n'est pas résiduel, comme il ne coupe pas l'ensemble $R\times A$. Démontrons encore que la fonction f n'est monotone dans aucun ensemble ouvert, non vide de l'espace X. Soit $U\subset X$ un ensemble ouvert, non vide de l'espace X. Soit $(x_0, y_0)\in U$ un point tel que $x_0\in C$. Remarquons que la coupe U_{x_0} de l'ensemble U est un ensemble ouvert dans R. La coupe $f_{x_0}(y) = f(x_0, y)$ n'est pas monotone dans l'ensemble ouvert U_{x_0}, il existe donc un nombre $z\in R$ tel que $(f_{x_0})^{-1}(z)$ n'est pas connexe. De plus, comme la fonction f_{x_0} est continue, l'ensemble $(f_{x_0})^{-1}(z)$ est fermé. Soit (α, β) une composante du complémentaire $U_{x_0}-(f_{x_0})^{-1}(z)$. L'ensemble $(f_x)^{-1}(z)$ étant non dense pour tout $x\in C$ et l'ensemble C étant dénombrable, l'ensemble $\bigcup_{x\in C}(f_x)^{-1}(z)$ est de première catégorie. Il en résulte que l'ensemble $((\alpha, \beta)\cap B)-\bigcup_{x\in C}(f_x)^{-1}(z)$ est non vide. Soit

$$y_1\in((\alpha, \beta)\cap B)-\bigcup_{x\in C}(f_x)^{-1}(z).$$

On a donc

$$U\cap f^{-1}(z)\cap\{(x, y)\in X\colon y = y_1\} = \emptyset$$

et par conséquent l'ensemble $U\cap f^{-1}(z)$ n'est pas connexe dans U, ce qui termine la démonstration.

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The closure of the space of homeomorphisms on a manifold. The piecewise linear case

by

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Abstract. Let $\overline{H}(M)$ denote the space of all continuous functions on a compact p.l. manifold M which can be approximated by homeomorphisms and $\overline{\mathrm{PLH}}(M)$ the space of all p.l. mappings which can be approximated by p.l. homeomorphisms. The pair $(\overline{H}(M), \overline{\mathrm{PLH}}(M))$ is studied and it is shown that $\overline{\mathrm{PLH}}(M)$ is an l_2^f-manifold for compact p.l. manifolds M of dim $\neq 4, 5$.

Let M be a compact piecewise linear manifold and $\mathrm{PLH}(M)$ denote the space of all piecewise linear homeomorphisms of M onto itself. We shall study the space, $\overline{\mathrm{PLH}}(M)$, of all piecewise linear mappings which can be approximated arbitrarily closed by elements of $\mathrm{PLH}(M)$. All function spaces on compact manifolds will be assumed to have the supremum metric ϱ; i.e., if X and Y are manifolds with d the metric on Y and f and g are mappings from X into Y, then

$$\varrho(f, g) = \sup_{x\in X}\{d(f(x), g(x))\}.$$

Note: Suppose $f_0, f_1\in\overline{\mathrm{PLH}}(M)$. In this topology a homotopy from f_0 to f_1 is a map $F\colon M\times[0, 1]\to M$ such that $F_0 = f_0$, $F_1 = f_1$ and for each $t\in[0, 1]$, $F_t\in\overline{\mathrm{PLH}}(M)$. In particular, it is not required that F be a p.l. map from $M\times[0, 1]$ into M.

This paper is a sequel to [12] where the author studied $\overline{H}(M)$, the space of all continuous functions on M which can be approximated by homeomorphisms of M onto itself. In many cases $\overline{H}(M)$ has been identified by Siebenmann [18] with the space of cellular maps of M onto itself.

