The metric confluent images of half-lines and lines

by

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Abstract. The metric spaces which are confluent images of $[0, +\infty)$ and $(-\infty, +\infty)$ are completely delineated. Thus, the first time, the confluent images of specific non-compact spaces have been determined.

1. Introduction. Let X and Y be metric spaces. A mapping (= continuous function) $f\colon X \overset{\text{onto}}{\to} Y$ is said to be *confluent* provided that if K is any closed connected subset of Y and C is any component of $f^{-1}[K]$, then $f[C] = K$ (cf., [1]). The first paper on confluent mappings is [1], where confluent mappings between continua (= compact connected metric spaces) were studied. A weaker type of confluence for general spaces was introduced and studied in [4].

Though some classes of spaces are known to be invariants of confluence (see, for example, [2, Theorem 13, p. 33]), the confluent images of specific spaces are, by and large, not precisely known even for continua. It is known that the confluent image of an arc is an arc or point [2, Corollary 20, p. 32] and, in [7], the confluent images of the sinusoidal curve are determined.

The simplest connected metric spaces which are not continua are half-lines (i.e., homeomorphs of $[0, +\infty)$) and lines (i.e., homeomorphs of $(-\infty, +\infty)$). *The purpose of this paper is to determine precisely all the metric confluent images of half-lines and of lines*. This is done in three basic steps. First, I determine all locally compact metric spaces which are confluent images of half-lines or lines (Section 3). Then I determine all the metric confluent images of a half-line (Section 5). Third, I determine all the metric confluent images of a line (Section 6).

The results in Sections 2 and 4 are used to obtain the main results. However, many of them are of interest in themselves. For example, some results (resp., Section 4) reveal a good bit of information about the behavior of confluent mappings on half-lines and lines. Other results give more details about structure than is in the main results.

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One of the main tools which is employed in this paper is one-to-one mappings of half-lines and lines. Use of such mappings is made possible by Lemma 2.2. In fact, it is the structure of locally compact one-to-one continuous images of half-lines and lines, contained in [8] and [10], which is the basic ingredient in the proofs of (3.1) and (3.2). The following aspect of the results in Section 3 seems interesting. For continua, confluent mappings are a generalization of open mappings [12, (7.5), p. 148]. However, in general there is no direct relationship between open mappings and confluent mappings. For example, it is easy to find open (resp., confluent) mappings of $[0, +\infty)$ onto an arc which are not confluent (resp., open). Despite these facts, it ends up (see (3.1) and (3.2) below and see [11, p. 39]) that the locally compact metric spaces which are confluent images of $[0, +\infty)$ (resp., of $(-\infty, +\infty)$) are *precisely* the open images of $[0, +\infty)$ (resp., of $(-\infty, +\infty)$)!

Throughout this paper, the word *nondegenerate* means consisting of more than one point. A *simple triod* is a continuum which is homeomorphic to a figure "**T**". The symbol S^1 will denote the unit circle in the plane. By a *circle* we mean any homeomorph of S^1. The symbol "cl" denotes closure and the slash "\" denotes complementation for sets.

2. Some preliminary lemmas. A metric space Y is said to be *a-triodic in the generalized sense*, or simply *a-triodic*, provided Y does not contain three closed connected subsets A_1, A_2, and A_3 such that

$$A_1 \cap A_2 \cap A_3 \neq \emptyset$$

and such that if $\{i, j, k\} = \{1, 2, 3\}$, then

$$A_i \not\subset [A_j \cup A_k].$$

The following lemma, though not stated in the literature, seems to be known at least for continua (see the paragraph immediately following Problem 5 of [5, p. 94]). We include a proof of it for completeness.

