and X, there exist two open subsets U and V of Y such that $K_0 \subset U$, $K_1 \subset V$, $F \subset U \cup V$ and $U \cap V = \emptyset$. Setting $U_i = \pi_i^{-1}(U)$ and $V_i = \pi_i^{-1}(V)$ we obtain two open and disjoint subsets of Z_i such that $\{x^i_{t0}\colon A_t \ni s_1\} \subset U_i$, $\{x^i_{t1}\colon A_t \ni s_1\} \subset V_i$ and $F_i \subset U_i \cup V_i$. Let U'_i and V'_i be open subsets of M_i such that $U_i = U'_i \cap Z_i$ and $V_i = V'_i \cap Z_i$; U'_i and V'_i are disjoint because Z_i is dense in M_i. By the theorem of M. Bockstein (see [4], P. 2.7.12) there exists a countable set $A \subset S$ such that $p_A(U'_i) \cap p_A(V'_i) = \emptyset$. We can assume that $s_0, s_1 \in A$. Let $t_0 \in T$ be such that $A = A_{t_0} \cup \{s_0\}$ and let $B_i = p^i_A(\{x^i_{t_0p}\colon p \in C_i\})$. Then

(12) the set $B_i \cap p^i_A(U_i)$ is open-and-closed in B_i,

$$p^i_A(x^t_{t_00}) \in B_i \cap p^i_A(U_i) \quad \text{and} \quad p^i_A(x^i_{t_01}) \in B_i \setminus p^i_A(U_i)\,.$$

Indeed, we have $B_i \cap p^i_A(U_i) \subset B_i \cap p^i_A(U'_i)$, $B_i \cap p^i_A(V_i) \subset B_i \cap p^i_A(V'_i)$, $p^i_A(U'_i) \cap p^i_A(V'_i) = \emptyset$ and $B_i \subset p^i_A(U_i) \cup p^i_A(V_i)$; thus $B_i \cap p^i_A(U_i) = B_i \cap p^i_A(U'_i)$ and $B_i \cap p^i_A(V_i) = B_i \cap p^i_A(V'_i)$. Hence the sets $B_i \cap p^i_A(U_i)$ and $B_i \cap p^i_A(V_i)$ are open in B_i (because $p^i_A(U'_i)$ and $p^i_A(V'_i)$ are open in $p^i_A(M_i)$), disjoint and they cover B_i. This proves (12).

Let us observe that

(13) if $p_i \in C_i$ for $i = 1, 2$ and $f(p_1) = f(p_2)$ then $p^1_A(x^1_{t_0p_1}) \in p^1_A(U_1)$ iff $p^2_A(x^2_{t_0p_2}) \in p^2_A(U_2)$.

In fact, since $U_i \cap F_i = (p^i_A)^{-1} p^i_A(U_i) \cap F_i$ we have

$$\begin{aligned} [p^1_A(x^1_{t_0p_1}) \in p^1_A(U_1)] &\Leftrightarrow [x^1_{t_0p_1} \in U_1] \Leftrightarrow [\pi(x^1_{t_0p_1}) = \pi(x^2_{t_0p_2}) \in U] \\ &\Leftrightarrow [x^2_{t_0p_2} \in U_2] \Leftrightarrow [p^2_A(x^2_{t_0p_2}) \in p^2_A(U_2)]\,. \end{aligned}$$

Let $h\colon B_i \to C_i$ be the function $h\big(p^i_A(x^i_{t_0p})\big) = p$; it is easy to see that h is a homeomorphism. Put $W_i = h\big(B_i \cap p^i_A(U_i)\big)$. Then, by (12) and (13) we have

(14) W_i is open-and-closed subset of C_i, $0 \in W_i$, $1 \notin W_i$ and

(15) if $p_i \in C_i$ for $i = 1, 2$ and $f(p_1) = f(p_2)$ then $p_1 \in W_1$ iff $p_2 \in W_2$.