For some years now there has been considerable interest in the question of whether $H(M)$, the space of homeomorphisms of M onto itself, is locally homeomorphic to l_2, the Hilbert space of square summable sequences. See [9] for a summary of what is known about $H(M)$ and the pair $(H(M), \mathrm{PLH}(M))$. In the appendix of this note we make an observation of a new criterion for determining if $H(M^n)$ is an l_2-manifold. The major portion of this note is devoted to proving the following:

THEOREM 1. *Let M^n be a compact p.l. n-manifold, $n\neq 4$; if $n = 5$, $\partial M^n = \emptyset$. Then given an open cover $\mathcal{U}$ of $\overline{H}(M^n)$, there exists a homeomorphism of $\overline{H}(M^n)$ onto itself which is limited by $\mathcal{U}$ and carries* $\mathrm{PLH}(M^n)$ *onto* $\overline{\mathrm{PLH}}(M^n)$.

Let l_2^f be the (dense, incomplete) linear subspace of l_2 consisting of those sequences having only finitely many nonzero entries. A space that is separable, metrizable and locally homeomorphic to l_2^f is called an l_2^f-manifold. Keesling and Wilson [14] have shown, using results of Geoghegan [8], Toruńczyk [19] and Haver [11] that $\mathrm{PLH}(M)$ is an l_2^f-manifold. We therefore have the following corollary to our theorem.

COROLLARY 1. *$\overline{\mathrm{PLH}}(M)$ is an l_2^f-manifold and hence an* ANR.

COROLLARY 2. *Given an open cover $\mathscr{U}$ of $\overline{\mathrm{PLH}}(M)$, there is a map $\varphi_{\mathscr{U}}\colon \overline{\mathrm{PLH}}(M) \to \mathrm{PLH}(M)$ that is limited by $\mathscr{U}$; i.e., piecewise linear cellular maps can be canonically approximated by piecewise linear homeomorphisms.*

We note that the statements in the topological category analogous to Corollaries 1 and 2 are unresolved. The statement corresponding to Corollary 1 is discussed in [12] and Corollary 2 in [8].

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NOTATION. Let $A \subset X$; $X \setminus A$ will denote the complement of A in X; this complement will also be denoted $\tilde{A}$ when there is no possibility of confusion; 1_A will denote the inclusion of A in X.

If M is a manifold, ∂M will denote the boundary of M. π_i is the projection of $\prod_{i=1}^{n} X_i$ onto X_i, $i = 1, \ldots, n$.

Let $\mathscr{U}$ be a collection of open subsets of X and $f\colon X \to X$ a function; f is limited by $\mathscr{U}$ if for each $x \in X$, $f(x) = x$ or there exists $U \in \mathscr{U}$ with $\{x\} \cup \{f(x)\} \subset U$.

We start our proof of the theorem with two lemmas concerning properties of function spaces on manifolds. Throughout this paper we assume that M is a compact p.l. manifold of dimension $n \neq 4$ and if $n = 5$, then $\partial M = \emptyset$. Let $H^*(M)$ be the subset of $H(M)$ consisting of those homeomorphisms which are isotopic to p.l. homeomorphisms. Let $\overline{H^*}(M)$ be the space of all continuous functions on M which can be approximated by elements of $H^*(M)$.

LEMMA 1. a) *$H(M)$ is uniformly locally contractible and hence $\overline{H}(M)$ is* LC^∞;

b) *each of* $\mathrm{PLH}(M)$ *and* $\overline{\mathrm{PLH}}(M)$ *is the countable union of finite dimensional compacta*;

c) $\mathrm{PLH}(M)$ *is an l_2^f-manifold*;

d) $\mathrm{PLH}(M)$ *is dense in $H^*(M)$ and hence $\overline{\mathrm{PLH}}(M)$ is dense in $\overline{H^*}(M)$.*

Proof. a) In Edwards–Kirby [6] and Chernavskii [5] it is shown that $H(M)$ is uniformly locally contractible. It then follows from Eilenberg and Wilder [7] that $\overline{H}(M)$ is LC^∞ (see also [12]). b) Was shown by Geoghegan in [8]. c) Was proved in [14]. Part d) is proved in detail in [9]; for an indication of proof, see Remark 2 at the end of this paper. The dimension restrictions are necessary only for part d).

