[11] J. Mycielski and S. Świerczkowski, *On the Lebesgue measurability and the axiom of determinateness*, Fund. Math. 54 (1964), pp. 67–71.
[12] J. C. Oxtoby, *Homeomorphic measures in metric spaces*, Proc. Amer. Math. Soc. 24 (1970), pp. 419–423.
[13] W. Rudin, *Real and complex analysis*, McGraw Hill, N. Y., 1966.
[14] D. D. Shochat, *Capacitability of Σ_2^1 sets*, Ph. D. thesis, University of California, Los Angeles.
[15] J. R. Shoenfield, *Mathematical Logic*, Addison-Wesley, Reading, Mass., 1967.
[16] J. H. Silver, *Measurable cardinals and Δ_3^1 well-orderings*, Ann. of Math. 94 (1971), pp. 414–446.
[17] R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. 92 (1970), pp. 1–56.
[18] — *On the cardinality of Σ_2^1 sets of reals*, in Foundations of Mathematics, Symposium Papers Commemorating the Sixtieth Birthday of Kurt Gödel, Springer Verlag, Berlin (Heidelberg) N. Y. 1969, pp. 58–73.

SCHOOL OF HISTORY, PHILOSOPHY AND POLITICS
MACQUARIE UNIVERSITY
North Ryde, N.S.W.
Australia

Accepté par la Rédaction le 13. 9. 1976

A theorem of Borsuk–Ulam type for multifunctions

by

M. Ehrlich (Berlin)

Abstract. We prove a generalization of a theorem of Borsuk–Ulam type of Joshi to multifunctions mapping a non necessarily symmetric subset of a locally convex space into a closed hyperplane.

§ 1. Introduction. We shall proove the following generalization of a theorem of Borsuk–Ulam type of Joshi [3]:

THEOREM. *Let E be a separated locally convex topological vector space, X a closed and finitely bounded subset of E such that $O \notin X$ and that the component of O in $E \setminus X$ is finitely bounded. Then, for every compact multivalued vector field F mapping X into a closed hyperplane E^0 of E there are points $x, y \in X$ and a number $\mu > 0$ such that*

$$y = -\mu x \quad \text{and} \quad F(x) \cap F(y) \neq \emptyset .$$

We start by explaining the terminology.

In the following let E be a separated locally convex vector space. A subset of E is called *finitely bounded* iff it has a bounded intersection with every finite-dimensional subspace of E.

If X is a topological space, a mapping f from X into the set 2^E of all nonvoid compact convex subsets of E is called a *multivalued function*, or *multifunction*, of X into E, written $f\colon X \to E$. Moreover f is called *compact* iff

(1) f is upper semicontinuous (u.s.c.), i.e. for every $x \in X$ and every open set $O \supset f(x)$ there is a neighborhood N of x such that

$$f(N) = \bigcup_{x \in N} f(x) \subset O ,$$

(2) $f(X)$ is relatively compact in E.

A multifunction $F\colon X \to E$ is called a *multivalued compact field* iff $f(x) = x - F(x)$ is a compact multifunction.

We shall extend the following result of Joshi [3]:

PROPOSITION 1. *Let E be a Banach space and let X be a closed, bounded subset of E such that $O \notin X$ and that the component of O in $E \setminus X$ is bounded. Then for any* *single-valued compact field F mapping X into a closed hyperplane of E there are points $x, y \in X$ and a number $\mu>0$ such that*

$$y = -\mu x \quad \text{and} \quad F(x) = F(y).$$

Before proving our theorem we want to discuss the condition that X and the component of O in $E\setminus X$ be finitely bounded. At first we remark that even in Banach spaces there exist finitely bounded sets which are not bounded: if $E = l^2$, the set X of all $x = (x_i)_{i=1} \in l^2$ such that $\frac{1}{2} \leqslant x_i \leqslant 2$ or $-1 \leqslant x_i \leqslant -\frac{1}{2}$ for all i is an example. Moreover the component of O in $E\setminus X$ is finitely bounded, but not bounded.

Therefore, introducing finitely bounded sets instead of bounded sets gives more general results even in Banach spaces. But if we want to consider non-normable locally convex spaces we are forced to introduce finitely bounded set, because, if X is closed $O \notin X$, and the component C of O in $E\setminus X$ is bounded, E must be normable. In fact there exists a convex neighborhood of zero U such that $U \subset E\setminus X$. Since $U \subset C$, U is bounded.

If C is only finitely bounded we see by the same argument that there exists a convex finitely bounded neighborhood of zero U in E. By a result of Pallaschke and Pantelidis ([8], Lemma 3.2) a convex neighborhood of zero U is finitely bounded iff U is *radially bounded*, i.e. iff U has a bounded intersection with every 1-dimensional subspace of E. Landsberg [5] calls E *locally radially bounded* iff there exists a radially bounded neighborhood of O in E.

