A theorem of Borsuk–Ulam type for multifunctions

by

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Abstract. We prove a generalisation of a theorem of Borsuk–Ulam type of Joshi to multifunctions mapping a non necessarily symmetric subset of a locally convex space into a closed hyperplane.

§ 1. Introduction. We shall prove the following generalization of a theorem of Borsuk–Ulam type of Joshi [3]:

THEOREM. Let $E$ be a separated locally convex topological vector space, $X$ a closed and finitely bounded subset of $E$ such that $O \notin X$ and that the component of $O$ in $E \setminus X$ is finitely bounded. Then, for every compact multivalued vector field $F$ mapping $X$ into a closed hyperplane $E^0$ of $E$ there are points $x, y \in X$ and a number $\mu > 0$ such that

$$\gamma = -\mu x \quad \text{and} \quad F(x) \cap F(y) \neq \emptyset.$$  

We start by explaining the terminology.

In the following let $E$ be a separated locally convex vector space. A subset of $E$ is called finitely bounded iff it has a bounded intersection with every finite-dimensional subspace of $E$.

If $X$ is a topological space, a mapping $f$ from $X$ into the set $\mathbb{R}$ of all nonvoid compact convex subsets of $E$ is called a multivalued function, or multifunction, of $X$ into $E$, written $f : X \to E$. Moreover $f$ is called compact iff

1. $f$ is upper semicontinuous (us.c.), i.e. for every $x \in X$ and every open set $O \ni f(x)$ there is a neighborhood $N$ of $x$ such that $f(N) = \bigcup_{x \in N} f(x) = O$.

2. $f(X)$ is relatively compact in $E$.

A multifunction $F : X \to E$ is called a multivalued compact field iff $f(x) = x - F(x)$ is a compact multifunction.

We shall extend the following result of Joshi [3]:

PROPOSITION 1. Let $E$ be a Banach space and let $X$ be a closed, bounded subset of $E$ such that $O \notin X$ and that the component of $O$ in $E \setminus X$ is bounded. Then for any
single-valued compact field $F$ mapping $X$ into a closed hyperplane of $E$ there are points $x, y \in X$ and a number $\mu > 0$ such that

$$y = -\mu x$$ and

$$F(x) = F(y).$$

Before proving our theorem we want to discuss the condition that $X$ and the component of $O$ in $E \setminus X$ be finitely bounded. At first we remark that even in Banach spaces there exist finitely bounded sets which are not bounded: if $E = \ell_1$, the set $X$ of all $x = (x_i)_{i=1}^\infty \in \ell_1$ such that $\frac{1}{i} \leq x_i \leq \frac{1}{i-1}$ for all $i$ is an example. Moreover the component of $O$ in $E \setminus X$ is finitely bounded, but not bounded.

Therefore, introducing finitely bounded sets instead of bounded sets gives more general results even in Banach spaces. But if we want to consider non-normable locally convex spaces we are forced to introduce finitely bounded sets, because, if $X$ is closed $O \not= X$, and the component $C$ of $O$ in $E \setminus X$ is bounded, $E$ must be normable. In fact there exists a convex neighborhood of zero $U$ such that $U \supseteq E \setminus X$. Since $U \subseteq C$, $U$ is bounded.

If $C$ is only finitely bounded we see by the same argument that there exists a convex finitely bounded neighborhood of zero $U \subseteq E$. By a result of Pallasschke and Pantelidis [8], Lemma 3.2] a convex neighborhood of zero $U$ is finitely bounded iff $U$ is radially bounded, i.e. iff $U$ has a bounded intersection with every 1-dimensional subspace of $E$. Landsberg [5] calls $E$ locally radially bounded iff there exists a radially bounded neighborhood of $O$ in $E$.

By the preceding discussion we see that if $E$ contains a subset $X$ subject to the conditions of the theorem, $E$ must be locally radially bounded. Landsberg [5] gives some examples of non-normable locally radius bounded spaces.

§ 2. Proof of the theorem. We first consider the finite-dimensional case. If $f : X \to E$ is a multifunction we let

$$\text{gr}(f) = \{(x, y) : x \in X, y \in E, y \in f(x)\}.$$  

We shall use the following result of Cellina [2]:

PROPOSITION 2. Let $E$ be a normed linear space, $X \subseteq E$ and $f : X \to E$ a compact multifunction. Then, given $\delta > 0$, there exists a continuous single-valued function $q : X \to \text{Co}(X)$ such that $\inf\{\|x-x\| + \|q(x)-z\| : (x, z) \in \text{gr}(f)\} \leq \delta$ for each $x \in X$.