Consistent with our previous notation, let $H(\overline{H}(M))$ denote the space of all homeomorphisms of $\overline{H}(M)$ onto itself, under the compact open topology. In [12] the author proved that $\overline{H}(M)$ is homogeneous. The following lemma is a parametrized version of that result and can easily be proved by the same methods with the following modifications:

1) In the proof of Lemma 2.5 of [12] we are given positive numbers b and c and $g \in \overline{H}(M)$ with $\varrho(g, 1_M) < b$. Then $h \in H(M)$ is chosen with $\varrho(h, g) < \min(b, c)$ and $\varrho(h, 1_M) < b$. Using Lemma 1a) it is possible to make the choice of h depend continuously on g. To be more precise: let b and c be positive numbers and D be a finite-dimensional compactum. Then if $g\colon D \to \overline{H}(M)$ is a mapping with $\varrho(g(d), 1_M) < b$ there exists a mapping $h\colon D \to H(M)$ with $\varrho(h(d), g(d)) < \min(b, c)$ and $\varrho(h(d), 1_M) < b$ for all $d \in D$.

2) In the proof of Lemma 2.4 of [12] let the map H depend continuously on h (see Lemma 2.2 of [12] and the paragraph preceding its statement).

LEMMA 2 (parametrized homogeneity). *Given $\varepsilon > 0$, there is a $\delta > 0$ such that if D is a finite-dimensional compacta and if $f\colon D \to \overline{H}(M)$ satisfies $\varrho(f(d), 1_M) < \delta$ for all $d \in D$, then there is a map $F\colon D \to H(\overline{H}(M))$ so that $F(d)(f(d)) = 1_M$ and for all $d \in D$, if $g \in \overline{H}(M)$ with $\varrho(g, 1_M) \geqslant \varepsilon$, then $F(d)(g) = g$. If $\varepsilon = \infty$, δ can be taken to be ∞.*

Remark. We will use only the case $\varepsilon = \infty$ in the following.

LEMMA 3. *Let E denote l_2 or l_2^f and Y be an E-stable space (i.e., $Y \approx Y \times E$) with D an arbitrary finite-dimensional compactum in Y. Then*

a) *there is a homeomorphism $\mu\colon Y \to Y \times E$ with $\mu(D) \subset Y \times \{0\}$;*

b) *given compacta $D_0 \subset D$ and $f\colon D \to Y$ there is a sequence $\{f_n\colon D \to Y\}_{n=1}^{\infty}$ such that f_n converges to f and for each n, $f_n | D_0 = f | D_0$, f_n is injective on $D \setminus D_0$ and $f_n(D \setminus D_0) \cap f_n(D_0) = \emptyset$.*

Proof. a) Follows immediately from the special case where $Y = E$ which is well known (cf. [1]); to obtain b), let $\mu\colon Y \to Y \times E$ be a homeomorphism with $\mu(D) \subset Y \times \{0\}$ and let $\varphi\colon D \to \{t = (t_i) \in E \mid t_1 = 1\}$ be an embedding. Then define $g_n\colon D \to Y \times E$ by

$$g_n(x) = \big(\pi_1 \mu f(x), (1/n\varrho(x, D_0))\, \varphi(x)\big).$$

Then for each n, $f_n = \mu^{-1} g_n$ is the desired map.

The following is a special case of a theorem of Toruńczyk that is formulated in a manner convenient for our purposes. For a proof see Theorem 4.2 and Proposition 4.1 of Chapter IV of [3] (see also [20, 21]).

