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First countable and countable spaces all compactifications of which contain βN

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Abstract. We construct the following examples.

EXAMPLE 1. A first countable Lindelöf (even cosmic) space Δ all compactifications of which contain βN.

EXAMPLE 2. A countable space Σ with one non-isolated point all compactifications of which contain βN.

Since βN has cardinality $2^{\mathfrak{c}}$, uncountable tightness and is neither first countable nor scattered, the above examples in particular yield:

(1) A first countable Lindelöf space with no first countable compactification.

(2) A countable space all compactifications of which have cardinality $2^{\mathfrak{c}}$ and uncountable tightness.

(3) A scattered space with no scattered compactification.

1. Introduction. Throughout this paper all spaces are assumed to be regular, a cardinal is an (von Neumann) ordinal, $\mathrm{cf}(\varkappa)$ is the cofinality of $\varkappa$, and $\mathfrak{c}$ is 2^{ω}. For undefined terms we refer to [E].

In this paper we construct the following two examples.

EXAMPLE 1.1. A first countable Lindelöf (even cosmic) space Δ all compactifications of which contain a homeomorph of βN.

EXAMPLE 1.2. A countable space Σ with one non-isolated point all compactifications of which contain a homeomorph of βN.

Recall that a space X is *cosmic* [Mi] if it has a countable network, i.e. a countable family $\mathscr{A}$ of subsets such that for each open $U\subset X$ and each $x\in U$ there is an $A\in\mathscr{A}$ with $x\in A\subset U$. Every cosmic space is hereditarily Lindelöf and hereditarily separable. Also recall that the *tightness* $\tau(X)$ of a space X, [AP], is the smallest cardinal $\varkappa$ such that, whenever $A\subset X$ and $x\in\overline{A}$, there exists a $B\subset A$ such that $x\in\overline{B}$ and $|B|\leqslant\varkappa$. It is known that βN has cardinality $2^{\mathfrak{c}}$ [E, Theorem 3.6.12] and $\tau(\beta N)=\mathfrak{c}$

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(since we do not know a convenient reference to the latter fact we record it as Proposition 2.6). A space is *scattered* if it contains no dense-in-itself subspace, so Σ is scattered since it has only one non-isolated point, and no compactification of Σ is scattered since $\beta N \setminus N$ is dense-in-itself. Finally, recall that the cardinality of a separable space is not greater than $2^{\mathfrak{c}}$ [E, Theorem 1.5.3].

It follows from the above discussion that Examples 1.1 and 1.2 also yield the following examples.

EXAMPLE 1.3. A first countable Lindelöf space with no first countable compactification.

EXAMPLE 1.4. A countable space all compactifications of which have cardinality $2^{\mathfrak{c}}$ and tightness equal to $\mathfrak{c}$.

EXAMPLE 1.5. A scattered space with no scattered compactification.

These examples answer questions of V. I. Ponomarev, A. V. Arhangel'skiĭ and Z. Semadeni respectively. Earlier V. M. Ul'janov, [U], had constructed a first countable Lindelöf space with no first countable compactification, B. A. Efimov, [Ef_2], had constructed a countable space all compactifications of which have cardinality $2^{\mathfrak{c}}$ and subsequently V. I. Malyhin, [M_2], had constructed a countable space all compactifications of which have cardinality $2^{\mathfrak{c}}$ and uncountable tightness, and P. J. Nyikos, [N], had constructed a scattered space with no scattered compactification. Our examples are much simpler, have better properties and are constructed in a unified way: Δ is the union of the closed unit interval and a countable set of isolated points and Σ is the quotient space Δ/I. Consequently Σ is a Fréchet space, being the closed image of a Fréchet space, [E, Exercise 2.4.G].

It is worth noting that the verification of the most interesting statements about all compactifications of Δ and Σ is very simple and does not require the fact that βN can be embedded in every compactification of either space: in Remark 2.3 we show that if K is any compactification of either Δ or Σ then K is not first countable, in fact $\tau(K) = \mathfrak{c}$, that $|K| > \mathfrak{c}$ and that K is not scattered.

Examples 1.1 and 1.2 are constructed in Section 2, the technique we use is the same as the technique we used in [vDP] to construct a (consistent) example of a first countable space of cardinality $\mathfrak{c}$ which cannot be embedded in any separable first countable space (such an example cannot exist under CH). In Section 3 we discuss a problem related to the construction of our examples.

