holds. We can assume that α_n is a refinement both of $\varphi(\beta_n)$ and of $\psi(\beta_n)$, so that one gets

$$\varphi_n(\pi^{\alpha_n}_{\varphi(\beta_n)})_* = \psi_n(\pi^{\alpha_n}_{\psi(\beta_n)})_* .$$

Then one concludes by means of Lemma 3.1 that $f_* = g_*$.

Now Theorem 1 follows immediately from Propositions 3.1, 3.2 and 3.3. As regards Theorem 2, it follows from the same propositions, accordingly reformulated for double-uniform shape maps. Their proof remains essentially the same — it suffices to remark that standard similar ε-extension $\tilde{\beta}$, constructed in §2, of a uniform covering β of a subspace always covers uniform neighbourhood of that subspace.

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Paracompact box products in forcing extensions

by

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Abstract. In an iterated ccc extension with length $\varkappa$, where $\varkappa$ has uncountable cofinality, $\square_{i<\omega} X_i$ is paracompact if each X_i is compact metrizable; if in addition $\varkappa$ is regular and no bigger than the cardinality of the continuum in the ground model, then $\square_{i<\omega} X_i$ is paracompact if each X_i is compact first countable.

§ 0. Introduction. The question of when box products are normal is an old one (see, e.g., [R_1]). Van Douwen and Kunen each showed that the box product of countably many spaces need not be normal if the spaces are not compact or are of large character. Known positive results are all consistency results and proceed by proving paracompactness. Thus attention has focused between the parameters of: is there an absolute proof that at least $\square^{\omega}(\omega+1)$ is paracompact? Is it consistent that some $\square_{i<\omega} X_i$ is not normal where each X_i is compact first countable?

The positive consistency results have been: that MA $\Rightarrow$ the box product of countably many compact first countable spaces is paracompact (Rudin, Kunen); that $\exists$ a λ-scale $\Rightarrow \square^{\omega}(\omega+1)$ is paracompact (Williams); that $\exists$ a λ-scale $\Rightarrow \square_{i<\omega} X_i$ is paracompact if each X_i is compact metrizable (Van Douwen. This is an improvement of Williams' result using a different technique).

Using a criterion inspired by Williams' method, we show that the converse of Van Douwen's result is false, and that, in fact, in many models of set theory both with and without λ-scales, the box product of countably many compact first countable spaces is paracompact.

More precisely, we have that if $\mathrm{cf}(\varkappa)>\omega$, then in a forcing extension by a ccc iterated algebra of length $\varkappa$ in a ground model M, the following hold:

$\square_{i<\omega} X_i$ is paracompact if each X_i is compact first countable of weight $\leqslant \mathrm{cf}(\varkappa)$; hence if each X_i is compact metrizable.

$\square_{i<\omega} X_i$ is paracompact if each X_i is compact first countable and $M \models \varkappa = \mathrm{cf}(\varkappa) \geqslant \mathfrak{c}$; in particular if $M \models (\varkappa$ regular uncountable and $\mathfrak{c} = \omega_1)$.

Many thanks are due to both Scott Williams and Eric van Douwen whose patience and insightful questions were invaluable; and to Ken Kunen for mentioning Lemma 5 at just the right time.

§ 1. Notation and preliminaries. All spaces are assumed Hausdorff unless they explicitly have to be proven so. A space is *paracompact* iff for every open cover there is a locally finite open cover refining it (locally finite = each point has a neighborhood intersecting at most finitely many members). Paracompact implies normal, as is well known.

Let X be the set $\prod_{i<\omega} X_i$ where each X_i is a topological space. Then $\square_{i<\omega} X_i$ (also written $\square X$) is the space whose basic open sets are all $\prod_{i<\omega} u_i$ where each u_i is open in X_i. If $x \in X$ or $u \subset X$, then x_i, u_i are their respective projections on the ith coordinate. For $x \in X$, we define $\bar{x} = \{y \in X\colon y_i = x_i$ for all but finitely many $i\}$. $\bar{x}$ is an equivalence class, and $\nabla_{i<\omega} X_i$ is the quotient space on $\{\bar{x}\colon x \in X\}$.

The relation between $\square$ and ∇ and the important properties of ∇ were discovered by Kunen. They are:

THEOREM 1 (Kunen). (a) *If each X_i is compact, then $\square_{i<\omega} X_i$ is paracompact iff $\nabla_{i<\omega} X_i$ is paracompact.*

(b) *G_δ's in any $\nabla_{i<\omega} X_i$ are open; hence $\nabla_{i<\omega} X_i$ is 0-dimensional if each X_i is regular.*

(c) *$\nabla_{i<\omega} X_i$ is paracompact iff every open cover has an open disjoint covering refinement (such a space is called ultraparacompact).*

Thus in asking about $\square_{i<\omega} X_i$ where each X_i is compact, we only have to look at $\square_{i<\omega} X_i$; and given an open cover we will be trying to refine it by disjoint open sets — a process fairly easily controlled.