By the preceeding discussion we see that if E contains a subset X subject to the conditions of the theorem, E must be locally radially bounded. Landsberg [5] gives some examples of non-normable locally radially bounded spaces.

§ 2. Proof of the theorem. We first consider the finite-dimensional case. If $f\colon X \to E$ is a multifunction we let

$$\operatorname{gr}(f) = \{(x, y)\colon x \in X, y \in E, y \in f(x)\}.$$

We shall use the following result of Cellina [2]:

PROPOSITION 2. *Let E be a normed linear space, $X \subset E$ and $f\colon X \to E$ a compact multifunction. Then, given $\delta>0$, there exists a continuous single-valued function $q\colon X \to \overline{\operatorname{co}}f(X)$ such that $\inf\{\|x-\bar{x}\|+\|q(x)-z\|\colon (\bar{x}, z) \in \operatorname{gr}(f)\} \leqslant \delta$ for each $x \in X$.*

Now we are able to prove

LEMMA 1. *Let E be finite-dimensional and let X be a compact subset of E such that $O \notin X$ and that the component of O in $E\setminus X$ is bounded. Then for every compact multivalued function f mapping X into a hyperplane E^0 of E there are points $x, y \in X$ and a number $\mu>0$ such that*

$$y = -\mu x \quad \text{and} \quad f(x) \cap f(y) \neq \varnothing.$$

Proof. We may suppose that the topology of E is generated by a norm $\|.\|$. Let (δ_i) be a sequence of positive numbers converging to O. By Proposition 2 there are continuous single-valued functions $q_i\colon X \to E^0$ such that

$$\inf\{\|x-\bar{x}\|+\|q_i(x)-z\|\colon (\bar{x}, z) \in \operatorname{gr}(f)\} \leqslant \delta_i \quad \text{for each } x \in X. \tag{1}$$

By the result of Joshi (Prop. 1) there are points $x_i, y_i \in X$ and numbers $\mu_i>0$ such that

$$y_i = -\mu_i x_i, \quad q_i(x_i) = q_i(y_i). \tag{2}$$

Since X is compact we may suppose (taking subsequences if necessary) $x_i \to x \in X$, $y_i \to y \in X$. Since $O \notin X$ we infer from (2):

$$\mu_i = \frac{\|y_i\|}{\|x_i\|} \to \mu = \frac{\|y\|}{\|x\|} > 0 \quad \text{and} \quad y = -\mu x.$$

Since X is compact $f(X)$ is also compact ([1], p. 110) and so is $\operatorname{gr}(f)$, being a closed subset of $X \times f(X)$. Hence there exist points $\bar{x}_i \in X$, $z_i \in f(\bar{x}_i)$ such that

$$\|x_i-\bar{x}_i\|+\|q_i(x_i)-z_i\| = \inf\{\|x_i-\bar{x}\|+\|q_i(x_i)-z\|\colon (\bar{x}, z) \in \operatorname{gr}(f)\}.$$

Taking a subsequence if necessary we may assume $z_i \to z$. Because of (1) this implies

$$\bar{x}_i \to x, \quad q_i(x_i) \to z.$$

Since $\operatorname{gr}(f)$ is closed ([1], p. 112) $\bar{x}_i \to x$ and $z_i \in f(\bar{x}_i)$, $z_i \to z$, imply $z \in f(x)$. In a similar way we can show

$$q_i(y_i) \to z' \in f(y).$$

Since $q_i(x_i) = q_i(y_i)$ it follows

$$z = z' \in f(x) \cap f(y).$$

We want to prove our theorem by approximating the given field F by fields mapping the intersection of X with some finite-dimensional subspace E'' of E into a hyperplane of E'' and then applying Lemma 1.

We first improve a result of Ma ([6], (3.1)).

PROPOSITION 3. *Let E^0 be a closed linear subspace of E of finite codimension and let*

$$E = E^0 \oplus E^1.$$

Let p_0, p_1 be the projections of E onto E^0 and E^1, respectively. Moreover, let X be a topological space and $f\colon X \to E$ a compact multifunction.

Then for every neighborhood of zero V there is a compact multifunction $f_V\colon X \to E$ such that $f_V(X)$ is contained in a finite dimensional subspace of E and that

$$f_V(x) \subset f(x)+V, \quad f(x) \subset f_V(x)+V, \quad p_1(f_V(x)) = p_1(f(x))$$

for every $x \in X$.

Proof. We modify the proof of Ma [6], (3.1).