(2.1) Lemma. *If X is an a-triodic metric space and if f is a confluent mapping of X onto a metric space Y, then Y is a-triodic.*

Proof. Let $f\colon X \xrightarrow{\text{onto}} Y$ be a confluent mappings. Assume Y contains three closed connected subsets A_1, A_2, and A_3 such that

$$A_1 \cap A_2 \cap A_3 \neq \emptyset$$

and such that, if $\{i, j, k\} = \{1, 2, 3\}$, then

$$A_i \not\subset [A_j \cup A_k].$$

Let $p \in [A_1 \cap A_2 \cap A_3]$. Let B be a component of $f^{-1}(p)$. For each $n \in \{1, 2, 3\}$, let B_n be the component of $f^{-1}[A_n]$ containing B. Since f is confluent,

$$f[B_n] = A_n \quad \text{for each } n \in \{1, 2, 3\}.$$

Hence, it follows that if $\{i, j, k\} = \{1, 2, 3\}$,

$$B_i \not\subset [B_j \cup B_k].$$

Since $B \subset B_n$ for each $n \in \{1, 2, 3\}$,

$$B_1 \cap B_2 \cap B_3 \neq \emptyset.$$

It now follows that X is not a-triodic.

The following lemma indicates how a-triodicity will be used.

(2.2) Lemma. *If an arcwise connected metric space Y is a confluent image of an a-triodic metric space X, then Y is a one-to-one continuous image of a connected subset of $(-\infty, +\infty)$.*

Proof. By (2.1), Y does not contain a simple triod. In [6, (3.2)] I proved that an arcwise connected metric space which contains no simple triod is a one-to-one continuous image of a connected subset of $(-\infty, +\infty)$.

Next, I give two technical lemmas concerning certain one-to-one images.

(2.3) Lemma. *If a locally compact metric space Y is a one-to-one continuous image of $[0, +\infty)$, then one (and only one) of the following holds:*

(2.3.1) *Y is a half-line or a circle;*

(2.3.2) *Y contains a simple triod or a subcontinuum which is not arcwise connected:*

Proof. If Y is not compact, then Y is a half-line by Theorem 7.1 of [10, p. 69]. So, for the purpose of proof, assume Y is compact. Then, by the Structure Theorem in [8, p. 128], Y contains an arcwise connected circle-like continuum Σ. Assume Σ is not a circle. Then, by Theorem 6 of [9, pp. 230–231], Σ contains a (chainable) continuum which is not arcwise connected. Next assume Σ is a circle. Then, if $\Sigma \neq Y$, it follows from the arcwise connectivity of Y that Y contains a simple triod.

(2.4) Lemma. *If a locally compact metric space Y is a one-to-one continuous image of $(-\infty, +\infty)$, then one (and only one) of the following holds:*

(2.4.1) *Y is a line;*

(2.4.2) *Y contains a triod or a closed connected subset which is not arcwise connected.*

Proof. First assume Y is compact. Then, by part (1) of the Structure Theorem for Real Curves [10, p. 9], it follows easily that Y contains a triod. Next assume Y is not compact and not a line. Then, using Remark 7.1 of [10, p. 72], it follows easily that:

(1) If Y is as in (1) of Remark 7.1, then Y contains a closed connected subset which is not arcwise connected.

(2) If Y is as in (2) of Remark 7.1, then Y contains a triod.

(2.5) LEMMA. *If a metric space Y is a confluent image of a connected subset L of $(-\infty, +\infty)$, then each closed connected subset of Y is arcwise connected.*

Proof. Since any connected subset of L is arcwise connected, the lemma is a simple consequence of the definition of confluence.

The final lemma of this section gives a simple fact about the behavior of confluent maps of $[0, +\infty)$. An especially interesting application of it occurs in the proof of (4.6).

(2.6) LEMMA. *If f is a confluent mapping of $[0, +\infty)$ onto a metric space Y, then $f(0)$ is not a cut point of any arc in Y.*

Proof. Assume $f(0) = p$ is a cut point of an arc A in Y. Let K_0 be the component of $f^{-1}(p)$ containing zero, and let $s_0 = \text{l.u.b.}[K_0]$, i.e., $K_0 = [0, s_0]$. Let e_1 and e_2 denote the two noncut points of A. For each $i \in \{1, 2\}$, let A_i denote the subarc of A with noncut points p and e_i. By the continuity of f, there exists $\delta > 0$ such that $e_i \notin f([s_0, s_0+\delta]) = f([0, s_0+\delta])$ for any $i \in \{1, 2\}$. From the definition of s_0 it follows that there exists t_0 such that $s_0 < t_0 \leqslant s_0+\delta$ and $f(t_0) \neq p$. Now, choose an A_i such that $f(t_0) \notin A_i$, and denote such a choice by A_j. Let M_0 be the component of $f^{-1}[A_j]$ containing s_0. Clearly, $M_0 = [0, s_1]$ for some $s_1 < t_0$. Therefore, since $e_j \in [A_j \setminus f([0, s_1])]$, we have that $f[M_0] \neq A_j$. Hence, f is not confluent.