As the Williams and van Douwen results indicate, positive consistency results all depend on the inner structure of ${}^\omega\omega$. The following notion is the fundamental one: for $f, g \in {}^\omega\omega$, A an infinite subset of ω, we say $f<g$ on A iff $\{i \in A\colon f(i) \nless g(i)\}$ is finite. We say $f<g$ if $f<g$ on ω, and denote the infinite subsets of ω by $P^*(\omega)$.

Although we will not use scales, for the curious reader we define a scale as a subset of ${}^\omega\omega$ cofinal in the ordering $<$. If $\langle L, <_L\rangle$ is a partial order, an L-scale is a scale $\{f_a\colon a \in L\}$ where $a <_L b \Rightarrow f_a < f_b$. Thus a λ-scale is a scale isomorphic to the ordinal λ, and a $(\varkappa, \lambda)$-scale is one isomorphic to $\varkappa \times \lambda$ under the lexographic order. If $\mathrm{cf}(\lambda) > \varkappa$, then $\exists$ a $(\varkappa, \lambda)$-scale $\Rightarrow \exists$ no γ-scale for any ordinal γ.

We will use $\nearrow$ and $\searrow$ to mean convergence respectively upward or downward in the set-theoretic sense.

The connection between countable box products of first countable spaces and ${}^\omega\omega$ is hinted at by the following notation: Suppose, for each $i<\omega$, X_i is regular first countable and for each $x_i \in X_i$ there is a fixed countable neighborhood basis $u_j^{x_i} \searrow \{x_i\}$ such that $\mathrm{Cl}(u_{j+1}^{x_i}) \subset u_j^{x_i}$. Then if $x \in \square_{i<\omega} {}_iX$, $f \in {}^\omega\omega$, we define $u_f^x = \prod_{i<\omega} u_{f(i)}^{x_i}$, and $\nabla u_f^x = \{\bar{y}\colon y \in u_f^x\}$. A trivial observation we will need is that if $\nabla u_f^x \cap \nabla u_f^y = \emptyset$ and $g \nless f$ on $\{i\colon u_{f(i)}^{x_i} \cap u_{f(i)}^{y_i} = \emptyset\}$ then $\nabla u_g^x \cap \nabla u_g^y = \emptyset$.

§ 2. Criteria for paracompactness. We give two criteria for ultraparacompactness, the second of which is explicitly designed for what we want to prove. The first criterion is purely topological and bears a strong resemblance to $\varkappa$-metrizability and the technique of van Douwen.

CRITERION I. X is a topological space and for some ordinal $\varkappa$

$\exists$ a $\varkappa$-sequence $S_\alpha \nearrow X$,

$\exists$ a $\varkappa$-sequence $V_\alpha \nearrow V$ a clopen basis for X,

$\exists \{V_\alpha^*\colon \alpha<\varkappa\}$ with each $V_\alpha^* \subset V_{\alpha+1}$,

such that

(a) V_α^* is a discrete open cover of S_α (i.e. S_α discrete under V_α^*) by sets disjoint in X,

(b) $\forall x \in S_\alpha \exists v \in V_\alpha^* (x \in v$ and $\forall u \in V_\alpha (x \notin u \Rightarrow v \cap u = \emptyset))$.

PROPOSITION 2. *If X satisfies Criterion I, then X is hereditarily ultraparacompact.*

Proof. Let $Y \subset X$, U be an open cover of Y. For $y \in Y$, define α_y as the least γ for which $\exists v \in V_\gamma \exists u \in U (y \in v \subset u)$; and if $\alpha_y = \alpha$, pick u_y to be such a $v \in V_\alpha$. Let $Y_\alpha = \{y\colon \alpha_y = \alpha\}$. We construct a disjoint refinement by induction. Let $U_0 = \{u \cap u_y\colon y \in Y_0, u \in V_0^*\}$. By (a), U_0 is a disjoint family. Suppose for each $\beta<\alpha$ $\exists U_\beta$ a disjoint cover of $\bigcup_{\gamma<\beta} Y_\gamma$ refining U, $U_\beta \subset V_\alpha$, and $\gamma<\beta \Rightarrow U_\gamma \subset U_\beta$. Let $Y_\alpha^* = Y_\alpha - \bigcup_{\beta<\alpha} U_\beta$ and define $U_\alpha = \bigcup_{\beta<\alpha} U_\beta \cup \{u \cap u_y\colon u \in V_\alpha^*, y \in Y_\alpha^*\}$. By (a) the new sets added are mutually disjoint, and by (b) the new sets are disjoint from the old ones. Since V is a basis, every point in Y is eventually covered, and so $\bigcup_{\alpha<\varkappa} U_\alpha$ is the desired disjoint refinement.