LEMMA 4 (Toruńczyk). *Let X be a complete metric space and $W = \bigcup_{n=1}^{\infty} W_n$ where each W_n is a finite-dimensional compactum in X. Suppose that given a finite-dimensional compactum $A \subset X$, $\varepsilon > 0$ and open $V \supset A$,*

a) *there exists a homeomorphism $F\colon X \to X$ such that $F/V = 1_V$, $\varrho(F, 1_M) < \varepsilon$ and $F(A) \cap A = \emptyset$,*

b) *there is a* $\delta>0$ *such that if* $f\colon A\to X$ *satisfies* $\varrho(f, 1_A)<\delta$, *there exists a homeomorphism* $F\colon X\to X$ *such that* $F|A=f$, $\varrho(F, 1_M)<\varepsilon$ *and* $F|\tilde{V}=1_{\tilde{V}}$, *and*

c) *given an integer* m *there exists an embedding* $f\colon A\to W$ *such that*

$$\varrho(f, 1_A)<\varepsilon\,, \quad f|A\cap W_m = 1_{A\cap W_m}\,.$$

Then if $\{A_n\}_{n=1}^{\infty}$ *is any collection of finite-dimensional compacta, and* $\mathcal{U}$ *is any open cover of* X, *there is a homeomorphism* $F\colon X\to X$ *such that* $F(W)=W\cup\bigcup_{n=1}^{\infty}A_n$ *and* F *is limited by* $\mathcal{U}$.

The following lemma makes use of a technique for extending homeomorphisms due originally to Klee.

Lemma 5. *If* $D\subset\overline{H}(M)$ *is a finite-dimensional compactum, then there exists a homeomorphism* $g\colon \overline{H}(M)\to\overline{H}(M)\times l_2$ *with* $g(D)\subset 1_M\times l_2$.

Proof. In [10] it was shown that $\overline{H}(M)$ is l_2-stable. Therefore by Lemma 3a) there is a homeomorphism $\mu\colon \overline{H}(M)\to\overline{H}(M)\times l_2$ with $\mu(D)\subset\overline{H}(M)\times\{0\}$. Let $\alpha\colon \pi_1\mu(D)\to l_2$ be an embedding with $\alpha(\pi_1\mu(D))$ contained in a subset B of l_2 that is homeomorphic to a finite-dimensional cube. We shall construct homeomorphisms g_1 and g_2 of $\overline{H}(M)\times l_2$ such that $g_2g_1(\mu(D))\subset 1_M\times l_2$. Then $g=g_2g_1\mu$ will be the required homeomorphism.

Since l_2 is an AR we can choose a mapping $\beta\colon \overline{H}(M)\to l_2$ extending $\alpha\colon \pi_1\mu(D)\to l_2$. Then define the homeomorphism $g_1\colon \overline{H}(M)\times l_2\to\overline{H}(M)\times l_2$ by $g_1(x,y)=(x, y+\beta(x))$.

Since $\alpha(\pi_1\mu(D))$ is contained in a finite-dimensional cube B and $\overline{H}(M)$ is uniformly LC^∞ there exists a map $f\colon B\to\overline{H}(M)$ extending α^{-1}. Then by Lemma 2, there is a map $F\colon B\to H(\overline{H}(M))$ with $F(b)(f(b))=1_M$ for all $b\in B$. Let r be any retraction of l_2 onto B and define the homeomorphism $g_2\colon \overline{H}(M)\times l_2\to\overline{H}(M)\times l_2$ by $g_2(x,y)=(F(r(y))(x), y)$. We check that for $d\in D$, $g_2g_1\mu(d)\in 1_M\times l_2$:

$$\begin{aligned} g_2g_1\mu(d) &= g_2g_1(\pi_1\mu(d), 0) = g_2(\pi_1\mu(d), \alpha\pi_1\mu(d))\\ &= (F(\alpha\pi_1\mu(d))(\pi_1\mu(d)), \alpha\pi_1\mu(d))\\ &= (F(\alpha\pi_1\mu(d))(f\alpha\pi_1\mu(d)), \alpha\pi_1\mu(d)) = (1_M, \alpha\pi_1\mu(d))\in 1_M\times l_2\,. \end{aligned}$$

In the following let $\mathrm{st}^n(\mathcal{U})$ denote the nth star of $\mathcal{U}$ (cf. [2]).