Let

$$r_0 = \inf\nolimits_I \{p \in C_1\colon p \notin W_1\}\,.$$

Of course $r_0 \in C$. Since the set W_1 is closed in C_1, r_0 must be the right end of some contiguous interval; let $r'_0 \in C_1$ be the left end of this interval. Then $r'_0 \in W_1$ by the difinition of r_0. Since $f(r_0) = f(r'_0)$, $r_0 \in W_2$ by virtue of (15). Because W_2 is open in C_2, there exists $r_1 > r_0$ such that $C_2 \cap [r_0, r_1) \subset W_2$. Hence, again by (15), we have $P \cap [r_0, r_1) \subset W_1$ and because W_1 is closed, $C_1 \cap [r_0, r_1) \subset W_1$, contrary to the definition of r_0.

The contradiction we have just obtain proves (11) and thereby finishes the proof of (8).

3.3. Now we shall construct Example 3.A. Let us assume that in the construction of the spaces Z_i given in Subsection 3.2 we have taken a set E_i which is dense in M_i, disjoint with the set $(p^i_{s_0})^{-1}(0)$ and, in addition, countable. Then the set $G_1 \cup G_2$, is a countable dense subset of X.

3.4. We obtain Example 3.B by assuming that the set E_i is dense in M_i, disjoint with $p_{s_0}^{-1}(0)$ and, in addition, every point of E_i has all but finite coordinates equal to 0 (compare [4], Example 5.1.3 and P. 5.5.3(c)). In order to show that the space X is then weakly paracompact, take an arbitrary open covering $\mathscr{U}$ of X. For each $t \in T$ and $p_1 \in C_1$ let us choose a neighbourhood V_{tp_1} of the point $\pi(x^1_{tp_1})$ in Y such that $V_{tp_1} \subset U$ for some $U \in \mathscr{U}$ and $\pi_i^{-1}(V_{tp_1}) \subset \{x \in M_i\colon |x(w_{tp_i}) - 2| < 1\}$ for $i = 1, 2$, where $p_2 \in C_2$ is such that $f(p_1) = f(p_2)$. Then the family

$$\mathscr{V} = \{V_{tp_1}\colon t \in T, p_1 \in C_1\} \cup \{\{x\}\colon x \in X \setminus \bigcup_{t \in T} \bigcup_{p_1 \in C_1} V_{tp_1}\}$$

is a point-finite open covering of X which refines $\mathscr{U}$ ($\mathscr{V}$ is point-finite, because each point $x \in G_i$ belongs only to the finite number of the sets V_{tp_1} for $p_1 \in C_1$).

3.5. Let us notice that the space Y defined above provides also the example having properties (a) and (b), simpler then the space X. The proof of this fact follows from (2), (3), (4), (10) and from the following condition, which follows immediately from (11):

(11′) there is no open-and-closed subset of Y containing K_0 and disjoint with K_1

(let us add that the direct proof of (11′) is a little simpler than that of (11)).

4. A scattered space X which is not zerodimensional and a weakly paracompact space X' with $\operatorname{locdim} X' \neq \dim X'$.

4.1. In this section we shall give an example of a space X having the following properties:

(a) $X = E \cup F$, where E is a functionally open discrete subspace of X and the subspace F contains only two non-isolated points a and b,

(b) X is connected between the points a and b.

Let us notice that from (a) it follows that the space X is scattered (see (17)) and zerodimensional at each point $x \neq a, b$.

Next, we shall modify the construction of X in order to obtain the following two examples:

EXAMPLE 4.A. A space X having the properties (a) and (b) and the property

(c) X is separable.

EXAMPLE 4.B. A space X having the properties (a) and (b) and the property

(d) X is weakly paracompact.

Let us add that in Subsection 5.5 we give an example of a space X having the properties (a), (b) and (c) or (d) which is in addition realcompact.

Let us notice that the first example of a scattered space which is not zerodimensional was given by R. C. Solomon [14] (this space is not weakly paracompact).