This first criterion, while handy, makes no mention of ${}^\omega\omega$ and gives no hint of the way it hooks up with the S_α's in picking the V_α^*'s. We will therefore develop a second criterion which implies the first. This second criterion is quite long, so we provide the motivation for it by sketching the construction by which, given a λ-scale, Williams refines an arbitrary open cover of $\nabla^\omega(\omega+1)$.

Given the λ-scale $\{g_\alpha\colon \alpha<\lambda\}$ and an open cover of $\nabla^\omega(\omega+1)$, Williams uses an induction of length λ to cover, at the αth stage, all points $\bar{f}$ such that f sits below g_α wherever f is finite; furthermore, the new open set ∇u^f covering $\bar{f}$ comes from some u^f which sits above g_α on the values where f is infinite. These salient features of g_α can be re-expressed in a clumsy form which provides the essential insight:

(*) If $\bar{f}$ is newly covered at stage α, then g_α dominates f on infinitely many values of any infinite set on which f is finite; and every function in u^f dominates g_α on infinitely many values of any infinite set on which f is infinite, and equals f where f is finite.

For an iterated forcing extension, we simply modify the phrase "infinite set" in (∗) to "infinite set in M_α" where M_α is the αth intermediate model of the iteration (M_α is defined in § 3). Then we note that if g_α is Cohen over M_α, we can cover $\nabla^\omega(\omega+1)\cap M_\alpha$ so the modified (∗) holds. An easy adaptation of Williams' proof then gives paracompactness of $\nabla^\omega(\omega+1)$ in the Cohen extension adding λ Cohen reals, for $\lambda\geqslant\omega_1$.

Unfortunately, we are not working only with $\nabla^\omega(\omega+1)$, so things are a bit more complicated.

First we need a standard collection of open sets to dip into. So let each X_i be regular first countable for $i<\omega$, $X=\nabla_{i<\omega} X_i$ and let $\nabla u_f^{\bar x}$ be as in § 1. We define the function $(n+f)\colon \omega\to\omega$ by $n+f(k)=n+f(k)$ and the clopen neighborhood $u_{\omega+f}^{\bar x}=\bigcap_{n<\omega}\nabla u_{n+f}^{\bar x}$. Naturally, $\bar x\in u_{\omega+f}^{\bar x}$, and all the $u_{\omega+f}^{\bar x}$'s form a basis.

Now suppose at the αth stage we want to cover $S\subset M_\alpha\cap X$. We want to avoid the open sets chosen at earlier stages for our refinement; this will fall out of our construction if we avoid all $\nabla u_f^{\bar x}$ where $\bar x\in S$ and $f\in{}^\omega\omega\cap M_\alpha$. This motivates the following definition: given $S\subset X$ and $W\subset{}^\omega\omega$, the set $V(S,W)$ of neighborhoods coded by S, W is $\{u\colon \exists\bar x\in S\,\exists f\in W(u=\nabla u_f^{\bar x})\}$.

Criterion II. Each X_i is regular first countable $\forall i<\omega$, and for some $\varkappa$

$\exists$ a $\varkappa$-sequence $S_\alpha\nearrow\nabla_{i<\omega} X_i$,

$\exists$ a $\varkappa$-sequence $W_\alpha\nearrow{}^\omega\omega$,

$\exists$ a $\varkappa$-sequence $A_\alpha\nearrow P^*(\omega)$,

$\exists\{g_\alpha\colon \alpha<\varkappa\}\subset{}^\omega\omega$,

such that

(1) $g_\alpha\in W_{\alpha+1}$,

(2) $f,f'\in W_\alpha\Rightarrow\sup(f,f')\in W_\alpha$ and $\forall n((n+f)\in W_\alpha)$,

(3) $f\in W_\alpha$, $A\in A_\alpha\Rightarrow g_\alpha\nless f$ on A,

(4) S_α is Hausdorff under $V(S_\alpha,W_\alpha)$,

(5) if $\bar x\in S_\alpha-\mathrm{Cl}(u)$ for some $u\in V(S_\alpha,W_\alpha)$, then $\exists f\in W_\alpha(\nabla u_f^{\bar x}\cap u=\varnothing)$,

(6) $u,v\in V(S_\alpha,W_\alpha)\Rightarrow\{i\colon u_i\cap v_i=\varnothing\}\in A_\alpha$.

Proposition 3. *If* $\nabla_{i<\omega} X_i$ *satisfies Criterion* II, *then it satisfies Criterion* I.