3. The locally compact confluent images of a half-line and a line. The two theorems below are the first main results of this paper.

(3.1) THEOREM. *A locally compact metric space Y is a confluent image of $[0, +\infty)$ if and only if Y is a one-point space, an arc, or a half-line.*

Proof. Assume Y is a confluent image of $[0, +\infty)$. Then, by (2.2), Y is a one-to-one continuous image of a connected subset L of $(-\infty, +\infty)$. For the purpose of proof, assume L is not a compact interval. Then Y satisfies the hypotheses of (2.3) or (2.4), depending on whether L is a half-open interval or an open interval. Also, by (2.1) and (2.5), neither (2.3.2) nor (2.4.2) holds. Therefore, (2.3.1) or (2.4.1) holds. By (2.6), Y is not a circle and (2.4.1) does not hold. Hence, Y is a half-line (the other possibility in (2.3.1)). This proves the first half of (3.1). The converse is easy since $f\colon [0, +\infty) \overset{\text{onto}}{\to} [-1, +1]$ given by $f(t) = \cos[t]$ is an example of a confluent mapping onto an arc.

(3.2) THEOREM. *A locally compact metric space Y is a confluent image of $(-\infty, +\infty)$ if and only if Y is a one-point space, an arc, a half-line, a line, or a circle.*

Proof. Assume Y is a confluent image of $(-\infty, +\infty)$. Using (2.1) through (2.5) as in the proof of (3.1), we see that Y is a one-point space or an arc (if L is a compact interval), or that (2.3.1) or (2.4.1) holds. Hence, Y must be one of the five spaces in the statement of (3.2). Conversely, it is easy to see that each of the five spaces is a confluent image of $(-\infty, +\infty)$. For example: The function

$$g\colon (-\infty, +\infty) \overset{\text{onto}}{\to} [-1, +1]$$

given by $g(t) = \sin[t]$ is a confluent mapping onto an arc, the absolute-value function defined on $(-\infty, +\infty)$ is a confluent mapping onto a half-line, and the function $k\colon (-\infty, +\infty) \overset{\text{onto}}{\to} S^1$ given by $k(t) = (\cos[t], \sin[t])$ is a confluent mapping onto a circle.

4. More preliminary lemmas. This section is devoted to giving information which will be used to obtain the characterization in Section 5 of all the confluent images of $[0, +\infty)$. For this purpose and *throughout this section let Z denote any metric space for which there is a confluent mapping, denoted by f, of $[0, +\infty)$ onto Z. Also, let*

$$Z_+(f) = \bigcap_{n=1}^{\infty} \text{cl}[f([n, +\infty))].$$

It is simple to prove, using an argument involving sequences, that

(4.0) $Z_+(f)$ is always a closed subset of Z, and if Z is compact, then $Z_+(f)$ is connected.

Now, I determine some other facts about $Z_+(f)$.

(4.1) LEMMA. *If there exist $s_1 < s_2$ such that $s_1 \in f^{-1}[Z_+(f)]$ and $s_2 \notin f^{-1}[Z_+(f)]$, then Z is an arc.*

Proof. Assume that $s_1 < s_2$ such that $s_1 \in f^{-1}[Z_+(f)]$ and $s_2 \notin f^{-1}[Z_+(f)]$. Then, $f^{-1}[f(s_2)]$ has an upper bound. Let t_0 be an upper bound for $f^{-1}[f(s_2)]$ such that $t_0 \notin f^{-1}[f(s_2)]$. Let

(a) $M = \text{cl}[f([t_0, +\infty))]$ or equivalently,

(b) $M = f([t_0, +\infty)) \cup Z_+(f)$.