Lemma 6. *Let* A *be a finite-dimensional compactum and let* $h=(h_t)\colon A\times I\to\overline{H}(M)$ *be a homotopy such that each of* h_0 *and* h_1 *is an embedding and* h *is limited by a given open (in* $\overline{H}(M)$*) cover* $\mathcal{U}$ *of* $h(A\times I)$. *Then*

a) *there is a homeomorphism* $f\colon \overline{H}(M)\to\overline{H}(M)$ *which is limited by* $\mathrm{st}^8(\mathcal{U})$ *and satisfies* $fh_0=h_1$, *and*

b) *if* $h|A\times(0,1)$ *is 1-1, then the* f *above can be chosen to be limited by* $\mathrm{st}^4(\mathcal{U})$.

Proof. We shall first prove b). If $h|A\times(0,1)$ is 1-1, then the dimension of $h(A\times I)=h_0(A)\cup h_1(A)\cup\bigcup_{n\in N}h(A\times[1/n, 1-1/n])$ is bounded by $1+\dim A$ and hence by Lemma 5 there is a homeomorphism $g\colon \overline{H}(M)\to\overline{H}(M)\times l_2$ with $gh(A\times I)\subset 1_M\times l_2$. Passing to a refinement if necessary, we may assume that $g(\mathcal{U})$ is of the form $\{N_\varepsilon(1_M)\times U \mid U\in\mathcal{U}'\}$ where $\mathcal{U}'$ is an open cover of $\pi_2gh(A\times I)$ in l_2 and $N_\varepsilon(1_M)$ is a ball in $\overline{H}(M)$ of a positive radius ε centered at 1_M.

By Theorem 4.2 of [2] there is an isotopy $(f_t)\colon l_2\times I\to l_2$ which is limited by $\mathrm{st}^4(\mathcal{U}')$ and satisfies $f_0\pi_2gh_0=\pi_2gh_1$ and $f_t=1_{l_2}$ for $t\geqslant\varepsilon$. We define $f'\colon \overline{H}(M)\times l_2\to\overline{H}(M)\times l_2$ by $f'(x,y)=(x, f_{\varrho(x,1_M)}(y))$. Then $f=g^{-1}fg\colon \overline{H}(M)\to\overline{H}(M)$ is the desired homeomorphism.

Proof of a). Since $\overline{H}(M)$ is l_2-stable, it follows from Lemma 3b) that there exists a homotopy $h'\colon A\times I\to\overline{H}(M)$ with $h'=h$ on $A\times\{0,1\}$, $h'/A\times(0,1)$ is injective and h' is limited by $\mathrm{st}(\mathcal{U})$. Thus part a) follows from b) applied to h' and $\mathrm{st}(\mathcal{U})$.

Lemma 7. *Let* $A\subset\overline{H^*}(M)$ *be a finite-dimensional compactum,* $A_0\subset A$ *be closed with* $A_0\subset\mathrm{PLH}(M)$ *and* $\varepsilon>0$ *be given. Then there exists an embedding* $f\colon A\to\mathrm{PLH}(M)$ *with* $\varrho(f, 1_A)<\varepsilon$ *and* $f/A_0=1_{A_0}$.

Proof. By Lemma 1 $\mathrm{PLH}(M)$ is a dense uniformly locally contractible subspace of $\overline{H^*}(M)$ and hence [7] there is a map $f'\colon A\to\mathrm{PLH}(M)$ with $\varrho(f', 1_A)<\frac{1}{2}\varepsilon$ and $f'/A_0=1_{A_0}$. Since $\mathrm{PLH}(M)$ is an l_2^f-manifold, by Lemma 3b) there is an embedding $f\colon A\to Y$ with $\varrho(f', f)<\frac{1}{2}\varepsilon$ and $f/A_0=1_{A_0}$. Then f has the required properties.