Proof. Let $V_\alpha=\{u_{\omega+f}^{\bar x}\colon \bar x\in S_\alpha,\ f\in W_\alpha\}$ and let $V_\alpha^*=\{u_{\omega+g_\alpha}^{\bar x}\colon \bar x\in S_\alpha\}$. We show $I(a)$ — that V_α^* is a discrete cover of S_α by sets disjoint in X. By (4), if $\bar x,\bar y\in S_\alpha$ then $\exists f,f'\in W_\alpha$ with $\nabla u_f^{\bar x}\cap\nabla u_{f'}^{\bar y}=\varnothing$. Hence by (6), $A=\{i\colon u_{f(i)}^{x_i}\cap u_{f'(i)}^{y_i}\}=\varnothing\in A_\alpha$. By (2) and (3), $g\nless\sup(f,f')$ on A. Then by the remark at the end of § 1, $\nabla u_{g_\alpha}^{\bar x}\cap\nabla u_{g_\alpha}^{\bar y}=\varnothing$, so $u_{\omega+g_\alpha}^{\bar x}\cap u_{\omega+g_\alpha}^{\bar y}=\varnothing$.

I(b) is proved similarly. Suppose $\bar x\in S_\alpha$, $u\in V_\alpha$, $\bar x\notin u$. We show $u_{\omega+g_\alpha}^{\bar x}\cap u=\varnothing$.

Since $u\in V_\alpha$, $\exists\bar y\in S_\alpha\exists f'\in W_\alpha(u=u_{\omega+f'}^{\bar y}=\bigcap_{n<\omega}u_{n+f'}^{\bar y}=\bigcap_{n<\omega}\mathrm{Cl}(u_{n+f'}^{\bar y}))$. Hence $\exists n(\bar x\notin\mathrm{Cl}(u_{n+f'}^{\bar y}))$, so by (5) $\exists f\in W_\alpha(u_f^{\bar x}\cap u_{n+f'}^{\bar y}=\varnothing)$. Since $u_{n+f'}^{\bar y}\supset u$, we proceed as in (a) above and the proof is complete.

§ 3. Three forcing facts. We now consider the class of forcing extensions we will use. In this section and the next we will be looking carefully at the inner structure of these extensions, and some familiarity with forcing is assumed. We omit standard sorts of proof, referring the reader to the excellent references [J] and [S]. We do try, however, to give enough motivation so the non-set-theorist can get some feel for why these proofs go the way they do.

As usual we abuse notation to ignore the distinction between term and object, saying, e.g., $\exists f\in M^{\boldsymbol B}\cap{}^\omega\omega$ instead of $\exists\tau(M^{\boldsymbol B}\models\tau$ a function from $\check\omega$ to $\check\omega)$; if we are working in $M^{\boldsymbol B}$ we shall just say $\exists f\in{}^\omega\omega$ instead to $\exists f\in M^{\boldsymbol B}\cap{}^\omega\omega$. Also, if $\boldsymbol P\in M$ is dense in $\boldsymbol B\in M$, we define $M^{\boldsymbol P}=M^{\boldsymbol B}$. Models of set theory are assumed transitive.

For the rest of this paper we fix M a model of set theory and in M fix both $\varkappa$ an ordinal of cofinality $>\omega$ and $\boldsymbol B$ a complete Boolean algebra such that the following statement is true in M:

$\exists$ a partial order $\langle\boldsymbol P,<\rangle$ dense in $\boldsymbol B$ and a $\varkappa$-sequence $\langle\boldsymbol P_\alpha,<_\alpha\rangle$ $\langle\boldsymbol P,<\rangle$ such that

(i) λ a limit $<\varkappa\Rightarrow\langle\boldsymbol P_\lambda,<_\lambda\rangle={}_{\alpha<\lambda}\langle\boldsymbol P_\alpha,<_\alpha\rangle$,

(ii) if $\beta<\alpha<\varkappa$ and $p,q\in\boldsymbol P_\beta$ are incompatible under $<_\beta$, then they are incompatible under $<_\alpha$,

(iii) $\forall\alpha\exists\langle\boldsymbol P^\alpha,<^\alpha\rangle$ such that $\boldsymbol P=\boldsymbol P_\alpha\times\boldsymbol P^\alpha$ under the order $\langle p,q\rangle\leqslant\langle p',q'\rangle$ iff $p\leqslant_\alpha p'$ and $p\Vdash q\leqslant^\alpha q'$,

(iv) $\beta<\alpha<\varkappa\Rightarrow M^{\boldsymbol B}\models M^{\boldsymbol P_\alpha}-M^{\boldsymbol P_\beta}=\varnothing$,

(v) $\boldsymbol P$ has the countable chain condition (abbr. ccc). I.e. any mutually incompatible set of elements of $\boldsymbol P$ is countable.