Taking the space $X' = X \setminus \{a\} \setminus \{b\}$, where X is the space from Example 4.B, we shall obtain the following

EXAMPLE 4.C. A space X' having the following properties:

(a') $X' = E \cup F'$, where E is a functionally open discrete subspace of X' and F' is a discrete subspace of X' (thus, $\operatorname{locdim} X' = 0$),

(b') $\dim X' > 0$,

(d') X' is weakly paracompact.

Let us observe that the spaces described in Examples 3.B and 5.B have also the properties (a'), (b') and (d'), but their construction is more complicated.

Let us notice that the space having the properties (a'), (b') and (d') can not be normal, as $\operatorname{locdim} = \dim$ in the class of normal weakly paracompact spaces (the equality $\operatorname{locdim} = \dim$ for paracompact spaces was proved by Dowker [3] and Nagami [9], the proof for normal weakly paracompact spaces is given in [5]).

4.2. Let us construct a space X satisfying the conditions (a) and (b); our construction is similar to that given in Section 3.

Let S be a set of cardinality $\mathfrak{c}$. Let us fix a point $s_0 \in S$ and let $\{A_t\}_{t \in T}$ be the family of all countable subsets of $S \setminus \{s_0\}$. For each pair (t, p), where $t \in T$ and $p \in (0, 1)$ let us choose an index $w_{tp} \in S \setminus (A_t \cup \{s_0\})$ so that $w_{tp} \neq w_{t'p'}$ for $(t, p) \neq (t', p')$ and let us define a point $x_{tp} \in R^S$ in the following way:

$$x_{tp}(s) = \begin{cases} p & \text{if} \quad s \in A_t, \\ 0 & \text{if} \quad s = s_0, \\ 2 & \text{if} \quad s = w_{tp}, \\ 3 & \text{otherwise}. \end{cases}$$

Let $a, b \in R^S$ be such that $a(s) = 0$ for $s \in S$, $b(s_0) = 0$ and $b(s) = 1$ for $s \in S \setminus \{s_0\}$. Put

$$F = \{x_{tp}\colon\ t \in T, p \in (0, 1)\} \cup \{a\} \cup \{b\}.$$

Let us take an arbitrary dense subset E of R^S such that $E \cap p_{s_0}^{-1}(0) = \emptyset$. Let $Y = E \cup F$ be the subspace of R^S and $X = Y_F$. Of course Y and X are Tychonoff spaces.

First let us notice that

(16) each point x_{tp} is isolated in F.

Indeed, a set

$$U_{tp} = \{x \in Y\colon\ |x(w_{tp}) - 2| < 1\}$$

is an open neighbourhood of x_{tp} in Y (and hence in X) such that $U_{tp} \cap F = \{x_{tp}\}$.

(17) The space X is scattered.

Indeed, let us take an arbitrary nonempty subspace A of X. If $A \cap E \neq \emptyset$ then a point from $A \cap E$ is isolated. If $A \cap E = \emptyset$ and some point x_{tp} belongs to A, then that point is isolated. If neither of the previous cases holds, then $A \subset \{a\} \cup \{b\}$ and a or b is isolated.

Now we shall prove that the subspace F of R^S has the following property:

(18) if $A \subset S$ is countable then the projection $p_A(F)$ is connected between the points $p_A(a)$ and $p_A(b)$.

Since $p_{s_0}(F) = \{0\}$, it suffices to consider the case when $A = A_{t_0}$ for some $t_0 \in T$. Let

$$\begin{aligned} B &= \{p_{A_{t_0}}(x_{t_0 p})\colon\ p \in (0, 1)\} \cup \{p_{A_{t_0}}(a)\} \cup \{p_{A_{t_0}}(b)\} \\ &= \{x \in R^{A_{t_0}}\colon\ x(s) = p \text{ for } s \in A_{t_0}, \text{ where } p \in I\} \subset p_{A_{t_0}}(F). \end{aligned}$$

The set B is homeomorphic to I, thus $p_{A_{t_0}}(F)$ is connected between $p_A(a)$ and $p_A(b)$.