2. Construction of the examples. Let I be the unit interval, let N be the set of positive integers and let Q be the set of rational numbers in I. Enumerate the family of all subsets of N as $\{A_s: s \in I\}$, and for each $s \in I$ and $m \in N$ choose a $q_s(m) \in Q$ such that $0 < |s - q_s(m)| < 1/m$, and put $Q_s = \{q_s(m): m \in N\}$.

We topologize the set $\Delta = I \times \{0\} \cup Q \times N$ as follows. Points of $Q \times N$ are isolated, and basic neighborhoods of a point $(s, 0) \in I \times \{0\}$ have the form

$$B_m(s) = \{(x, y) \in \Delta: |s - x| < 1/m\} \setminus (Q_s \times A_s \cup \{s\} \times N) \quad \text{for } m \in N.$$

One easily checks that this definition is correct, and that Δ is regular. Also, the subspace topology on $I \times \{0\}$ coincides with the usual topology on $I \times \{0\}$. Since $Q \times N$ is countable it follows that Δ is cosmic.

Let Σ be the quotient space obtained from Δ by identifying $I \times \{0\}$ to a point. It is clear that Σ is a T_1-space with only one non-isolated point, hence Σ is regular. Moreover, Σ is a Fréchet space since the identification map is closed (even is perfect), [E, Excercise 2.4.G].

Let K be an arbitrary compactification of Δ or Σ.

For each $q \in Q$ the subspace $c_q N = \overline{\{q\} \times N}$ of K will be considered as a compactification of N (i.e. we identify $\langle q, n \rangle$ and n for each $n \in N$). In order to show that K contains a homeomorph of βN is suffices to show that some $c_q N$ contains a homeomorph of βN.

DEFINITION 2.1. Let cN be a compactification of N, and let A be a subset of N. We say that *cN separates A and $N \setminus A$ if $\overline{A} \cap \overline{N \setminus A} = \emptyset$.*

LEMMA 2.2. *For each $A \subset N$ there is a $q \in Q$ such that $c_q N$ separates A and $N \setminus A$.*

Proof. Assume that K is a compactification of Δ. If K is a compactification of Σ the proof is analogous.

Choose an $s \in I$ such that $A = A_s$ and an open $U \subset K$ such that $U \cap \Delta = B_1(s)$. There are an open $V \subset K$ and an $m \in N$ such that

$$B_m(s) \subset V \subset \overline{V} \subset U.$$

If $q = q_s(m)$ then $\overline{\{q\} \times A} \subset K \setminus U$ and $\{q\} \times (N \setminus A) \subset B_m(s) \subset V$.

Hence $\overline{\{q\} \times A} \cap \overline{\{q\} \times (N \setminus A)} = \emptyset$ and therefore, in virtue of our identification, $c_q N$ separates A and $N \setminus A$. ■

Remark 2.3. At this stage we can show already that one of the $c_q N$'s, and hence K, must be quite "large". Let cN be the supremum of the family $\{c_q N: q \in Q\}$ of compactifications of N, [E, Theorem 3.5.9]. From Lemma 2.2 and the well-known characterization of the Čech–Stone compactification it follows that $cN = \beta N$. Now cN can be embedded into $\prod_{q \in Q} c_q N \subset K^{\omega}$, hence K^{ω} contains a homeomorph of βN. But βN is not first countable and $|\beta N| > \mathfrak{c}$, therefore K is not first countable either, and $|K| > \mathfrak{c}$ (1).

Malyhin [M_1] proved that a countable product of compact spaces with countable tightness has countable tightness, see also [E, Problem 3.12.8], so K does not have countable tightness. It is proved in ([RNT], Theorem 2) that for a compact scattered space X we have $|X| \leqslant w(X)$ the weight of X. But K is separable, hence $w(K) \leqslant \mathfrak{c}$ [E, Theorem 1.5.6] and therefore K is not scattered since $|K| > \mathfrak{c}$. ■

We now prove that one of the $c_q N$'s contains a homeomorph of βN. This

(1) It seems that we cannot prove that $|K| = 2^{\mathfrak{c}}$ in a similar way. For it is consistent with ZFC that $\mathfrak{c} = \omega_1$ and (*) $2^{\varkappa} = \omega_{\omega+1}$ for $\mathfrak{c} \leqslant \varkappa \leqslant \omega_{\omega}$. Since $\varkappa^{\mathrm{cf}(\varkappa)} > \varkappa$ for each cardinal $\varkappa$, (*) implies $(\omega_{\omega})^{\omega} = \omega_{\omega+1}$, so $|K| = \omega_{\omega} < 2^{\mathfrak{c}}$ would seem possible.

requires two more Lemmas. Lemma 2.5 is known (see e.g. [Ef$_1$]), however, for the sake of completeness, we include its short proof.