Now we are able to prove

LEMMA 1. Let $E$ be finite-dimensional and let $X$ be a compact subset of $E$ such that $0 \not= X$ and that the component of $O$ in $E \setminus X$ is bounded. Then for every compact multivalued function $f$ mapping $X$ into a hyperplane $E'$ of $E$ there are points $x, y \in X$ and a number $\mu > 0$ such that

$$y = -\mu x$$ and

$$f(x) \cap f(y) \not= \emptyset.$$  

Proof. We may suppose that the topology of $E$ is generated by a norm $\| \cdot \|$. Let $(x_n)$ be a sequence of positive numbers converging to $0$. By Proposition 2 there are continuous single-valued functions $q_i : X \to E'$ such that

$$\inf\{|x-x_n| + |q_i(x)-z| : (x, z) \in \text{gr}(f)\} \leq \frac{1}{i}$$ for each $x \in X$.

By the result of Joshi (Prop. 1) there are points $x_i, y_i \in X$ and numbers $\mu_i > 0$ such that

$$\mu_i = \frac{\|y_i\|}{\|x_i\|} > \mu \geq 0$$ and

$$y = -\mu x.$$  

Since $X$ is compact, $f(X)$ is also compact ([1], p. 110) and so is $\text{gr}(f)$, being a closed subset of $X \times f(X)$. Hence there exist points $x_i, y_i \in f(X)$ such that

$$\inf\{|x_i-x| + |q_i(x)-z| : (x, z) \in \text{gr}(f)\} \leq \frac{1}{i}$$

Taking a subsequence if necessary we may assume $x_i \to x$. Because of (1) this implies

$$x_i \to x$$ and

$$q_i(x_i) \to z.$$  

Since $g(f)$ is closed ([1], p. 112), $z \in f(X)$, and $x_i \to x$, imply $z \in f(x)$. In a similar way we can show

$$q_i(x_i) \to z \in f(x).$$  

Since $q_i(x_i) = q_i(y_i)$ it follows

$$z = z \in f(x) \cap f(y).$$  

We want to prove our theorem by approximating the given field $F$ by fields mapping the intersection of $X$ with some finite-dimensional subspace $E''$ of $E$ into a hyperplane of $E''$ and then applying Lemma 1.

We first improve a result of Ma ([6], (3.1)).

PROPOSITION 3. Let $E$ be a closed linear subspace of $E$ of finite codimension and let $E = E'' \oplus E'$.

Let $p_0, p_1$ be the projections of $E$ onto $E''$ and $E'$, respectively. Moreover, let $X$ be a topological space and $f : X \to E$ a compact multifunction.

Then for every neighborhood of zero $V$ there is a compact multifunction $f_V : X \to E$ such that $f_V(X)$ is contained in a finite dimensional subspace of $E$ and that

$$f_V(x) = f(x) + V,$$  

$$p_1(f_V(x)) = p_1(f(x))$$ for every $x \in X.$

Proof. We may suppose that the topology of $E$ is generated by a norm $\| \cdot \|$. Let $(x_n)$ be a sequence of positive numbers converging to $0$. By Proposition 2 there are continuous single-valued functions $q_i : X \to E''$ such that

$$\inf\{|x-x_n| + |q_i(x)-z| : (x, z) \in \text{gr}(f)\} \leq \frac{1}{i}$$ for each $x \in X$.

By the result of Joshi (Prop. 1) there are points $x_i, y_i \in X$ and numbers $\mu_i > 0$ such that

$$\mu_i = \frac{\|y_i\|}{\|x_i\|} > \mu \geq 0$$ and

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Taking a subsequence if necessary we may assume $x_i \to x$. Because of (1) this implies

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Then for every neighborhood of zero $V$ there is a compact multifunction $f_V : X \to E$ such that $f_V(X)$ is contained in a finite dimensional subspace of $E$ and that

$$f_V(x) = f(x) + V,$$  

$$p_1(f_V(x)) = p_1(f(x))$$ for every $x \in X.$
Proof. We modify the proof of Ma [6], (3.1).