Next we shall prove that

(19) X is connected between the points a and b.

Suppose on the contrary that the points a and b can be separated by open-and-closed subsets of X. Then there exist two open subsets U and V of the space Y such that $a \in U$, $b \in V$, $U \cap V = \emptyset$ and $F \subset U \cup V$. Let U' and V' be open subsets of R^S such that $U = U' \cap Y$ and $V = V' \cap Y$. Because Y is dense in R^S, U' is disjoint with V'. Hence by the theorem of M. Bockstein (see [4], P. 2.7.12) there exists a countable set $A \subset S$ such that $p_A(U') \cap p_A(V') = \emptyset$. Then

$$p_A(U) \cap p_A(F) = p_A(U') \cap p_A(F), \quad p_A(V) \cap p_A(F) = p_A(V') \cap p_A(F)$$

and $p_A(U) \cap p_A(V) \supset p_A(F)$. Thus the sets $p_A(U) \cap p_A(F)$ and $p_A(V) \cap p_A(F)$ are open-and-closed disjoint subsets of $p_A(F)$ containing $p_A(a)$ and $p_A(b)$ respectively, which contradics (18). This finishes the proof of (19).

4.3. In order to obtain Example 4.A it suffices to take in the definition of Y an arbitrary dense subspace E of R^S, disjoint with the set $p_{s_0}^{-1}(0)$, which is, in addition, countable. Then E is a dense countable subset of X.

4.4. We obtain Example 4.B by taking in the definition of Y an arbitrary dense subset E of R^S, disjoint with the set $p_{s_0}^{-1}(0)$, such that every point of E has all but finite coordinates equal to 0. Let us show that X is then weakly paracompact. Let $\mathcal{U}$ be an arbitrary open covering of X. For each $t \in T$ and $p \in (0, 1)$ let us choose an open subset V_{tp} of Y such that $x_{tp} \in V_{tp} \subset U_{tp}$, where $U_{tp} = \{x \in Y: |x(w_{tp})-2| < 1\}$ and $V_{tp} \subset U$ for some $U \in \mathcal{U}$. Take $U_a, U_b \in \mathcal{U}$ such that $a \in U_a$, $b \in U_b$. Then the family

$$\mathcal{V} = \{V_{tp}: t \in T, p \in (0, 1)\} \cup \{U_a\} \cup \{U_b\} \cup \{\{x\}: x \notin \bigcup_{t \in T} \bigcup_{p \in (0,1)} V_{tp}\}$$

is a point-finite open refinement of $\mathcal{U}$.

4.5. Now we shall construct Example 4.C. Let X be the space constructed above having the properties (a), (b) and (d). Let $X' = X \setminus \{a\} \setminus \{b\}$ be the subspace of the space X. Of course X' has the property (a'). Let $u: X \to I$ be a continuous function such that $u(a) = 0$ and $u(b) = 1$. Because the space X is connected between the points a and b, the sets $u^{-1}([0, \frac{1}{3}]) \cap X$ and $u^{-1}([\frac{2}{3}, 1]) \cap X'$ are two functionally closed disjoint subsets of X' which cannot be separated by open-and-closed sets; thus $\dim X' > 0$. By a slight modification of the proof of the property (d) of the space X given in 4.4 one can show that X' has the property (d').

5. A realcompact locally zero-dimensional not N-compact space.

5.1. The aim of this section is to give an example of a space X having the following properties:

(a) X is realcompact,

(b) $X = E \cup F$, where E is a functionally open discrete subspace of X and F is a discrete subspace of X,

(c) X is not N-compact.

Let us notice that the condition (b) implies that X is scattered and $\operatorname{locdim} X = 0$, whereas (a) and (c) imply that $\dim X > 0$ (see [13], p. 478).

Next, we shall slightly strengthen our construction in order to obtain the following two examples:

Example 5.A. A space X having the properties (a)-(c) and the property

(d) X is separable.

Example 5.B. A space X having the properties (a)-(c) and the property

(e) X is weakly paracompact.