It is easy to see by (a) that M is a closed connected subset of Z and, by (b), that $s_1 \in f^{-1}[M]$. Let C denote the component of $f^{-1}[M]$ such that $s_1 \in C$. By confluence of f we have $f[C] = M$. From the way t_0 was defined and from (b) it follows that $s_2 \notin f^{-1}[M]$. Hence, $s_2 \notin C$. Thus, since $s_1 < s_2$ and $s_1 \in C$, we have that $C \subset [0, s_2]$. Therefore,

$$M = f[C] \subset f([0, s_2])$$

which implies that

$$Z \subset f([0, t_0]).$$

This proves that $Z = f([0, t_0])$. Hence, Z is a compact confluent image of $[0, +\infty)$. Furthermore, since $f^{-1}[Z_+(f)] \neq [0, +\infty)$ (recall that $s_2 \notin f^{-1}[Z_+(f)]$), Z is not a one-point space. Therefore, by (3.1) Z is an arc.

(4.2) COROLLARY. *There exists a closed connected subset A of $[0, +\infty)$ such that $f[A] = Z_+(f)$. Furthermore, if Z is not compact, $Z_+(f) = \varnothing$ or $f^{-1}[Z_+(f)] = [r_0, +\infty)$ for some $r_0 \in [0, +\infty)$.*

Proof. First, assume Z is compact. Then, by (4.0), $Z_+(f)$ is a compact connected subset of Z. Thus, letting A be any component of $f^{-1}[Z_+(f)]$, we see using the confluence of f that A satisfies the conditions in (4.2). Second, assume Z is not compact. If $Z_+(f) = \emptyset$, then $A \neq \emptyset$ satisfies the conditions in (4.2). So, for the purpose of proof, assume $Z_+(f) \neq \emptyset$. Let

$$r_0 = \text{g.l.b.}(f^{-1}[Z_+(f)]).$$

Since $Z_+(f)$ is a closed subset of Z [see (4.0)], we have that $r_0 \in f^{-1}[Z_+(f)]$. Thus since Z is not compact, we have by (4.1) that $f^{-1}[Z_+(f)] \supset [r_0, +\infty)$. Hence, $f^{-1}[Z_+(f)] = [r_0, +\infty]$ by definition of r_0. Therefore, taking $A = [r_0, +\infty)$, we see that A satisfies the conditions in (4.2).

(4.3) Corollary. *The set $Z_+(f)$ is a closed arcwise connected subset of Z.*

Proof. By (4.0), $Z_+(f)$ is a closed subset of Z. It follows from (4.2) that $Z_+(f)$ is arcwise connected.

The next lemma determines Z when $Z_+(f)$ is degenerate. The first statement in the lemma is valid without f being confluent, and confluence is not used in the proof of it.

(4.4) Lemma. *If $Z_+(f) = \emptyset$, then Z is a half-line. If $Z_+(f)$ is a one-point space, then Z is a one-point space or an arc.*

Proof. First assume $Z_+(f) = \emptyset$. Let $p \in Z$ and let U be an open subset of Z such that $p \in U$. Since $Z_+(f) = \emptyset$, $f^{-1}(p)$ is bounded. By continuity of f, there is a bounded open subset W of $[0, +\infty)$ such that $f^{-1}(p) \subset W$ and $f[W] \subset U$. Suppose that $f[W]$ is not a neighborhood of p in Z. Then, there exist points $p_n \in (Z \setminus f[W])$, $n = 1, 2, \ldots$, such that $p_n \to p$ as $n \to \infty$. For each $n = 1, 2, \ldots$, let $t_n \in [0, +\infty)$ such that $f(t_n) = p_n$. Since $p_n \notin f[W]$ for any $n = 1, 2, \ldots$, $t_n \notin W$ for any $n = 1, 2, \ldots$ Also, since $p \notin Z_+(f)$ and $f(t_n) \to p$ as $n \to \infty$, the sequence $\{t_n\}_{n=1}^{\infty}$ is bounded. Thus, $\{t_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{t_{n(i)}\}_{i=1}^{\infty}$, and $\{t_{n(i)}\}_{i=1}^{\infty}$ converges to a point $t_0 \notin W$. Hence, it follows from continuity that $f(t_0) = p$. This contradicts the fact that $f^{-1}(p) \subset W$. Hence, $f[W]$ is a neighborhood of p in Z. Therefore, since W is bounded, $f[\text{cl}(W)]$ is a compact neighborhood of p in Z. We have now proved Z is locally compact. Since $Z_+(f) = \emptyset$, Z is not compact. Hence, by (3.1), Z is a half-line. This proves the first part of the lemma. Next, assume $Z_+(f) = \{z_0\}$. Let