Let V be a convex neighborhood of zero, $K = \overline{f(X)}$. Since p_0 is continuous ([4], p. 156) there is a neighborhood of zero U such that $p_0(U) \subset V$. By a result of Nagumo [7] there is a continuous (single-valued) function P_U: $K \to E$ such that $x - P_U x \in U$ for all $x \in K$ and that $P_U(K)$ is contained in some finite-dimensional subspace E_U of E. Now we define a continuous map Q: $K \to E$ by

$$Qy = p_0(P_U y) + p_1(y),$$

and we let $f_V(x) = \operatorname{co} Q(f(x))$ for $x \in X$.

Since $Q(f(x))$ is a compact subset of the finite-dimensional space $p_0(E_U) \oplus E^1$, $\operatorname{co}(Q(f(x)))$ is also compact ([9], Satz 3.10). By the same argument the compactness of $\overline{f_V(X)}$ can be shown. The upper semicontinuity of f_V can be shown as in the proof of (3.1) in [6].

For each $x \in X$ we have

$$p_1(f_V(x)) = \operatorname{co}\big(p_1(Q(f(x)))\big) = \operatorname{co}(p_1(f(x))) = p_1(f(x)).$$

If $z \in f_V(x)$ there exist $\mu_i \geqslant 0$ and $y_i \in f(x)$ $(i = 1, 2, \ldots, n)$ such that

$$\sum_{i=1}^{n} \mu_i = 1 \quad \text{and} \quad z = \sum_{i=1}^{n} \mu_i Q y_i .$$

This implies

$$\begin{aligned} z = \sum \mu_i(p_0(P_U y_i) + p_1(y_i)) &= \sum \mu_i p_0(P_U y_i - y_i) + \sum \mu_i y_i \\ &\in f(x) + \sum \mu_i p_0(U) \subset f(x) + V. \end{aligned}$$

Therefore $f_V(x) \subset f(x) + V$, and in a similar way we can show $f(x) \subset f_V(x) + V$.

Lemma 2. *Let E^0 be a closed hyperplane in E, $X \subset E$ and F: $X \to E^0$ a multivalued compact field.*

Then for every neighborhood of zero U there exist a finite-dimensional space $E_U \subset E$ and a multivalued compact field F_U: $X \to E^0$ such that $F_U(x) \subset F(x) + U$ for each $x \in X$ and $F_U(X \cap E_U) \subset E_U^0$, where E_U^0 is a hyperplane in E_U.

Proof. Let E^1 be a 1-dimensional subspace of E such that

$$E = E^0 \oplus E^1$$

and let p_1 be the projection of E onto E^1. Moreover, let U be a convex neighborhood of zero.

Since $f(x) = x - F(x)$ is a compact multifunction, there are by Proposition 3 a finite dimensional space E_U and a compact multifunction f_U: $X \to E_U$ such that

$$(+) \qquad f_U(x) \subset f(x) + U \quad \text{and} \quad p_1(f_U(x)) = p_1(f(x))$$

for every $x \in X$. Obviously we may suppose $E_U \supset E^1$.

Let $F_U(x) = x - f_U(x)$. (+) implies $F_U(x) \subset F(x) + U$ for each $x \in X$. Moreover

$$p_1(F_U(x)) = p_1(x) - p_1(f_U(x)) = p_1(x) - p_1(f(x)) = p_1(F(x)) = 0.$$

Therefore $F_U(X) \subset E^0$.

Let $E_U^0 = E_U \cap E^0$. E_U^0 is clearly a hyperplane in E_U, and we have for each $x \in X \cap E_U$:

$$F_U(x) = x - f_U(x) \in E_U + E_U = E_U.$$

Therefore $F_U(X \cap E_U) \subset E_U \cap E^0 = E_U^0$.

Proof of the theorem. Given a neighborhood of zero V we choose a symmetric neighborhood U such that $U + U \subset V$. From Lemma 2 we get a finite-dimensional space E_V and a multivalued compact field F_V: $X \to E^0$ such that

$$F_V(x) \subset F(x) + U \quad \text{for all } x \in X \text{ and } F_V(X \cap E_V) \subset E_V^0,$$

where E_V^0 is a hyperplane in E_V.

Our hypotheses on X imply that

(i) $X \cap E_V$ is compact,

(ii) the component of O in $E_V \setminus (X \cap E_V)$ is bounded.