Let us notice that a realcompact, not N-compact (and hence not strongly zero-dimensional), weakly paracompact space with $\operatorname{locdim} = 0$ can not be normal (see the final remark of Subsection 4.1).

Let us add that if we want to construct a space having only the properties mentioned in the title, then the construction is simpler (see Subsection 5.4).

The first example of a zerodimensional realcompact (metrizable) not N-compact space was given by P. Nyikos [10]. For other examples of normal spaces with this phenomenon see [11] and [12] (see also [2]).

5.2. Now we shall construct a space X having the properties (a)-(c).

Let S be a set of cardinality $\mathfrak{c}$, s_0 — a fixed point of S and $\{A_t\}_{t \in T}$ — the family of all countable subsets of $S \setminus \{s_0\}$. Put $M = (R^2)^S$. Let $\{C_t\}_{t \in T}$ be a family of disjoint, connected and dense subsets of the square $I^2 \subset R^2$ (it is not hard to verify that such a family exists). Let us choose a bijection $t \to a_t$ of the set T onto the interval $J = \{(x_1, x_2) \in R^2: x_1 = -1, 0 \leqslant x_2 \leqslant 1\}$. For each pair (t, p), where $t \in T$ and $p \in C_t$ let us choose an index $w_{tp} \in S \setminus (A_t \cup \{s_0\})$ so that $w_{tp} \neq w_{t'p'}$ if $(t, p) \neq (t', p')$ and let us define a point $x_{tp} \in M$ as follows:

$$x_{tp}(s) = \begin{cases} p & \text{if} \quad s \in A_t, \\ (0, 0) & \text{if} \quad s = s_0, \\ (0, 2) & \text{if} \quad s = w_{tp}, \\ a_t & \text{otherwise}. \end{cases}$$

Let $F = \{x_{tp}: t \in T, p \in C_t\}$ and E be an arbitrary dense subset of the space M such that $E \cap p_{s_0}^{-1}((0, 0)) = \emptyset$ and for every point $x_0 \in E$ the set $E \setminus \{x_0\}$ is G_δ-closed in M (such a set E exists, because an arbitrary dense and countable subset of $p_{s_0}^{-1}(R^2 \setminus \{(0, 0)\})$ satisfies the required conditions). Let $Y = E \cup F$ be the subspace of M and let $X = Y_F$.

Let us observe that

(20) F is a functionally closed discrete subspace of Y.

Indeed, $F = p_{s_0}^{-1}((0, 0)) \cap Y$ and for every $t \in T$ and $p \in C_t$ the set

$$U_{tp} = p_{w_{tp}}^{-1}(\{(x_1, x_2) \in R^2: \sqrt{x_1^2 + (x_2-2)^2} < 1\}) \cap Y$$

is an open neighbourhood of the point x_{tp} in Y such that $U_{tp} \cap F = \{x_{tp}\}$.

Because the topology of X is finer than the topology of Y, the set F is a discrete functionally closed subspace of X. Thus, since each point of E is isolated in X, the condition (b) is satisfied.

Observe that

$$\operatorname{locdim} X = 0. \tag{21}$$

In fact, for $t \in T$ and $p \in C_t$ the set U_{tp} defined above is a strongly zerodimensional open-and-closed neighbourhood of the point x_{tp} in X, whereas the points of E are isolated.

Let us prove now that

(22) the space Y is hereditarily realcompact.

For the purpose, it suffices to prove that for every $y \in Y$ the set $Y\setminus\{y\}$ is realcompact (see [4], Exercise 3.11.B). Since, by the Mrówka's theorem ([4], P. 3.12.15), every G_δ-closed subset of a realcompact space is realcompact, it suffices to show that the set $Y\setminus\{y\}$ is G_δ-closed in M for every $y \in Y$.

First, we shall show that

(23) the set F is G_δ-closed in M.