$$s_1 = \text{g.l.b.}[f^{-1}(z_0)].$$

If there exists $s_2 > s_1$ such that $s_2 \notin f^{-1}(z_0)$, then Z is an arc by (4.1). Hence, we assume $f^{-1}(z_0) = [s_1, +\infty)$. Then, $Z = f([0, s_1])$ and, hence, Z is compact. Therefore, by (3.1), Z is a one-point space or an arc.

Recall that a space X is said to be *indecomposable* if and only if X is connected and X is not the union of two closed connected proper subsets [3, p. 204].

(4.5) Lemma. *If Z is not locally compact, then $Z_+(f)$ is indecomposable and nondegenerate.*

Proof. Assume Z is not locally compact. To prove $Z_+(f)$ is indecomposable first note that, by (4.3), $Z_+(f)$ is connected. Suppose there exist two closed connected proper subsets B_1 and B_2 of $Z_+(f)$ such that $B_1 \cup B_2 = Z_+(f)$. For each $i \in \{1, 2\}$, let C_i be a component of $f^{-1}[B_i]$. Since $Z_+(f)$ is a closed subset of Z (see (4.0)), B_1 and B_2 are each closed (and connected) subsets of Z. Hence, by confluence of f, we have that $f[C_i] = B_i$ for each $i \in \{1, 2\}$. Suppose that C_1 and C_2 were each unbounded. Then one of them, say C_1, would be contained in the other, C_2. Hence, $B_1 \subset B_2$ which implies $B_2 = Z_+(f)$, a contradiction. Therefore, C_1 or C_2 is bounded, say C_1. Suppose C_2 were also bounded. Note that $Z_+(f) \neq \emptyset$, otherwise B_1 and B_2 would not exist. Thus, since Z is not compact, we have by (4.2) that

(a) $f^{-1}[Z_+(f)] = [r_0, +\infty)$ for some $r_0 \in [0, +\infty)$.

Also we have:

(b) $C_1 \cup C_2$ is bounded;

(c) $f[C_1 \cup C_2] = Z_+(f)$.

Hence, by (a) and (c),

(d) $f[C_1 \cup C_2] = f([r_0, +\infty))$.

It now follows easily from (b) and (d) that Z is the image under f of a closed and bounded interval. This proves Z is compact, a contradiction. Therefore, C_2 is unbounded. For some $s_0 \in [0, +\infty)$, $C_2 = [s_0, +\infty)$. Since $B_1 \not\subset B_2$ and f is confluent, no component of $f^{-1}[B_1]$ is contained in C_2. Also, since $B_2 \not\subset B_1$ and f is confluent, no component of $f^{-1}[B_1]$ contains C_2. It now follows that $f^{-1}[B_1] \subset [0, m]$ for some $m \in [0, +\infty)$. Therefore, since $B_1 \subset Z_+(f)$, it follows from the definition of $Z_+(f)$, that B_1 is nowhere dense in $Z_+(f)$, a contradiction (because $B_1 \setminus B_2$ is a nonempty open subset of $Z_+(f)$). We have now proved that $Z_+(f)$ is indecomposable. Since Z is not locally compact, (4.4) implies that $Z_+(f)$ is nondegenerate. This completes the proof of (4.5).