Conditions (i) through (iii) define an iteration by direct limits; (iv) ensures that at no point do the algebras become trivial; (v) ensures that cofinalities and hence cardinals are preserved, and that we add new reals. We say $\varkappa$ is a length of $\boldsymbol B$.

Let $M_\alpha=M^{\boldsymbol P_\alpha}$. We will rely heavily on the fact that if $M^{\boldsymbol B}\models G$ a generic filter on $\boldsymbol P$ over M, then $M^{\boldsymbol B}\models G\cap\boldsymbol P^\alpha$ a generic filter on $\boldsymbol P^\alpha$ over M_α, and $M_\alpha\models G\cap\boldsymbol P_\alpha$ a generic filter on $\boldsymbol P_\alpha$ over M. Hence $M^{\boldsymbol B}=M_\alpha^{\boldsymbol P^\alpha}$.

What algebras fit this description? Ccc iterations were first developed by Solovay and Tennenbaum to destroy Suslin trees, and then generalized by Solovay and Martin to prove MA+$\neg$CH consistent. The algebra adding $\varkappa$ many Cohen reals, and the algebra adding $\varkappa$ many random reals by iteration (not the same as simultaneously) are two others widely studied; in the model given by the former there are neither λ-scales nor (γ,λ)-scales if $\varkappa\geqslant\omega_2$. Hechler's algebras for adding $\varkappa$-scales and $(\varkappa,\lambda)$-scales are two more examples.

We need some facts to control the relation between our spaces X_i and the models M_α, and to ensure a good sequence of g_α's. Our main instrument for the first task is the following standard fact.

LEMMA 4. *Let* $f\in M^B$ *be a function with* $M^B\models(\operatorname{dom} f\subset\bigcup_{\alpha<\varkappa} M_\alpha$ *and* $|\operatorname{dom} f|<\operatorname{cf}(\varkappa))$. *Then* $f\in M_\alpha$ *for some* $\alpha<\varkappa$.

The inexperienced reader is warned that the restriction of f's domain is crucial. $M\neq\bigcup_{\alpha<\varkappa} M_\alpha$.

The next fact is less well-known and is due to Kunen. It will ensure that the sequence $\{g_\alpha:\ \alpha<\varkappa\}$ of Criterion II exists.

LEMMA 5. *Let* $\beta<\alpha<\varkappa$, $\operatorname{cf}(\alpha)=\omega$. *Then* $\exists g\in M_\alpha\neq{}^{\omega}\omega$ *with g Cohen over* M_β. (*By g Cohen over* M_β *we mean that g is in no* M_β*-coded first-category subset of* ${}^{\omega}\omega$.)

Sketch of proof. We are done if we can produce a partial order $Q\in M_\beta$, Q embedded in P_α, Q isomorphic to the Cohen partial order, such that if G is M_α-generic on P_α, and $D\subset Q$ is dense with respect to Q, then $G\cap D\neq\emptyset$. The required function g will then be the generic object forced by Q.

So let $\alpha_i\nearrow\alpha$ with $\beta\leqslant\alpha_0$. For $i<\omega$, let Q_i be an infinite subset of $P_{\alpha_{i+1}}$ such that $M_{\alpha_i}\models Q_i$ a maximal partition of $P_{\alpha_{i+1}}$. Let $F=\{s:\ \operatorname{dom} s$ is a finite subset of ω, $s(i)\in Q_i\}$ and let $Q=\{\inf E:\ s\in F\ (E=\text{range } s)\}$. Then Q is the desired partial order.

The last forcing lemma needed, ensuring that Cohen reals are what we want, is again standard.

LEMMA 6. *Let* $g\in M^B\cap{}^{\omega}\omega$ *be Cohen generic over some model* $N\subset M^B$. *Then* $f\in N\cap{}^{\omega}\omega$, $A\in N\cap P^*(\omega)\Rightarrow g\not<f$ *on* A.

§ 4. **Satisfying Criterion II.** Given M, $\varkappa$, B as in § 3, and compact first countable spaces $X_i\in M^B$ of weight $\leqslant\operatorname{cf}(\varkappa)$ (weight = minimal cardinality of a basis, and in first countable spaces is the same as the cardinality of the space), we produce A_α, W_α, $\{g_\alpha:\ \alpha<\varkappa\}$ satisfying (1), (2) and (3) of Criterion II, and after some analysis of the X_i's manage to construct the S_α's which will work for the rest of Criterion II. All work is done in M^B.

Let $t:\ \varkappa\to\varkappa$ be an increasing function with range $t\subset\{\alpha:\ \operatorname{cf}(\alpha)=\omega\}$. We let $W_\alpha=M_{t(\alpha)}\cap{}^{\omega}\omega$; let $A_\alpha=M_{t(\alpha)}\cap P^*(\omega)$; and let g_α be the function in $M_{t(\alpha+1)}$ guaranteed by Lemma 5 relative to $M_{t(\alpha)}$. Then (1) is immediate; that $M_{t(\alpha)}$ is a model of set theory gives us (2); and Lemma 6 ensures (3). All we have to do is define such a t against whose background we can define the S_α's.