The proof of the main theorem now follows easily.

Proof of Theorem 1. Since $\overline{H^*}(M)$ is a separable metric space, we can apply Lemma 4, letting $W=\mathrm{PLH}(M)$. By Lemma 1b), $\mathrm{PLH}(M)$ is the union of finite-dimensional compacta. Condition a) is satisfied trivially since $\overline{H^*}(M)$ is l_2-stable ([10]). Lemma 6 implies that condition b) is satisfied since two sufficiently close maps of a finite-dimensional compacta into a locally contractible space are homotopic. Finally, Lemma 7 shows that condition c) is satisfied. Since $\overline{\mathrm{PLH}}(M)$ contains $\mathrm{PLH}(M)$ and is the countable union of finite-dimensional compacta, given a cover $\mathcal{U}'=\{U\cap\overline{H^*}(M) \mid U\in\mathcal{U}\}$ of $\overline{H^*}(M)$ there is a homeomorphism $F'\colon \overline{H^*}(M)\to\overline{H^*}(M)$ limited by $\mathcal{U}'$ and taking $\mathrm{PLH}(M)$ onto $\overline{\mathrm{PLH}}(M)$. Finally, extend F' to a homeomorphism $F\colon \overline{H}(M)\to\overline{H}(M)$ by $F/\overline{H}(M)\setminus H^*(M)=1_{\overline{H}(M)\setminus\overline{H^*}(M)}$.

The proofs of Lemmas 2, 5 and 6 follow exactly the same if $\overline{H}(M)$ is replaced by $H(M)$. In [9] it was shown that $\mathrm{PLH}(M)$ has the "finite-dimensional compact absorption property" in $H^*(M)$ and hence that $(H^*(M), \mathrm{PLH}(M))$ is an (l_2, l_2^f)-manifold pair if and only if $H(M)$ is an l_2-manifold. The following corollary is a strengthening of the main result of [9]. It follows immediately from the suggested modifications of Lemma 6 and Lemma 7.

Corollary 3. *Let* $(A, A_0)\subset(H^*(M), \mathrm{PLH}(M))$ *be a pair of finite-dimensional compacta. Given* $\varepsilon>0$, *there exists a homeomorphism* $\varphi\colon H^*(M)\to H^*(M)$ *with* $\varphi(A)\subset\mathrm{PLH}(M)$, $\varphi/A_0=1_{A_0}$ *and* $\varrho(\varphi(f), f)<\varepsilon$ *for all* $f\in H^*(M)$.

Appendix. One reason for studying $\overline{H}(M)$ and the pair $(\overline{H}(M), \overline{\mathrm{PLH}}(M))$ is in order to gain insight into the question of whether $H(M)$ is an ANR (and hence by results of Toruńczyk and Geoghegan an l_2-manifold (cf. [19])). In particular, it is easy to see that if $\overline{H}(M)$ is an ANR, then so is $H(M)$ (cf. [12]). In this section we observe that for a given integer n, it suffices to study the most simple case: $H_\partial(B^n)$. (Here $H_\partial(B^n) = \{h \in H(B^n) | \; h|\partial B^n = 1_{\partial B^n}\}$ and $\mathrm{PLH}_\partial(B^n) = H_\partial(B^n) \cap \mathrm{PLH}(B^n)$. Also $N_\delta(1_M) = \{h \in H(M) | \; \varrho(h, 1_M) < \delta\}$.) Our proof makes use of the following fact which appears in the proof of Corollary 1.3 on p. 79 of [6].