(4.6) Lemma. *If Z is not locally compact, then $Z_+(f)$ is a one-to-one continuous image of $[0, +\infty)$.*

Proof. By (2.1) and (4.3), $Z_+(f)$ is an arcwise connected metric space which contains no simple triod. Hence, by (3.2) of [6], there is a one-to-one continuous function g from a connected subset L of $(-\infty, +\infty)$ onto $Z_+(f)$. Since g is one-to-one and continuous, it follows using (4.5) that L is not compact. By the second part of (4.2),

$$f^{-1}[Z_+(f)] = [r_0, +\infty) \quad \text{for some } r_0 \in [0, +\infty).$$

Clearly, then,

$$f([0, r_0]) \cap Z_+(f) = \{f(r_0)\}.$$

Now, suppose that L is an open interval. Let $t_0 \in L$ such that $g(t_0) = f(r_0)$. Choose $\varepsilon > 0$ such that $[t_0 - \varepsilon, t_0 + \varepsilon] \subset L$; such an ε exists since L is an open interval. Suppose $r_0 \neq 0$. Then,

$$f([0, r_0]) \cup g([t_0 - \varepsilon, t_0 + \varepsilon])$$

contains a simple triod, which is a contradiction to (2.1). Hence, $r_0 = 0$ and we have

$$Z = Z_+(f) = g[L].$$

This proves that Z is a one-to-one continuous image of an open interval. Thus, each point of Z is a cut point of an arc in Z. This contradicts (2.6). Therefore, L is not an open interval. The lemma now follows.

(4.7) Lemma. *If Z is not locally compact, then each nondegenerate closed connected proper subset of $Z_+(f)$ is an arc.*

Proof. Let K be a nondegenerate closed connected proper subset of $Z_+(f)$. By (4.0), K is a closed connected subset of Z. Let C be a component of $f^{-1}[K]$. Then, by confluence of f, $f[C] = K$. By (4.2) we have that

$$f^{-1}[Z_+(f)] = [r_0, +\infty) \quad \text{for some } r_0 \in [0, +\infty).$$

Now, suppose that C is unbounded. Then, since C is a closed subset of $[0, +\infty)$ and since $Z_+(f) \neq K \subset Z_+(f)$, it follows that $C = [s_0, +\infty)$ for some $s_0 > r_0$. Let $A = f([r_0, s_0])$. Then,

$$A \cup K = A \cup f[C] = f([r_0, +\infty)) = Z_+(f).$$

Therefore, by (4.5) and the fact that A is a locally connected continuum (hence, decomposable or a one-point set), we have that $K = Z_+(f)$. This is a contradiction. Therefore, C is a closed and bounded interval. Hence, since $f[C] = K$, K is a locally connected continuum. By (2.1), K does not contain a simple triod. Hence, K is an arc or a circle. Suppose K is a circle. Then, since $K \neq Z$ and Z is arcwise connected, it would follow that there is a simple triod in Z, a contradiction to (2.1). Therefore, K is an arc.

Using (2.3), (2.4), (3.1) and (3.2) it is easy to see that *no* one-to-one continuous function from $[0, +\infty)$, or from $(-\infty, +\infty)$, onto a *compact* metric space can be confluent. At the other end of the spectrum are the *indecomposable* metric spaces which are one-to-one continuous images of $[0, +\infty)$ or of $(-\infty, +\infty)$. For many of these, as well as certain decomposable spaces, the next result will be used to show (see the proofs of the converse parts of (5.1) and (6.1)) such one-to-one continuous functions *must* be confluent.

(4.8) Lemma. *Let Y be a metric space such that*

(a) *Y is not a circle;*

(b) *Y contains no simple triod;*

(c) *each closed connected proper subset of Y is arcwise connected.*

If g is a one-to-one continuous function from a connected subset L of $(-\infty, +\infty)$ onto Y, then g is confluent.

Proof. Let K be a closed connected subset of Y. Since g is onto, it suffices to prove that $g^{-1}[K]$ is connected. Suppose $g^{-1}[K]$ is not connected. Then, there exist $r_0 < s_0 < t_0$ in L such that

(*) $$[r_0, t_0] \cap g^{-1}[K] = \{r_0, t_0\}.$$

Since $g^{-1}[K]$ is not connected, $K \neq Y$. Hence, by (c), there is an arc A in K such that A has noncut points $g(r_0)$ and $g(t_0)$ (note that, since g is one-to-one, $g(r_0) \neq g(t_0)$). Let $B = g([r_0, t_0])$. Since g is a homeomorphism on $[r_0, t_0]$, B is an arc with noncut points $g(r_0)$ and $g(t_0)$. Since $A \subset K$, it follows easily from (*) that

$$A \cap B = \{g(r_0), g(t_0)\}.$$

It now follows that $A \cup B$ is a circle. Thus, by (a), $A \cup B \neq Y$. Therefore, since Y is arcwise connected (because $g[L] = Y$), it follows that Y contains a simple triod. This contradicts (b) and completes the proof of (4.8).