Roughly, we want to let $S_\alpha=\nabla_{i<\omega} X_i$ relative to $M_{t(\alpha)}$. But what does this means? To each X_i is associated a lattice L_i of basic open sets; since X_i is first countable we may identify points in X_i with certain countably generated maximal filters on L_i. Two problems may present themselves if L_i under its partial order sits in no M_α. First of all, points may split — some filter that M_α thinks defines a point may be split in some higher M_β, so that what M_α thinks is X_i may not even be a subset

of X_i. Secondly the partial orders at different stages may disagree about empty infs. Unless a dense subset of X_i sits in some M_α, we may have for each $\alpha<\varkappa$ a pair $u,v\in L_i\cap M_\alpha$ with $M_\alpha\models u\cap v=\emptyset$ and yet some higher $M_\beta\models u\cap v\neq\emptyset$. This means no M_α can recognize disjoint open sets of X_i, making satisfaction of (6) quite unlikely.

Our first task, then, is to find a condition under which empty pairwise infs at early levels of a lattic L are preserved. This is assured if the lattice is not large; hence the condition on the weight of the X_i's. In order to use Lemma 4, we note that a lattice L under its partial order is really a function $h:\ |L|^2\to 2$; and a sublattice a subset of h.

LEMMA 7. *Let L be a lattice*, $|L|\leqslant\operatorname{cf}(\varkappa)$. *Then* $\exists s:\ \varkappa\to\varkappa$ *and a* $\varkappa$*-sequence* $L_\alpha\nearrow L$ *such that* $L_\alpha\in M_{s(\alpha)}$ *and if* $u,v\in M_{s(\alpha)}\cap L$ *and* $u\cap v\neq\emptyset$, *then*

$$M_{s(\alpha)}\models\exists w\in L_\alpha(w\subset u\cap v)\,.$$

Proof. We say a subset L' of L is good if $\forall u,v\in L'$,

$$u\cap v\neq\emptyset\Rightarrow\exists w\in L'(w\subset u\cap v)\,.$$

By a countable process it is easily seen that $L'\subset L\Rightarrow\exists L^*(L'\subset L^*\subset L$, L^* good, and $|L'|=|L^*|)$. So let $L=\{u_\alpha:\ \alpha<\lambda\leqslant\operatorname{cf}(\varkappa)\}$. By induction construct good $L_\alpha\subset L$ such that $u_\alpha\in L_\alpha$; $|L_\alpha|=|\alpha|<\operatorname{cf}(\varkappa)$. By Lemma 4 $\exists$ least γ with $L_\alpha\in M_\gamma$. Let $s(\alpha)$ be this γ. If $\sup\{s(\alpha):\ \alpha<\varkappa\}=\delta=\varkappa$, this is the s we want. If $\delta<\varkappa$, let $L_\alpha=L$ and $s(\alpha)=\delta+\alpha$.

What are the compact first countable spaces of weight $\leqslant\varkappa$? The following are some examples: compact metric spaces; compact Suslin lines; any compact first countable space if $M\models 2^\omega\geqslant\operatorname{cf}(\varkappa)=\varkappa$ (since, by a theorem of Arhangel'skiĭ, such a space has cardinality c, and hence weight $\leqslant c$).

Now we use good sub-lattices to define good subspaces. Suppose X is a first countable space with the lattice L of basic open sets, and L' is a good subset of L, $L'\in M_\alpha$. We say $Y\subset X$ is the set good for α, L' if Y is the set of all points $x\in X$ for which $\exists\, U_x$ a filter on L' with:

A. $U_x\in M_\alpha$,

B. $\exists\{u_i^x:\ i<\omega\}\in M_\alpha\cap U_x$ with $M_\alpha\models\{u_i^x:\ i<\omega\}$ a base for U_x,

C. $u_i^x\searrow\{x\}$ and is a base for x,

D. $\operatorname{Cl}(u_{i+1}^x)\subset u_i^x$.

In other words, Y is the set of points defined with reference to L' that do not split. Since M_α does not actually know that U_x will not split, Y may not itself be an element of M_α. But we are working in M^B, so this does not matter. By Lemma 4, if $L_\alpha\subset L$, L_α good $\in M_{s(\alpha)}$, then $X=\bigcup\{Y_\alpha:\ Y_\alpha$ good for $L_\alpha, s(\alpha)\}$.

The next lemma tells us that good subspaces under the topology of the associated good lattices are well-behaved, which will enable us to put them together in ∇-products.