Therefore there exist by Lemma 1 points $x_V, y_V \in X \cap E_V$ and a number $\mu_V > 0$ such that

$$y_V = -\mu_V x_V \quad \text{and} \quad F_V(x_V) \cap F_V(y_V) \neq \emptyset.$$

We conclude that

$$\text{(iii)} \qquad F(x_V) \cap (F(y_V) + V) \neq \emptyset.$$

Now, since F is a compact field,

$$f(x) = x - F(x)$$

is a compact multifunction. Because of (iii) there are points $a_V \in f(x_V)$, $b_V \in f(y_V)$ such that

$$\text{(iv)} \qquad x_V - a_V \in y_V - b_V + V.$$

Let $\mathscr{V}$ be the filterbase of all symmetric neighborhoods of zero and consider the nets $(a_V)_{V \in \mathscr{V}}$ and $(b_V)_{V \in \mathscr{V}}$. Since f is compact we may suppose (taking subnets if necessary) that

$$a_V \to a \quad \text{and} \quad b_V \to b.$$

From (iv) we get

$$(1 + \mu_V) x_V = x_V - y_V \in a_V - b_V + V.$$

Therefore $(1 + \mu_V) x_V \to a - b$. This implies $\mu_V \not\to +\infty$, because otherwise $x_V \to 0$ which is impossible since X is closed and $O \notin X$. Hence there exists a subnet of $(\mu_V)_{V \in \mathscr{V}}$ converging to some $\mu \geqslant 0$. To simplify the notation, this subnet is also denoted by (μ_V).

We conclude

$$x_V \to \frac{a-b}{1+\mu} =: x \in X ,$$

$$y_V = -\mu_V x_V \to -\mu x =: y \in X .$$

Since $\mathrm{gr}(F)$ is closed we have $x-a \in F(x)$, $y-b \in F(y)$. From (iv) we get

$$x-a = y-b \in F(x) \cap F(y) ,$$

and the theorem is proved.

References

[1] C. Berge, *Topological Spaces*, New York 1963.

[2] A. Cellina, *A theorem on the approximation of compact multivalued mappings*, Lincei — Rend. Sc. fis. mat. e nat. 47 (1969), pp. 430–433.

[3] K. D. Joshi, *Infinite dimensional non-symmetric Borsuk–Ulam theorem*, Fund. Math. 89 (1975), pp. 45–56.

[4] G. Köthe, *Topological Vector Spaces* I, Berlin–Heidelberg–New York 1969.

[5] M. Landsberg, *Lineare Abbildungen von einem Produkt in einen lokal radial beschränkten Raum und ihre Filter*, Math. Annalen 146 (1962), pp. 232–248.

[6] T.-W. Ma, *Topological degrees of set-valued compact fields in locally convex spaces*, Dissertationes Math. 92 (1972), pp. 1–47.

[7] M. Nagumo, *Degree of mapping in convex linear topological spaces*, Amer. J. Math. 73 (1951), pp. 479–511.

[8] D. Pallaschke und G. Pantelidis, *Homotopieeigenschaften von Sphären in Φ-Räumen und approximative Kompaktheit*, Archiv der Math. 20 (1969), pp. 176–185.

[9] F. A. Valentine, *Konvexe Mengen*, Mannheim 1968.

Accepté par la Rédaction le 14. 9. 1976

Uniform shape and uniform Čech homology and cohomology groups for metric spaces

by

D. Doitchinov (Sofia)

Abstract. It is proved that the uniform Čech homology and cohomology groups of a metric space, namely those defined by means of all uniform coverings of the space, are invariant with respect to uniform shape equivalence understood in the sense of the paper [3] as well as in the sense of the paper [4].

In paper [3] a concept of uniform shape for metric spaces was introduced. Another notion of uniform shape equivalence, called uniform fundamental equivalence, for complete metric spaces was defined earlier in [4]. The purpose of this paper is to establish two theorems showing that the uniform Čech homology and cohomology groups, namely those defined by means of all uniform coverings of the space, are invariant with respect to uniform shape equivalence understood as either of the notions mentioned above. As these two theorems are proved in essentially the same way, only the proof of one of them will be given in detail.

§ 1. Uniform and double-uniform shapes for metric spaces. Let us recall the definition of uniform shape for metric spaces given in [3]. Every metric space X can be considered as uniformly embedded in a complete metric space M which is an absolute uniform neighbourhood extensor for metric spaces — such a space will be called a UANE-space. The family of all open neighbourhoods of X in M will be denoted by $\underline{U}(X, M)$.

If X and Y are subsets of the UANE-spaces M and N, respectively, then a *uniform shape map*

$$f\colon \underline{U}(X, M) \to \underline{U}(Y, N)$$

is a collection of uniformly continuous maps $f\colon U \to V$, where $U \in \underline{U}(X, M)$, $V \in \underline{U}(Y, N)$, provided that the following conditions are satisfied:

a) for every $V \in \underline{U}(Y, N)$ there exist a $U \in \underline{U}(X, M)$ and a $f\colon U \to V$ with $f \in \underline{f}$;

b) if $f \in \underline{f}$, $f\colon U \to V$, $U' \subset U$, $V' \supset V$, $U' \in \underline{U}(X, M)$, $V' \in \underline{U}(Y, N)$, and if $f' = jf|_{U'}$, where $j\colon V \to V'$ is the inclusion map, then $f' \in \underline{f}$;