proper subset of A is an arc. Hence, by (6.1), A is a confluent image of $(-\infty, +\infty)$. However, as is known, A is not embeddable in the plane. It would be interesting to know exactly which confluent images of a half-line, or of a line, are embeddable in the plane. However, this is not known for one-to-one images of $[0, +\infty)$ as in (4) of (5.1).

(2) By (4) of (5.1), each nondegenerate closed connected proper subset of an indecomposable confluent image of $[0, +\infty)$ is an arc. The corresponding statement, for confluent images of $(-\infty, +\infty)$, is false. For example, let Γ denote the composant "you see" of the continuum in Example 1 of [3, pp. 204–205]. Then, Γ satisfies (4) of (5.1). Now, delete the origin from Γ and denote the new space by Γ_0,

$$\Gamma_0 = \Gamma\setminus\{(0,0)\}.$$

Then, it can be seen that each closed connected proper subset of Γ_0 is an arc or a half-line. Hence, since Γ_0 is arcwise connected, Γ_0 is a confluent image of $(-\infty, +\infty)$ by (6.1). Also, Γ_0 has a closed subset which is a half-line.

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Capacitability and determinacy

by

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Abstract. We establish a conjecture of Mycielski and show that the Axiom of Determinacy implies that every set in three-dimensional Euclidean space is capacitable. The capacitability of projective sets follows from projective determinacy, but the case of Σ_2^1 sets requires no determinacy at all, but only some such weaker assumption as the existence of a measurable cardinal.

Introduction. This paper explores the analogy between (Newtonian) *capacity* and Lebesgue measure. Mycielski and Świerczkowski proved that the Axiom of Determinacy (AD) implies that every set of real numbers is Lebesgue measurable [11] (1). Of course this has to be in the absence of the Axiom of Choice (AC), in view of the classical derivation due to Vitali of a non-measurable set from AC. Also, Solovay showed that even without AD, it is at least consistent that every set of reals be Lebesgue measurable [17]. The model that he obtains by forcing which satisfies this property, also satisfies the principle of Dependent Choice (DC):

$$(\forall \alpha \in X)(\exists \beta)\langle \alpha, \beta\rangle \in A \Rightarrow (\exists f)(\forall n)\langle f(n), f(n+1)\rangle \in A .$$

Now the notion of (Newtonian) *capacitability* (of sets in three-dimensional Euclidean space R^3) has certain analogies with Lebesgue measurability. For example, Choquet [1] showed that analytic sets are capacitable. Accordingly, Mycielski conjectured that in Solovay's model, all sets in R^3 might be capacitable ([17], p. 2, Remark 5), and also indicated how this might be proved, via the proposition BC (2) (for "Borel Choice"):

Let X and T be complete separable metric spaces, μ a non-negative, non-atomic Borel measure on X. If $U \subseteq X\times T$, then there is a Borel set $B\subseteq X$ and a Borel measur-

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(1) We have not given a statement of AD in the paper, since we do not use it directly. [3] is a recent comprehensive survey article.

(2) This formulation of BC was conveyed to the author privately by Professor Mycielski. The version stated in [17] occurs as Proposition 5(a) of Theorem 1 on page 1. This is the special case for $X = T =$ the real numbers, and $\mu =$ Lebesgue measure. However, as Mycielski has pointed out, this case and the general case are both equivalent to the special case when X is the Baire space, in view of the reduction mentioned just before Theorem 1 in this paper.

able function f: $B \to T$ such that $f \subseteq U$ and $\pi(U) \setminus B$ has (exterior) μ-measure 0, where π: $X \times T \to X$ is the projection map on X.

We shall establish Mycielski's conjecture by showing that BC does imply that all sets in R^3 are capacitable. However Solovay has also shown that AD implies BC. Since it has not been published elsewhere we reproduce his proof in the appendix to this paper. Thus our main result Theorem 1 follows once we show that BC implies all sets in R^3 are capacitable.