LEMMA 8. *Let X be Hausdorff and regular with the lattice L of basic open sets. Suppose $L' \in M_\alpha$ is a good subset of L, and $Y \subset X$ is the set good for α, L'. Then Y is Hausdorff under the topology given by L', and if $x \in Y$, $Z \subset Y$, and $x \notin \mathrm{Cl}(Z)$ then $\exists u \in L'(x \in u$ and $u \cap \mathrm{Cl}(Z) = \emptyset)$.*

Proof. For Hausdorff, we note that if U is as above, then

$$x, y \in Y, x \neq y \Rightarrow \exists u, v \in L(x \in u, y \in v, u \cap v = \emptyset)$$
$$\Rightarrow \exists u' \in U_x, v' \in U_y(u' \subset u, v' \subset v)$$

and hence $x \in u'$, $y \in v'$, $u' \cap v' = \emptyset$. Similarly, if $u \in L$ separates x from some closed Z, then $\exists u' \in U_x$ with $u' \cap Z = \emptyset$.

Lemma 8 will, of course, be used in the proof of (4) and (5) of Criterion II.

The final construction and the lemmas which follow it essentially use the fact that if M_α is smart enough to recognize a bunch of sequences, then it is smart enough to use them.

So the last thing we have to do is: Given a countable collection of compact first countable spaces of small weight, we simultaneously construct good partial bases and good subsets relative to the same sequence on M_α's; we use these to define the S_α's of criterion II; and we prove, using the fact that the M_α's are smart, that II(4), (5), and (6) hold.

CONSTRUCTION 9. For $i<\omega$ let X_i be a first countable Hausdorff space of weight $\leqslant \mathrm{cf}(\varkappa)$, with L_i a basis of minimal cardinality. Let $L_{i,\alpha} \nearrow L$, s_i be as in Lemma 7. We define $t: \varkappa \to \varkappa$ by $t(\alpha) =$ least $\gamma > t(\beta)$ for $\beta < \alpha$, γ of cofinality ω, and $\gamma \geqslant s_i(\alpha)$ for all $i<\omega$. Using t, we define the subspaces $X_{i,\alpha}$ of X_i as: $X_{i,\alpha}$ is the set good for $L_{i,\alpha}$, $t(\alpha)$; and for $x_i \in X_{i,\alpha} - \bigcup_{\beta<\alpha} X_{i,\beta}$ assign the same sequence $\{u_j^{x_i}: j<\omega\}$ that $M_{t(\alpha)}$ does as in the definition of good subspaces. Finally, we let $S = \{\bar{x}: x \in (\prod_{i<\omega} X_{i,\alpha}) \cap M_{t(\alpha)}\}$.

LEMMA 10. *S_α, $V(S_\alpha, W_\alpha)$ satisfy* II(4).

Proof. Recall that $W = M_{t(\alpha)} \cap {}^\omega\omega$. To prove Hausdorff, suppose $\bar{x}, \bar{y} \in S_\alpha$, $\bar{x} \neq \bar{y}$. Since some infinite tail of x and some infinite tail of y are in $M_{t(\alpha)}$, wlog $x, y \in M_{t(\alpha)}$. In $M_{t(\alpha)}$ we define $f \in {}^\omega\omega$ by: $f(i) =$ least k such that $u_k^{x_i} u_k^{y_i} = \emptyset$ if $x_i \neq y_i$; $f(i) = 0$ otherwise. Lemma 8 ensures that we can do this; and since f is defined within $M_{t(\alpha)}, f \in M_{t(\alpha)}$. Hence u_f^x, $u_f^y \in M_{t(\alpha)}$ which is all we need.

LEMMA 11. *S_α, $V(S_\alpha, W_\alpha)$ satisfy* II(5).

Proof. To prove this version of regularity, if $\bar{x} \in S_\alpha$, $u = u_{f'}^y \in V(S_\alpha, W_\alpha)$, $\bar{x} \notin \mathrm{Cl}(u)$, we again assume $x \in M_{t(\alpha)}$ and now define $A = \{i: x_i \notin \mathrm{Cl}(u_i)\}$. Since $\mathrm{Cl}(u) = \nabla_{i<\omega} \mathrm{Cl}(u_i)$, this suffices; and since A is defined in $M_{t(\alpha)}$, $A \in M_{t(\alpha)}$. Then defining f by $f(i) =$ least k such that $u_k^{x_i} \cap \mathrm{Cl}(u_i) = \emptyset$ if $i \in A$; $f(i) = 0$ otherwise, we note again that $f \in M_{t(\alpha)}$, so u_f^x is the desired neighborhood.

LEMMA 12. *$V(S_\alpha, W_\alpha)$ satisfies* II(6).