Suppose that a point $y \in M$ belongs to the G_δ-closure of F. Then

(24) for every set $A \subset S$ of cardinality $\leqslant \aleph_0$ there exists a point $x \in F$ such that $p_A(x) = p_A(y)$

(because if $|A| \leqslant \aleph_0$ then the set $p_A^{-1}p_A(y)$ is a G_δ-set in M). Clearly, the point y belongs to the closure of the set F, hence $y \in (I^2 \cup J \cup \{(0,2)\})^{S\setminus\{s_0\}} \times \{(0,0)\}$. Suppose that $y(s) \in I^2$ for every $s \in S\setminus\{s_0\}$. Let us fix an arbitrary $s_1 \in S\setminus\{s_0\}$. Then, by (24), $y(s_1) = x(s_1) \in I^2$ for some $x \in F$, hence $y(s_1) = p \in C_t$ for some $t \in T$. We shall show that there is no point $x \in F$ such that $p_{\{s_1, w_{tp}\}}(x) = p_{\{s_1, w_{tp}\}}(y)$. Indeed, if $x = x_{tp}$ then $x(w_{tp}) = (0,2) \neq y(w_{tp}) \in I^2$; if $x = x_{tp'}$ for $p' \neq p$ then $x(s_1) = p' \neq y(s_1)$; and if $x = x_{t'p'}$ for $t' \neq t$ then $x(s_1) \in C_{t'}$, $y(s_1) \in C_t$ and $C_t \cap C_{t'} = \emptyset$, hence $x(s_1) \neq y(s_1)$. Thus we have obtained a contradiction with the assumption that $y \in (I^2)^{S\setminus\{s_0\}} \times \{(0,0)\}$. Hence two cases are possible:

1° There exists $s_1 \in S\setminus\{s_0\}$ such that $y(s_1) = (0,2)$. Then $y(s_1) = x_{tp}(s_1)$, where $t \in T$ and $p \in C_t$ are such that $s_1 = w_{tp}$, and for every $x \in F\setminus\{x_{tp}\}$ we have $x(s_1) \neq y(s_1)$. Hence, by (24), $p_{\{s_1,s\}}(y) = p_{\{s_1,s\}}(x_{tp})$ for every $s \in S$, thus $y = x_{tp} \in F$.

2° There exists $s_1 \in S\setminus\{s_0\}$ such that $y(s_1) = a_t$ for some $t \in T$. Then $p_{A_t}(y) = p_{A_t}(x_{tp}) = p$ for some $p \in C_t$. Since for every point $x_{t'p'} \neq x_{tp}$ we have $p_{A_t}(x_{t'p'}) \neq p_{A_t}(y)$, by (24) we obtain $y = x_{tp} \in F$.

Let us show now that

(25) for every $y \in Y$ the set $Y\setminus\{y\}$ is G_δ-closed in M.

If $y \in E$ then by the assumption on E the set $E\setminus\{y\}$ is G_δ-closed in M. Thus, in view of (23), the set $Y\setminus\{y\}$ is G_δ-closed in M. If $y = x_{tp} \in F$, then the set $\{y\}$ is functionally closed in Y (as $\{x_{tp}\} = p_{s_0}^{-1}((0,0)) \cap p_{w_{tp}}^{-1}((0,2)) \cap Y$, hence the set $F\setminus\{y\}$ (and thus $Y\setminus\{y\}$) is G_δ-closed in M. This shows (25) and thereby finishes the proof of (22).

Because X can be mapped by a continuous one-to-one mapping onto Y, from (22) it follows that

(26) X is realcompact (moreover, X is hereditarily realcompact)

(see [4], Exercise 3.11.B).

It remains to show that

(27) X is not N-compact.

Since, by (21), X is zerodimensional, it suffices to verify that there exists an ultrafilter $\mathcal{U}$ in the family of open-and-closed subsets of X having the countable intersection property such that $\bigcap \mathcal{U} = \emptyset$ (see [13], p. 478).