Technical preliminaries. We shall work throughout in ZF+DC ([3]). In view of Solovay's model this is demonstrably consistent with BC, and it is believed to be consistent with AD. All the results of potential theory and other "ordinary mathematics" that we draw upon require only DC and not the full AC. The projective hierarchy of sets in a complete separable metric space is defined in [9], but we shall usually use the modern notation Δ_1^1, Σ_1^1, Π_1^1, Δ_2^1 etc. The notion of capacity arises in potential theory. We sketch very briefly the background and terminology that we need. Details may be found in [5]. If D is an open subset of the Euclidean space R^n, $n \geqslant 2$, there is an associated Green function G, with certain exceptions in the case $n = 2$. For definiteness we take $n = 3$ and $D = R^3$ throughout. Thus our capacity will be the particular case of *Newtonian* capacity. Let μ be any non-negative measure on D. Then μ and G together determine a potential function, written $G\mu$, on D. For any compact $K \subseteq D$ there is an associated *equilibrium potential* V_K and *equilibrium distribution* μ_K related by the equation $V_K = G\mu_K$. The *capacity* of K, written $c(K)$, is simply the total "charge" of μ_K: $c(K) = \mu_K(K)$. For arbitrary $A \subseteq D$ the interior and exterior capacities are defined as:

$$c_*(A) = \sup\{c(K) |\ K \subseteq A,\ K \text{ compact}\},$$

$$c^*(A) = \inf\{c_*(U) |\ U \supseteq A,\ U \text{ open}\},$$

and A is *capacitable* if $c_*(A) = c^*(A)$. As long as A is capacitable, an equilibrium potential can still be defined: there are compact subsets and open supersets:

$$K_1 \subseteq K_2 \subseteq \ldots \subseteq A \subseteq \ldots \subseteq G_2 \subseteq G_1$$

such that $V_A(p) = \lim_{n\to\infty} V_{K_n}(p)$ for all p in R^3, and $V_A(p) = \lim_{n\to\infty} V_{G_n}(p)$ holds *quasi-everywhere*, i.e. everywhere except possibly on a set of exterior capacity 0 (and everywhere if A is σ-compact).

Now we can state the famous theorem of Doob [2] which gives a probabilistic interpretation of V_A. Let A be capacitable, so that V_A is defined, and let X be the space of Brownian trajectories issuing from some arbitrarily fixed p in R^3. X is the set of continuous functions x: $T \to R^3$, $T = [0, \infty)$, subject to the condition $x(0) = p$, endowed with the topology of uniform convergence on compact subsets. Let μ be the Wiener measure on X [7], [8]. Then if we write $\Lambda(A, p)$ for the set of Brownian trajectories issuing from p and meeting A at some time $t > 0$, then $\Lambda(A, p)$ is μ-measurable, and the equation $V_A(p) = \mu(\Lambda(A, p))$ holds quasi-everywhere with respect to p. Intuitively: the potential due to A at p = probability that a particle executing Brownian motion starting from p will meet A in a finite time. By [10] pp. 93–94 X is a complete separable metric space. μ is defined on Borel sets in X, and it is non-negative, non-atomic and normalized [7], [8]. Thus the hypotheses of [12] Theorem 2 are satisfied, and the measure μ on X can be reduced to the Lebesgue measure λ on the Baire space ${}^\omega\omega$ in the following sense: there is a homeomorphic embedding h: ${}^\omega\omega \to X$ and the measure induced on X by λ via h coincides with μ. For any $A \subseteq X$, A is μ-measurable in X iff $h^{-1}(A)$ is λ-measurable in ${}^\omega\omega$, and then $\mu(A) = \lambda(h^{-1}(A))$.

(3) A standard exposition of ZF may be found in [15].

THEOREM 1. AD *implies that any set in* R^3 *is capacitable.*

Proof. Solovay's proof that AD$\Rightarrow$BC is given in the appendix. We shall work from BC. Let A be any bounded set in R^3. We shall eventually remove this assumption of boundedness, but in this case we can easily arrange for A to be entirely contained in the interior of some closed ball S. Take p to be any point not in S, let X and T be as above. $X \times T$, with the product topology, is again a complete separable metric space, so projective and topological terms can be applied to its subsets. Let Θ: $X \times T \to R^3$ be the evaluation map $\Theta(x, t) = x(t)$. Θ is continuous, as is also the projection map π: $X \times T \to X$ where $\pi(x, t) = x$. Note that BC implies that all sets in ${}^\omega\omega$ are Lebesgue measurable [17]. Thus all sets in X are μ-measurable with respect to the Wiener measure μ, by the reduction of (X, μ) to $({}^\omega\omega, \lambda)$.