Proof. Recall that $A_\alpha = M_{t(\alpha)} \cap P^*(\omega)$. Suppose $x, y \in S_\alpha$, $f, f' \in W_\alpha$, and $u_{\omega+f}^x \cap u_{\omega+f'}^y = \emptyset$. $u_i \cap v_i = \emptyset$ iff $\exists n(u_{n+f(i)}^{x_i} \cap u_{n+f'(i)}^{y_i} = \emptyset)$; hence $A = \{i: u_i \cap v_i = \emptyset\} = \{i: \exists n(u_{n+f(i)}^{x_i} \cap u_{n+f'(i)}^{y_i} = \emptyset\}$ is defined in $M_{t(\alpha)}$ and thus is in $M_{t(\alpha)}$.

From all of this we derive as a corollary

THEOREM 13. *In $M^{\mathbf{B}}$, where $\mathbf{B}$ is a ccc iteration of length $\varkappa$ and $\mathrm{cf}(\varkappa) > \omega$, if each X_i is compact first countable of weight $\leqslant \mathrm{cf}(\varkappa)$, then $\nabla_{i<\omega} X_i$ is ultraparacompact, and hence $\square_{i<\omega} X_i$ is paracompact.*

And from the remarks following the definition of ccc iterations, we have

COROLLARY 14. *The box product of countably many compact first countable spaces is paracompact $\not\Rightarrow \exists$ a λ-scale.*

Noting that if each $L_i \in M$ then Lemma 7 is not needed, we have as a corollary to the method of proof:

Define X the completion of the lattice L if X is isomorphic to the set of maximal filters of L under the topology whose basic sets are, for each $u \in L$, $N_u = \{x: u \in x\}$.

COROLLARY 15. *In $M^{\mathbf{B}}$, where $\mathbf{B}$ a ccc iteration of length $\varkappa$ and $\mathrm{cf}(\varkappa) > \omega$ if each X_i is a compact first countable completion of some $L_i \in M$, then $\nabla_{i<\omega} X_i$ is ultraparacompact, and hence $\square_{i<\omega} X_i$ is paracompact.*

Finally, we note that these proofs extend to any iterated forcing extension with cofinally many reals which satisfy Lemma 6, where the length of the iteration ensures that Lemma 4 is satisfied. Thus, e.g.,

THEOREM 16. *In $M^{\mathbf{B}}$, where $\mathbf{B}$ is a λ-cc iteration of length $\varkappa$, $\mathrm{cf}(\varkappa) \geqslant \lambda$, $\lambda = \mathbf{c}^{M^{\mathbf{B}}}$, $\lambda > \mathbf{c}^M$ and $\mathbf{B}$ has cofinally many Mathias reals, if each X_i is compact first countable, then $\nabla_{i<\omega} X_i$ is ultraparacompact, and hence $\square_{i<\omega} X_i$ is paracompact.*

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First countable and countable spaces all compactifications of which contain βN

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Abstract. We construct the following examples.

EXAMPLE 1. A first countable Lindelöf (even cosmic) space Δ all compactifications of which contain βN.

EXAMPLE 2. A countable space Σ with one non-isolated point all compactifications of which contain βN.

Since βN has cardinality $2^{\mathfrak{c}}$, uncountable tightness and is neither first countable nor scattered, the above examples in particular yield:

(1) A first countable Lindelöf space with no first countable compactification.

(2) A countable space all compactifications of which have cardinality $2^{\mathfrak{c}}$ and uncountable tightness.

(3) A scattered space with no scattered compactification.

1. Introduction. Throughout this paper all spaces are assumed to be regular, a cardinal is an (von Neumann) ordinal, $\mathrm{cf}(\varkappa)$ is the cofinality of $\varkappa$, and $\mathfrak{c}$ is 2^{ω}. For undefined terms we refer to [E].

In this paper we construct the following two examples.

EXAMPLE 1.1. A first countable Lindelöf (even cosmic) space Δ all compactifications of which contain a homeomorph of βN.

EXAMPLE 1.2. A countable space Σ with one non-isolated point all compactifications of which contain a homeomorph of βN.

Recall that a space X is *cosmic* [Mi] if it has a countable network, i.e. a countable family $\mathscr{A}$ of subsets such that for each open $U\subset X$ and each $x\in U$ there is an $A\in\mathscr{A}$ with $x\in A\subset U$. Every cosmic space is hereditarily Lindelöf and hereditarily separable. Also recall that the *tightness* $\tau(X)$ of a space X, [AP], is the smallest cardinal $\varkappa$ such that, whenever $A\subset X$ and $x\in\overline{A}$, there exists a $B\subset A$ such that $x\in\overline{B}$ and $|B|\leqslant\varkappa$. It is known that βN has cardinality $2^{\mathfrak{c}}$ [E, Theorem 3.6.12] and $\tau(\beta N)=\mathfrak{c}$

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