For each countable set $A \subset S\setminus\{s_0\}$ let us put

$$F_A = \{x_{tp}\colon A_t \supset A,\, p \in C_t\}.$$

We define

$$\mathcal{U} = \{U\colon U \text{ is open-and-closed in } X \text{ and there exists a countable set } A \subset S \text{ such that } U \supset F_A\}.$$

(28) $\mathcal{U}$ is an ultrafilter in the family of open-and-closed subsets of X.

Indeed, $\mathcal{U}$ is a filter: if U and U' are open-and-closed in X, then the conditions $U' \supset U$ and $U \supset F_A$ imply that $U' \supset F_A$ and the conditions $U \supset F_A$ and $U' \supset F_{A'}$ imply that $U \cap U' \supset F_{A \cup A'}$. In order to show that the filter $\mathcal{U}$ is maximal, let us take an arbitrary open-and-closed subset U of X. Then there exist sets V and W open in Y such that

$$U \cap F \subset V \subset U \quad \text{and} \quad (Y\setminus U) \cap F \subset W \subset Y\setminus U.$$

Let V' and W' be open subsets of M such that $V = V' \cap Y$ and $W = W' \cap Y$ because Y is dense in M we have $V' \cap W' = \emptyset$. By the theorem of M. Bockstein (see [4], P. 2.7.12) there exists a countable set $A \subset S$ such that $p_A(V') \cap p_A(W') = \emptyset$. We can assume that $A = A_{t_0 \cup \{s_0\}}$ for some $t_0 \in T$. Then the set

$$\begin{aligned} p_A(F_{A_{t_0}}) &= p_A(\{x_{tp}\colon A_t \supset A_{t_0}, p \in C_t\}) \\ &= \{x \in (R^2)^A\colon x(s_0) = (0,0) \text{ and } x(s) = p \text{ for } s \in A\setminus\{s_0\}, \\ &\qquad \text{where } p \in C_t, A_t \supset A\} \end{aligned}$$

is homeomorphic to the set $\bigcup \{C_t\colon A_t \supset A_{t_0}\}$, which is connected (because the sets C_t are connected and dense in I^2). Since $p_A(V')$ and $p_A(W')$ are disjoint open subsets of $p_A(M)$ such that $p_A(V') \cup p_A(W') \supset p_A(V) \cup p_A(W) \supset p_A(F) \supset p_A(F_{A_{t_0}})$ and the set $p_A(F_{A_{t_0}})$ is connected, we have either $p_A(V') \supset p_A(F_{A_{t_0}})$ or $p_A(W') \supset p_A(F_{A_{t_0}})$. Thus we have $p_A^{-1}p_A(V') \supset F_{A_{t_0}}$ or $p_A^{-1}p_A(W') \supset F_{A_{t_0}}$. Because $V \cup W \supset F_{A_{t_0}}$ we have $p_A^{-1}p_A(V') \cap (V \cup W) = V \supset F_{A_{t_0}}$ or $p_A^{-1}p_A(W') \cap (V \cup W) = W \supset F_{A_{t_0}}$. Thus $V \in \mathcal{U}$ or $W \in \mathcal{U}$. This finishes the proof that $\mathcal{U}$ is an ultrafilter.

(29) The ultrafilter $\mathcal{U}$ has the countable intersection property.

In fact, suppose that $U_i \in \mathcal{U}$ for $i = 1, 2, \ldots$ Then $U_i \supset F_{A_i}$ for some countable $A_i \subset S$ and $\bigcap_{i=1}^{\infty} U_i \supset F_{\bigcup_{i=1}^{\infty} A_i} \neq \emptyset$.

It remains to show that

$$\bigcap \mathscr{U} = \emptyset. \tag{30}$$

If $x \notin F$ then the set $X\setminus\{x\}$ is open-and-closed in X and contains F, thus $X\setminus\{x\} \in \mathscr{U}$ and $x \notin \cap\mathscr{U}$. If $x = x_{tp}$, then there exists a neighbourhood U of the point x in X such that $x \in U \subset \overline{U} \subset X\setminus F$; thus $X\setminus U$ is an open-and-closed subset of X which contains $F_{A_t \cup \{w_{tp}\}}$, hence $X\setminus U \in \mathscr{U}$ and $x \notin \bigcap \mathscr{U}$.