LEMMA. *Let* $U \subseteq X \times T$. *Then there is σ-compact set* $H \subseteq U$ *such that* $\mu(\pi(U)) = \mu(\pi(H))$.

Proof. By BC there is a Borel set $B \subseteq \pi(U)$ and a Borel measurable function f: $B \to T$ such that $f \subseteq U$ and $\mu(\pi(U)) = \mu(B)$. By Lusin's Theorem [13], for any $\varepsilon > 0$ there is a compact set $F \subseteq B$ such that $\mu(B \setminus F) < \varepsilon$ and $f \restriction F$ is continuous. Then let $K = \{\langle x, f(x)\rangle |\ x \in F\}$. K is a compact set in $X \times T$, and $\mu(\pi(U)) - \mu(\pi(K)) < \varepsilon$. Now let ε run through some countable sequence diminishing to zero, and let H be the union of the associated K's.

Now $\pi(\Theta^{-1}(A))$ is just the set of Brownian trajectories starting at p which meet A at some time $t \geqslant 0$; or equivalently, $t > 0$, since p is separated by S from A. So $\pi(\Theta^{-1}(A))$ is the same set that we have also called $\Lambda(A, p)$. We are trying to show that A is capacitable, but suppose for a moment that we already had. Doob's Theorem would apply and we would have $V_A(p) = \mu(\pi(\Theta^{-1}(A)))$. However, even without the knowledge that A is capacitable the right hand side of this equation is defined, since $\pi(\Theta^{-1}(A))$ is measurable in any case. (The possibility that p might be an exceptional point where $V_A(p) \neq \mu(\pi(\Theta^{-1}(A)))$ can be shown to be precluded by the fact that p is separated from A by S.)

We now proceed to enclose A between capacitable approximating sets M, N. To get M we apply the lemma to $U = \Theta^{-1}(A)$, then set $M = \Theta(H)$, where H is the σ-compact subset of U given by the lemma. Clearly M is σ-compact, since the continuous image of a compact set is compact. Now $H \subseteq \Theta^{-1}(A)$, so $\Theta(H) = M \subseteq A$, so $\Theta^{-1}(A) \supseteq \Theta^{-1}(M) \supseteq H$. Thus

$$\mu(\pi(\Theta^{-1}(A))) = \mu(\pi(\Theta^{-1}(M))) = \mu(\pi(H)).$$

Next we define a G_δ set $N \supseteq A$ such that

$$\mu(\pi(\Theta^{-1}(A))) = \mu(\pi(\Theta^{-1}(N))).$$

Consider the set $C = X \setminus \pi(\Theta^{-1}(A))$. Since BC implies every subset of X is μ-measurable, or alternatively by applying the lemma to $C \times T$, C has a σ-compact subset P such that $\mu(P) = \mu(C)$. Supposing P is $\bigcup_{i=1}^{\infty} P_i$, each P_i compact, define $P_{ij} = P_i \times [0, j]$. Then $\langle P_{ij} \rangle_{i,j}$ is a sequence of compact subsets of $P \times T \subseteq C \times T$, and $P \times T = \bigcup_{i,j} P_{ij}$. Now put $Z = \Theta(P \times T)$. Z is σ-compact in R^3. Finally take $N = (R^3 \setminus Z) \cap \operatorname{Int} S$. (We chose S so that $A \subseteq \operatorname{Int} S$.) N is obviously a G_δ. Since $P \times T$ is disjoint from $\Theta^{-1}(A)$, $\Theta(P \times T) \subseteq R^3 \setminus A$, whence $N \supseteq A$. This implies immediately that