LEMMA 8 (Edwards–Kirby). *Let $\{B_1, \ldots, B_p\}$ be an open cover of M^n with $\overline{B}_i$ a closed n-ball for each i. Then there exists a $\delta > 0$ and a map*

$$\varphi\colon N_\delta(1_M) \to H_\partial(B_1) \times \ldots \times H_\partial(B_p)\,.$$

such that for each homeomorphism $h \in N_\delta(1_M)$, $h = [\pi_p(\varphi(h))]' \circ \ldots \circ [\pi_1(\varphi(h))]'$, where for each i, $[\pi_i(\varphi(h))]'\colon M \to M$ is the homeomorphism defined by

$$[\pi_i(\varphi(h))]'(x) = \begin{cases} \pi_i(\varphi(h))(x) & \text{for} \quad x \in B_i\,, \\ x & \text{for} \quad x \notin B_i\,. \end{cases}$$

THEOREM 2. *Let n be a fixed positive integer. If $H_\partial(B^n)$ is an* ANR, *then $H(M^n)$ is an* ANR *for any compact n-manifold, M^n. Hence, if $\overline{H_\partial}(B^n)$ is an* ANR, *then $H(M^n)$ is an* ANR *for any compact n-manifold, M^n.*

Proof. Let $\{B_1, \ldots, B_p\}$ be an open cover of M^n with $\overline{B}_i$ a closed n-ball for each i. Then let $N_\delta(1_M) \subset H(M)$ and $\varphi\colon N_\delta(1_M) \to M_\partial(B_1) \times \ldots \times H_\partial(B_p)$ be as in Lemma 8. Define $\psi\colon H_\partial(B_1) \times \ldots \times H_\partial(B_p) \to H(M)$ by $\psi(f_1, \ldots, f_p) = f_p' \circ \ldots \circ f_1'$, where

$$f_i'(x) = \begin{cases} f_i(x) & \text{for} \quad x \in B_i\,, \\ x & \text{for} \quad x \notin B_i\,. \end{cases}$$

Then $\psi/\psi^{-1}(N_\delta(1_M))\colon \psi^{-1}(N_\delta(1_M)) \to N_\delta(1_M)$ is an r-map; i.e., there exists a map $\varphi\colon N_\delta(1_M) \to \psi^{-1}(N_\delta(1_M))$ such that $(\psi/\psi^{-1}(N_\delta(1_M))) \circ \varphi\colon N_\delta(1_M) \to N_\delta(1_M)$ is equal to $1_{N_\delta(1_M)}$. But $\psi^{-1}(N_\delta(1_M))$ is an open subset of $H_\partial(B) \times \ldots \times H_\partial(B_p)$ and hence, by assumption, is an ANR. Therefore, being the r-image of an ANR [4], $N_\delta(1_M) \subset H(M^n)$ is an ANR. But then since $H(M^n)$ is a topological group, each point has an open ANR neighborhood and hence $H(M^n)$ is an ANR.

Remarks. 1) In [16] Mason's theorem [17] that $H_\partial(B^2)$ is an ANR was used to prove that $H(M^2)$ is an ANR for every compact 2-manifold, M^2. Theorem 2 thus provides an alternate path to this result.

2) In [15] it is shown that $\mathrm{PLH}_\partial(B^n)$ is dense in $H_\partial(B^n)$ for $n \neq 4$. The reader can easily see that this fact, combined with Lemma 8 shows that $\mathrm{PLH}(M^n)$ is dense in $H(M^n)$ with the proper dimensional restrictions (see [9] for details). Thus the methods of this paper provide an alternate proof of the results of [9] (see Corollary 3).

3) Theorem 2 suggests many possible methods for showing that $H(M^n)$ is an ANR. For example, since $\overline{\mathrm{PLH}_\partial}(B^n)$ is an ANR, to show that for any n-manifold M^n, $n \neq 4$, $H(M^n)$ is an ANR it suffices to show that for a given open cover $\mathcal{U}$ of $\overline{H_\partial}(B^n)$ there exists a map $\varphi\colon \overline{H_\partial}(B^n) \to \overline{\mathrm{PLH}_\partial}(B^n)$ limited by $\mathcal{U}$.

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