Let D be the two-point discrete space $\{0, 1\}$.

LEMMA 2.4. *There exists a $q \in Q$ such that c_qN can be mapped continuously onto $D^{\mathfrak{c}}$.*

Proof. Let $D^{\mathfrak{c}} = \prod_{\alpha\in\mathfrak{c}} D_\alpha$, where $D_\alpha = D$, for $\alpha<\mathfrak{c}$, and let $\pi_\alpha\colon D^{\mathfrak{c}}\to D_\alpha$ be the projection. Since $D^{\mathfrak{c}}$ is separable, there is a mapping $f\colon N\to D^{\mathfrak{c}}$ such that $f(N)$ is dense in $D^{\mathfrak{c}}$. For each $\alpha\in\mathfrak{c}$ let $B_\alpha = (\pi_\alpha\circ f)^{-1}(0)$. Since the cofinality of $\mathfrak{c}$ is greater than ω, it follows from Lemma 2.2 that there is a $q\in Q$ such that

$$S = \{\alpha\in\mathfrak{c}\colon\ c_qN \text{ separates } B_\alpha \text{ and } N\setminus B_\alpha\}$$

has cardinality $\mathfrak{c}$. Let $\pi\colon D^{\mathfrak{c}}\to\prod_{\alpha\in S} D_\alpha$ denote the projection.

We can define a map $g\colon c_qN\to\prod_{\alpha\in S} D_\alpha$ by

$$g(x)_\alpha = \begin{cases} 0 & \text{if}\quad x\in\overline{B}_\alpha \\ 1 & \text{if}\quad x\in\overline{N\setminus B_\alpha}\end{cases}$$

for $\alpha\in S$. It is clear that g is continuous, and also that $g(n) = \pi(f(n))$ for each $n\in N$. The latter fact implies that $g(N)$ is dense in $\prod_{\alpha\in S} D_\alpha$. Since c_qN is compact, g maps c_qN onto $\prod_{\alpha\in S} D_\alpha$.

Finally, note that $D^{\mathfrak{c}}$ and $\prod_{\alpha\in S} D_\alpha$ are homeomorphic. ■

LEMMA 2.5. *If a compact space X can be mapped onto $D^{\mathfrak{c}}$, then it contains a homeomorph of βN.*

Proof. Let $f\colon X\to D^{\mathfrak{c}}$ be a be a continuous surjection. We may assume that βN is a subspace of $D^{\mathfrak{c}}$. Choose an $x_n\in X$ such that $f(x_n) = n$ for each n. Then $\{x_n\colon n\in N\}$ is a discrete subspace of X, so we may assume that $x_n = n$ for each n. Let $bN = \mathrm{Cl}_X N$, and let $g = f|bN$. Then g maps bN onto βN, and is the identity on N. Consequently, $bN = \beta N$, [E, Theorem 3.5.7]. ■

It follows from Lemmas 2.4 and 2.5 that there exists a $q\in Q$ such that $\beta N\subset c_qN\subset K$. This completes the verification of the properties of the spaces Δ and Σ. ■

For the convenience of the reader we include below a short proof of the known fact that $\tau(\beta N) = \mathfrak{c}$.

PROPOSITION 2.6 $\tau(\beta N) = \mathfrak{c}$.

Proof. $\tau(\beta N)\leqslant\mathfrak{c}$ since $w(\beta N) = \mathfrak{c}$. Also, $\tau(\beta N)\geqslant\tau(D^{\mathfrak{c}})$ since βN can be mapped continuously onto $D^{\mathfrak{c}}$, [E, Theorem 3.6.11]. But $D^{\mathfrak{c}}$ contains a homeomorph of the ordinal space $\varkappa+1$, for each regular cardinal $\varkappa$ with $\omega\leqslant\varkappa\leqslant\mathfrak{c}$, and clearly $\tau(\varkappa+1) = \varkappa$ for each regular $\varkappa$. Therefore $\tau(\beta N)\geqslant\tau(D^{\mathfrak{c}})\geqslant\mathfrak{c}$. ■

If follows from Proposition 2.6 that $\tau(K) = \mathfrak{c}$.

3. Final remarks. In Section 2 we showed that if $\mathscr{C}$ is a countable collection of compactifications of N such that for each $A\subset N$ there is an $cN\in\mathscr{C}$ which separates A and $N\setminus A$, then

(1) $\sup\mathscr{C} = \beta N$, and

(2) some $cN\in\mathscr{C}$ contains a homeomorph of βN.