$$\mu(\pi(\Theta^{-1}(N))) \geqslant \mu(\pi(\Theta^{-1}(A))).$$

On the other hand, since $\Theta^{-1}(Z) \supseteq P \times T$, it follows that $\Theta^{-1}(N) \supseteq X \times T \setminus P \times T$. Thus $\pi(\Theta^{-1}(N)) \supseteq X \setminus P$. But although $X \setminus P$ may properly contain $\pi(\Theta^{-1}(A))$, it has the same μ-measure, since $\mu(P) = \mu(C)$, and $C = X \setminus \pi(\Theta^{-1}(A))$. It follows that

$$\mu(\pi(\Theta^{-1}(N))) \leqslant \mu(\pi(\Theta^{-1}(A))),$$

and thus

$$\mu(\pi(\Theta^{-1}(A))) = \mu(\pi(\Theta^{-1}(N))).$$

Thus $M \subseteq A \subseteq N$, and by applying Doob's Theorem to M and N,

$$V_M(p) = V_N(p) = \mu(\pi(\Theta^{-1}(A))).$$

Note that there are no exceptional points as regards the application of Doob's Theorem, since M is σ-compact, and p is separated from N as from A by S. Clearly in order to show that A is capacitable it will suffice to show $c(M) = c(N)$. This follows immediately from what we have done already using details from ordinary potential theory. One can show that the potentials V_M, V_N are each generated by unique measures μ_M and μ_N, just as for compact sets, using the Riesz Decomposition Theorem for superharmonic functions [5], so that $V_M = G\mu_M$ and $V_N = G\mu_N$. Then $c(M) = \mu_M(M)$ and $c(N) = \mu_N(N)$. Since $M \subseteq N \subseteq \operatorname{Int} S$, μ_M and μ_N each have their support inside S, and V_M and V_N are both harmonic throughout $R^3 \setminus S$. But since $V_M \geqslant V_N$, the difference is also a non-negative harmonic function throughout $R^3 \setminus S$. It takes the value 0 at p, a non-boundary point, and hence must be 0 throughout the domain, i.e. $V_M = V_N$ everywhere in $R^3 \setminus S$. Now apply Gauss's Theorem. Let Σ be a large ball of radius δ such that $\Sigma \subseteq R^3 \setminus S$. Then

$$\text{average of } V_M \text{ over } \Sigma = \frac{\mu_M(M)}{\delta} = \frac{\mu_N(N)}{\delta} = \text{average of } V_N \text{ over } \Sigma.$$

Thus $c(M) = c(N)$.

Finally, to remove the restriction that A be bounded, we use exactly the same argument as that used in [6] pp. 185–187 to get the capacitability of unbounded analytic sets from the bounded case. First one shows that unbounded open sets are capacitable. Then if $\langle A_n \rangle$ is a sequence of capacitable sets, and if for each n $G_n \supseteq A_n$ is an open set, one has

$$c\big(\bigcup_n G_n\big) - c_*\big(\bigcup_n A_n\big) \leqslant \Sigma_n\big(c(G_n) - c(A_n)\big).$$

Given an unbounded set A, put $A_n = A \cap S_n$, S_n being the closed ball of radius n centered at the origin. Then A_n is capacitable. Let G_n be an open set containing A_n such that $c(G_n) - c(A_n) < \varepsilon/2^n$. Then

$$c\big(\bigcup_n G_n\big) - c_*(A) \leqslant \varepsilon$$

and clearly A is capacitable.

We remark that in view of the proof of Theorem 1, projective determinacy implies the capacitability of projective sets, since it is a simple matter to keep track of the projective degree throughout the argument of Solovay's proof from AD to BC, and from BC to capacitability. But BC and hence AD can be dispensed with in the case of Σ_2^1 sets, since we can use the Kondo–Addison Theorem instead, together with the assumption that all Σ_2^1 sets in ${}^{\omega}\omega$ are Lebesgue measurable. This latter assumption seems rather unnatural as the hypothesis of a theorem, so we take instead the assumption that there is a measurable cardinal [16]. Solovay has shown [18] that this implies that Σ_2^1 sets are Lebesgue measurable.