Let $V$ be a convex neighborhood of zero, $K = f(X)$. Since $p_0$ is continuous ([4], p. 156) there is a neighborhood of zero $U$ such that $p_0(U) \subset V$. By a result of Nagumo [7] there is a continuous (single-valued) function $F_0: K \to E$ such that $x-p_0(x) \in U$ for all $x \in K$ and that $F_0(K)$ is contained in some finite-dimensional subspace $E_0$ of $E$. Now we define a continuous map $Q: K \to E$ by

$$Qy = p_0(F_0(y)) + p_0(y),$$

and we let $f_0(x) = co Q(f(x))$ for $x \in X$.

Since $Q(f(x))$ is a compact subset of the finite-dimensional space $p_0(E_0) \oplus E^1$, $co Q(f(x))$ is also compact (9), Satz 3.10. By the same argument the compactness of $f_0(X)$ can be shown. The upper semicontinuity of $f_0$ can be shown as in the proof of (3.1) in [6].

For each $x \in X$ we have

$$p_1(f_0(x)) = co p_1(Q(f(x))) = co (p_1(f(x))) = p_1(f(x)).$$

If $z \in f_0(x)$ there exist $\mu_i \geq 0$ and $y_i \in f(x)$ ($i = 1, 2, \ldots, n$) such that

$$\sum_{i=1}^n \mu_i = 1 \quad \text{and} \quad z = \sum_{i=1}^n \mu_i y_i.$$

This implies

$$z = \sum \mu_i (p_0(F_0(y_i)) + p_0(y_i)) = \sum \mu_i p_0(F_0(y_i)) + \sum \mu_i y_i \in f(x) + \sum \mu_i p_0(U) = f(x) + V.$$

Therefore $f_0(x) \subset f(x) + V$, and in a similar way we can show $f(x) \subset f_0(x) + V$.

**Lemma 2.** Let $E^0 \subset E$ be a closed hyperplane in $E$, $X \subset E$ and $F: X \to E^0$ a multivalued compact field.

Then for every neighborhood of zero $U$ there exist a finite-dimensional space $E_0 \subset E$ and a multivalued compact field $F_0: X \to E^0$ such that $F_0(x) \subset F(x) + U$ for each $x \in X$ and $F_0(X \cap E_0) \subset E_0^1$, where $E_0^1$ is a hyperplane in $E_0$.

**Proof.** Let $E^1$ be a 1-dimensional subspace of $E$ such that

$$E = E^0 \oplus E^1$$

and let $p_1$ be the projection of $E$ onto $E^1$. Moreover, let $U$ be a convex neighborhood of zero.

Since $f(x) = x - F(x)$ is a compact multifunction, there are by Proposition 3 a finite dimensional space $E_0$ and a compact multifunction $f_0: X \to E_0$ such that

$(+)$

$$f_0(x) \subset f(x) + U \quad \text{and} \quad p_1(f_0(x)) = p_1(f(x)).$$

for every $x \in X$. Obviously we may suppose $E_0 \supset E^1$.

Let $F_0(x) = x - f_0(x)$. $(\ast)$ implies $F_0(x) + U$ for each $x \in X$. Moreover

$$p_1(F_0(x)) = p_1(x) - p_1(f_0(x)) = p_1(x) - p_1(f(x)) = p_1(F(x)) = 0.$$

Therefore $F_0(x) \subset E^0$.

Let $E_0^2 = E_0 \cap E^0$. $E_0^2$ is clearly a hyperplane in $E_0$, and we have for each $x \in X \cap E_0$:

$$F_0(x) = x - f_0(x) \in E_0 + E_0 = E_0^2.$$

Therefore $F_0(X \cap E_0) \subset E_0^1 \cap E_0^2 = E_0^2$.

**Proof of theorem.** Given a neighborhood of zero $N$ we choose a symmetric neighborhood $U$ such that $U^\circ \cup U \subset N$. From Lemma 2 we get a finite-dimensional space $E_0$ and a multivalued compact field $F_0: X \to E^0$ such that

$$F_0(X) \subset F(x) + U \quad \text{for all} \quad x \in X \quad \text{and} \quad F_0(X \cap E_0) \subset E_0^2,$$

where $E_0^2$ is a hyperplane in $E_0$.

Our hypotheses on $X$ imply that

(i) $X \cap E_0$ is compact,

(ii) the component of $O$ in $E_0 \setminus (X \cap E_0)$ is bounded.