The proof that X has the properties (a)-(c) is completed.

5.3. We obtain Example 5.A by assuming additionally that the set E is countable.

5.4. In order to obtain Example 5.B it suffices to assume that the set E has the following properties: E is dense in M, $E \cap p_{s_0}^{-1}((0,0)) = \emptyset$, for every $x \in E$ the set $E\setminus\{x\}$ is G_δ-closed in M and, in addition, every point of E has all but finite coordinates equal to 0. Then the proof analogous to that given in Subsection 4.4 shows that X is weakly paracompact. Let us show that there exists a set E having all the properties mentioned above. Let $\{B_t\}_{t\in T}$ be the family of all finite subsets of the set $S\setminus\{s_0\}$ and $\{C'_t\}_{t\in T}$ — a family consisting of disjoint and dense subsets of $R^2\setminus\{(0,0)\}$. Put

$$E = \bigcup_{t\in T} \{x \in M\colon\ x(s) \in C'_t \text{ for } s \in B_t \cup \{s_0\} \text{ and } x(s) = (0,0) \text{ for } s \notin B_t \cup \{s_0\}\}.$$

First we shall show that the set E is G_δ-closed in M. For, let $y \in M$ belongs to the G_δ-closure of E. Then $y(s_0) \in C'_{t_0}$ for some $t_0 \in T$. There exists a point $x \in E$ such that $p_{B_{t_0}\cup\{s_0\}}(y) = p_{B_{t_0}\cup\{s_0\}}(x)$. Observe that if x' is a point of E different from x then $p_{B_{t_0}\cup\{s_0\}}(x') \neq p_{B_{t_0}\cup\{s_0\}}(y)$. Indeed, if $x'(s_0) \in C'_{t_0}$, then $x'(s) = x(s) = (0,0)$ for $s \notin B_{t_0\cup\{s_0\}}$ hence $p_{B_{t_0}\cup\{s_0\}}(x') \neq p_{B_{t_0}\cup\{s_0\}}(x) = p_{B_{t_0}\cup\{s_0\}}(y)$; and if $x'(s_0) \notin C'_{t_0}$, then $x'(s_0) \neq y(s_0) \in C'_{t_0}$. Thus $y = x \in E$ and E is G_δ-closed in M. Now, let x_0 be an arbitrary point of E and let $t_0 \in T$ be such that $x(s_0) \in C'_{t_0}$. Then

$$\{x_0\} = (p_{B_{t_0}\cup\{s_0\}})^{-1} p_{B_{t_0}\cup\{s_0\}}(x_0) \cap E$$

and so x_0 is functionally closed (and hence G_δ-open) in E. Thus $E\setminus\{x_0\}$ is G_δ-closed in M. It is easy to verify that the set E has also all the remaining properties mentioned above.

5.5. Remark. There exists a realcompact scattered space X' which is not zero-dimensional. Moreover, X' can be separable or weakly paracompact.

Let $Y' = Y \cup \{a\} \cup \{b\}$, where Y is the space constructed in 5.2 whereas a and b are two points of M such that $a(s) = (0,0)$ for $s \in S$, $b(s_0) = (0,0)$ and $b(s) = (0,1)$ for $s \in S\setminus\{s_0\}$. Let $X' = Y_{F\cup\{a\}\cup\{b\}}$.

Then the space X' is connected between the points a and b. The proof of this property of X' which is analogous to the proof of (19), follows from the fact that for every countable set $A \subset S$ the set $p_A(F \cup \{a\} \cup \{b\})$ is connected between the points $p_A(a)$ and $p_A(b)$. By a slight modification of the proof of (26) one shows that X' is realcompact and it is easy to verify that X' is scattered (compare the proof of (17)). By an appropriate choise of E (see 5.3 and 5.4) we obtain X' separable or weakly paracompact.

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