V. I. Malyhin proved (unpublished) that if $\prod_{n\in\omega} X_n$ contains a homeomorph of βN then some X_n contains a homeomorph of βN, from which it follows that (2) is in fact a consequence of (1). Below we sketch the proof of a slight generalization of Malyhin's result.

THEOREM (Malyhin). *If $\prod_{\alpha\in\varkappa} X_\alpha$ contains a homeomorph of βN and $\varkappa<\mathrm{cf}(\mathfrak{c})$ then some X_α contains a homeomorph of βN.*

Sketch of proof. It is easy to prove this by induction for $k<\omega$, using the fact that every infinite closed subspace of βN contains a homeomorph of βN [E, Theorem 3.6.14].

Next consider an arbitrary $\varkappa<\mathrm{cf}(\mathfrak{c})$. Assume that $\beta N\subset\prod_{\alpha\in\varkappa} X_\alpha$ and let $f\colon \beta N\to D^{\mathfrak{c}} = \prod_{\alpha\in\mathfrak{c}} D_\alpha$ be a continuos surjection. For $\alpha<\mathfrak{c}$ let $\pi_\alpha\colon D^{\mathfrak{c}}\to D_\alpha$ be the projection. For each $\alpha\in\mathfrak{c}$ the sets $(\pi_\alpha\circ f)^{-1}(0)$ and $(\pi_\alpha\circ f)^{-1}(1)$ are disjoint compact subsets of the Hausdorff space $\prod_{\alpha\in\varkappa} X_\alpha$. Since $\varkappa<\mathrm{cf}(\mathfrak{c})$ it easily follows that there is an $S\subset\mathfrak{c}$ with $|S| = \mathfrak{c}$ and a finite $F\subset\varkappa$ such that for each $\alpha\in S$ there are disjoint open U_0, U_1 in $\prod_{\alpha\in F} X_\alpha$ such that

$$(\pi_\alpha\circ f)^{-1}(i)\subset U_i\times\prod_{\alpha\in\varkappa\setminus F} X_\alpha .$$

Let $p\colon \prod_{\alpha\in\varkappa} X_\alpha\to\prod_{\alpha\in F} X_\alpha$ and $p\colon \prod_{\alpha\in\mathfrak{c}} D_\alpha\to\prod_{\alpha\in S} D_\alpha$ be the projections. Then one can define a continuous surjection $g\colon p(\beta N)\to\prod_{\alpha\in S} D_\alpha$ by requiring $g(x)$, for $x\in p(\beta N)$, to be the unique element of $\pi[f(p^{-1}(x)\cap\beta N)]$. Hence $\prod_{\alpha\in F} X_\alpha$ contains a homeomorph of βN by Lemma 2.5. But F is finite, so some X_α, $\alpha\in F$, contains a homeomorph of βN, as noted at the beginning of this sketch. ■

COROLLARY. *Let $\{c_\alpha N\colon \alpha\in\varkappa\}$ be a family of compactifications of N. If $\varkappa<\mathrm{cf}(\mathfrak{c})$ and $\sup\{c_\alpha N\colon \alpha\in\varkappa\} = \beta N$, then some $c_\alpha N$ contains a homeomorph of βN.*

Proof. Clearly $\beta N = \sup\{c_\alpha N\colon \alpha\in\varkappa\}$ can be embedded into $\prod_{\alpha\in\varkappa} c_\alpha N$ [E, Theorem 3.5.9] and hence some $c_\alpha N$ contains βN. ■

Remark. The condition $\varkappa<\mathrm{cf}(\mathfrak{c})$ is essential in the theorem. Indeed, otherwise one could write $D^{\mathfrak{c}}$ as $\prod_{\alpha\in\mathrm{cf}(\mathfrak{c})}\prod_{\beta\in A(\alpha)} D_\beta$ where $|A(\alpha)|<\mathfrak{c}$ and $\sum_{\alpha\in\mathrm{cf}(\mathfrak{c})}|A(\alpha)| = \mathfrak{c}$. Then no $\prod_{\beta\in A(\alpha)} D_\beta$ contains a homeomorph of βN since it has weight $|A(\alpha)|<\mathfrak{c}$, but of course $D^{\mathfrak{c}}$ contains a homeomorph of βN.

A similar argument shows that the condition $\varkappa<\mathrm{cf}(\mathfrak{c})$ is essential in the corollary, since βN is the supremum of all two-point compactifications of N.

Added in proof. The result of V. I. Malyhin referred to in the second paragraph of Section 3 will appear in his paper "βN is prime", Bull. Polon. Acad. Sci., in print.

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