THEOREM 2. *Assume that there is a measurable cardinal. Then all Σ_2^1 sets in R^2 are capacitable.*

Proof. Let A be a bounded Σ_2^1 set in X. (The boundedness can be removed just as before.) Then $\Theta^{-1}(A)$ is also Σ_2^1, since Θ is continuous, and so again is $\pi(\Theta^{-1}(A))$, since π is a projection. Thus $\pi(\Theta^{-1}(A)) = \pi(Q)$ for some Π_1^1 set $Q \subseteq X \times T$. Now apply the Kondo–Addison Theorem [15] to "uniformize" Q: there is an $f \subseteq X \times T$ such that

(i) $\pi(f) = \pi(Q) = \pi(\Theta^{-1}(A))$,

(ii) f is Π_1^1,

(iii) f is a function on $\pi(\Theta^{-1}(A))$.

Since f is Π_1^1, the f-preimage of an arbitrary open set in T will be Σ_2^1 and thus measurable, so f is a measurable function. All Σ_2^1 sets in X are μ-measurable, by the reduction of (X, μ) to $({}^{\omega}\omega, \lambda)$: the h-preimage of a Σ_2^1 set in X will be Σ_2^1 in ${}^{\omega}\omega$ as h is continuous, and such sets are all Lebesgue measurable since there is a measurable cardinal. $\pi(\Theta^{-1}(A))$ is Σ_2^1, so it is measurable. Lusin's Theorem applies to f and

$\pi(\Theta^{-1}(A))$ as before, and the construction of M is the same. As for N: since $\pi(\Theta^{-1}(A))$ is measurable $C = X\setminus\pi(\Theta^{-1}(A))$ is too, thus the construction of N is just as before, since it depends only on the fact that C is measurable.

Remarks.

(1) D. D. Shochat [14], also working from suggestions of Mycielski, has shown independently that Theorem 2 holds, not just for Newtonian capacity but also for the Choquet capacities which are "alternating of order ∞" (defined in [1] § 30.1). Mycielski has announced a proof of the corresponding generalization to Choquet capacities of Theorem 1 (Notices of the American Mathematical Society, Vol. 19, No. 7, pA-765, 72T-E 104), unfortunately without giving details.

(2) The analogy between capacitability and measurability fails in one important respect: the complement of a capacitable set need not be capacitable. Thus although Choquet showed that Σ_1^1 sets are capacitable, not only for Newtonian capacity but for his wider class of capacities alternating of order ∞, Π_1^1 sets need not be. In fact, assuming V = L [4], Choquet showed that there is a Π_1^1 set which is not f-capacitable for a certain Choquet capacity, and also that there is a Σ_2^1 set which is not Newtonian capacitable ([1] §§ 33.1, 34.) If V = L^D [16] the same constructions yield respectively a Π_2^1 and a Σ_3^1 set. It is an open question whether assuming V = L/V = L^D there is a Π_1^1/Π_2^1 set which is not Newtonian capacitable.

Appendix. With his kind permission we reproduce Solovay's proof that AD$\Rightarrow$BC. It is a generalization of the Mycielski–Świerczkowski proof that AD implies all sets are Lebesgue measurable. He takes BC in the form applying to the product space $[0,1]\times{}^{\omega}2$, but this is equivalent to the formulation in the paper, in view of Footnote (2).

LEMMA 1. *If* BC *fails, there is a set U with the following properties*:

(1) $U\subseteq[0,1]\times{}^{\omega}2$.

(2) *For each x in* $[0,1]$ *there is a* $y\in{}^{\omega}2$ *such that* $\langle x,y\rangle\in U$.

(3) *Let* $K\subseteq[0,1]$ *be compact and of positive measure.*

Let $f\colon K\to{}^{\omega}2$ *be continuous. Then for some* $x\in K$, $\langle x,f(x)\rangle\notin \underline{U}$.

We fix a U as provided in Lemma 1. Let $\langle r_i\colon i=1,2,3,\ldots\rangle$ be a decreasing sequence of positive rationals, less than $\frac{1}{4}$, such that $\sum_{i=1}^{\infty} r_i<\infty$. Let J_n be defined as in [11]. Let $S_0=[0,1]$.