Therefore there exist by Lemma 1 points $x_\gamma, y_\gamma \in X \cap E_0$ and a number $\mu_\gamma > 0$ such that

$$y_\gamma = -\mu_\gamma x_\gamma \quad \text{and} \quad F_0(x_\gamma) \cap F_0(y_\gamma) \neq \emptyset.$$

We conclude that

(iii) $F(x_\gamma) \cap (F(y_\gamma) + V) \neq \emptyset$.

Now, since $F$ is a compact field,

$$f(x) = x - F(x)$$

is a compact multifunction. Because of (iii) there are points $a_\gamma \in f(x_\gamma), b_\gamma \in f(y_\gamma)$ such that

(iv) $\gamma \to a_\gamma = b_\gamma + V$.

Let $\gamma$ be the filterbase of all symmetric neighborhoods of zero and consider the nets $(a_\gamma)_x^{\gamma} \gamma$ and $(b_\gamma)_y^{\gamma} \gamma$. Since $f$ is compact we may suppose (taking subnets if necessary) that

$$a_\gamma \to a \quad \text{and} \quad b_\gamma \to b.$$

From (iv) we get

$$(1 + \mu_\gamma)x_\gamma = x_\gamma - y_\gamma \in a_\gamma - b_\gamma + V.$$

Therefore $(1 + \mu_\gamma)x_\gamma \to a - b$. This implies $\mu_\gamma \to +\infty$, because otherwise $x_\gamma \to 0$ which is impossible since $X$ is closed and $O \notin X$. Hence there exists a subnet of $(\mu_\gamma)_\gamma \gamma$ converging to some $\mu \geq 0$. To simplify the notation, this subnet is also denoted by $(\mu_\gamma)$.

\footnote{Fundamenta Mathematicae t. CII}
We conclude

\[ x_{\gamma} - \frac{a - b}{1 + \mu} = : x \in X, \]
\[ y_{\gamma} = -\mu y_{x_{\gamma}} - \mu x = : y \in X. \]

Since \( \text{gr}(F) \) is closed we have \( x - a \in F(x), y - b \in F(y) \). From (iv) we get

\[ x - a = y - b \in F(x) \cap F(y), \]

and the theorem is proved.

References


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Uniform shape and uniform Čech homology and cohomology groups for metric spaces

by

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Abstract. It is proved that the uniform Čech homology and cohomology groups of a metric space, namely those defined by means of all uniform coverings of the space, are invariant with respect to uniform shape equivalence understood in the sense of the paper [3] as well as in the sense of the paper [4].

In paper [3] a concept of uniform shape for metric spaces was introduced. Another notion of uniform shape equivalence, called uniform fundamental equivalence, for complete metric spaces was defined earlier in [4]. The purpose of this paper is to establish two theorems showing that the uniform Čech homology and cohomology groups, namely those defined by means of all uniform coverings of the space, are invariant with respect to uniform shape equivalence understood as either of the notions mentioned above. As these two theorems are proved in essentially the same way, only the proof of one of them will be given in detail.

§ 1. Uniform and double-uniform shapes for metric spaces. Let us recall the definition of uniform shape for metric spaces given in [3]. Every metric space \( X \) can be considered as uniformly embedded in a complete metric space \( M \) which is an absolute uniform neighbourhood extensor for metric spaces — such a space will be called a UANE-space. The family of all open neighbourhoods of \( X \) in \( M \) will be denoted by \( \mathcal{U}(X, M) \).

If \( X \) and \( Y \) are subsets of the UANE-spaces \( M \) and \( N \), respectively, then a uniform shape map

\[ f: \mathcal{U}(X, M) \to \mathcal{U}(Y, N) \]

is a collection of uniformly continuous maps \( f: U \to V \), where \( U \in \mathcal{U}(X, M) \), \( V \in \mathcal{U}(Y, N) \), provided that the following conditions are satisfied:

a) for every \( V \in \mathcal{U}(Y, N) \) there exist a \( U \in \mathcal{U}(X, M) \) and a \( f: U \to V \) with \( f \in f \);

b) if \( f \in f, f: U \to V, U' \in U, V' \supset V \), \( U' \in \mathcal{U}(X, M) \), \( V' \in \mathcal{U}(Y, N) \), and if \( f' = ff |_{V'} \), where \( j: V' \to V \) is the inclusion map, then \( f' \in f \);