We define a game G:

If $n=2k+1$, player I picks $S_n\subseteq S_{n-1}$ with $S_n\in J_n$, and an integer $y_k\in 0,1$. If $n=2k+2$, player II picks $S_n\subseteq S_{n-1}$. A play of the game determines $x\in[0,1]$ such that $\{x\}=\bigcap_{n=1}^{\infty}S_n$ and $y=\langle y_0,y_1,\ldots\rangle\in{}^{\omega}2$. I wins if and only if $\langle x,y\rangle\in U$.

LEMMA 2. *I does not win G.*

Proof. As in [11] construct from a winning strategy for I a pair K, f contradicting property (3) of U.

LEMMA 3. *II does not win G.*

Proof. We fix a strategy σ for II. For each finite sequence s of 0's and 1's we can define a subset A_s of $[0,1]$ such that (1)-(3) below hold. (This is done as in [11].)

(1) $s\supseteq t$ implies $A_s\subseteq A_t$.

(2) If length(s) = length$(t)+1$, and length $(t)=n$

$$\mu(A_s)\geqslant(1-2r_k)\mu(A_t)$$

where $k=2n+1$.

(3) Take $y\in{}^{\omega}2$, and let $x\in\bigcap_{n=0}^{\infty}A_{y\restriction n}$. Then there is a play of the game G in which II plays according to the postulated strategy σ that he has, and the outcome is $\langle x,y\rangle$.

Now set $A=\bigcap_s A_s$. By (1) and (2),

$$\mu(A)\geqslant\prod_{k=1}^{\infty}(1-2r_k)>0\,.$$

By (3), if $x\in A$ and $y\in{}^{\omega}2$ then $\langle x,y\rangle$ is a possible outcome of a play of G with II playing according to his strategy σ. But then by property (2) of U, σ cannot be a winning strategy for II. For let $x\in A$ and y be given by property (2) of U so that $\langle x,y\rangle\in U$. Then $\langle x,y\rangle$ is a possible outcome of a play of G with II using σ, and if σ were a winning strategy we would have $\langle x,y\rangle\notin U$.

From Lemmas 1–3 the implication AD$\Rightarrow$BC is clear (Solovay has also proved the analogous result where the Baire property replaces Lebesgue measurability).

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A theorem of Borsuk–Ulam type for multifunctions

by

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Abstract. We prove a generalization of a theorem of Borsuk–Ulam type of Joshi to multifunctions mapping a non necessarily symmetric subset of a locally convex space into a closed hyperplane.

§ 1. Introduction. We shall proove the following generalization of a theorem of Borsuk–Ulam type of Joshi [3]:

THEOREM. *Let E be a separated locally convex topological vector space, X a closed and finitely bounded subset of E such that $O \notin X$ and that the component of O in $E \setminus X$ is finitely bounded. Then, for every compact multivalued vector field F mapping X into a closed hyperplane E^0 of E there are points $x, y \in X$ and a number $\mu > 0$ such that*

$$y = -\mu x \quad \textit{and} \quad F(x) \cap F(y) \neq \emptyset .$$

We start by explaining the terminology.

In the following let E be a separated locally convex vector space. A subset of E is called *finitely bounded* iff it has a bounded intersection with every finite-dimensional subspace of E.

If X is a topological space, a mapping f from X into the set 2^E of all nonvoid compact convex subsets of E is called a *multivalued function*, or *multifunction*, of X into E, written $f\colon X \to E$. Moreover f is called *compact* iff

(1) f is upper semicontinuous (u.s.c.), i.e. for every $x \in X$ and every open set $O \supset f(x)$ there is a neighborhood N of x such that

$$f(N) = \bigcup_{x \in N} f(x) \subset O ,$$

(2) $f(X)$ is relatively compact in E.

A multifunction $F\colon X \to E$ is called a *multivalued compact field* iff $f(x) = x - F(x)$ is a compact multifunction.

We shall extend the following result of Joshi [3]:

PROPOSITION 1. *Let E be a Banach space and let X be a closed, bounded subset of E such that $O \notin X$ and that the component of O in $E \setminus X$ is bounded. Then for any*