5. All the confluent images of a half-line. In (3.1) I determined all the locally compact metric spaces which are confluent images of $[0, +\infty)$. In the theorem below, I extend (3.1) to determine *all* metric confluent images of $[0, +\infty)$.

(5.1) Theorem. *A metric space Y is a confluent image of $[0, +\infty)$ if and only if Y is one of the following:*

(1) *a one-point space;*

(2) *an arc;*

(3) *a half-line;*

(4) *an indecomposable metric space, which is a one-to-one continuous image of $[0, +\infty)$, such that each nondegenerate closed connected proper subset is an arc;*

(5) *a metric space which is a one-to-one continuous image of $[0, +\infty)$ under a mapping g such that there exists $t_0 > 0$ such that $g([t_0, +\infty))$ is as in (4) and is a closed proper subset of Y.*

Proof. Assume $f\colon [0, +\infty) \xrightarrow{\text{onto}} Y$ is a confluent mapping onto a metric space Y. If Y is locally compact, then Y is as in (1), (2), or (3) by (3.1). So, assume Y is not locally compact. Then, by (4.5) through (4.7),

(*) $$Y_+(f) \text{ has all the properties in (4)}.$$

Hence, if $Y_+(f) = Y$, then Y is as in (4). So, assume $Y_+(f) \neq Y$. Then, since Y is not compact, we have by (4.2) that

$$f^{-1}[Y_+(f)] = [r_0, +\infty) \quad \text{for some } r_0 > 0.$$

Now, $f([0, r_0])$ is a nondegenerate (because $Y_+(f) \neq Y$) locally connected continuum which, by (2.1), contains no simple triod. Hence, $f([0, r_0])$ is an arc or a circle, Since $f([0, r_0]) \neq Y$ and Y is arcwise connected and contains no simple triod (by (2.1)), $f([0, r_0])$ must be an arc. By (4.6), there is a one-to-one continuous function $k\colon [1, +\infty) \xrightarrow{\text{onto}} Y_+(f)$. Since Y does not contain a simple triod [by (2.1)] and since $f([0, r_0])$ is an arc such that

(#) $$f([0, r_0]) \cap Y_+(f) = \{f(r_0)\},$$

it follows that

(**) $$k(1) = f(r_0)$$

and

(***) $$f(r_0) \text{ is a noncut point of } f([0, r_0]).$$

Let $h\colon [0, 1] \xrightarrow{\text{onto}} f([0, r_0])$ be a homeomorphism such that [see (**) and (***)]

(##) $$h(1) = f(r_0) = k(1).$$

Define $g\colon [0, +\infty) \xrightarrow{\text{onto}} Y$ be

$$g(t) = \begin{cases} h(t) & \text{for} \quad t \in [0, 1], \\ k(t) & \text{for} \quad t \in [1, +\infty). \end{cases}$$

Since $h(1) = k(1)$, g is a function. Since h and k are each continuous, g is continuous. Since h and k are each one-to-one, it follows using (#) and (##) that g is one-to-one. Therefore, by (*) and the fact that

$$g([1, +\infty)) = Y_+(f) \neq Y,$$

it follows that g has all the properties in (5) with $t_0 = 1$. Thus, Y is as in (5). This proves half of (5.1).

To prove the converse let Y be as in (4) or (5) (see (3.1)). Then, it is easy to see that (a), (b), and (c) of (4.8) each holds for Y. Hence, letting g be a one-to-one continuous function from $[0, +\infty)$ onto Y, we see by (4.8) that g is confluent. This completes the proof of (5.1).

6. All the confluent images of a line. I have determined all locally compact metric confluent images of $(-\infty, +\infty)$ (see (3.2)) and all metric confluent images of $[0, +\infty)$ (see (5.1)). In this section I complete the study of the structure of metric confluent images of half-lines and lines. The following result, combined with (5.1), delineates all metric confluent images of $(-\infty, +\infty)$.

(6.1) THEOREM. *A metric space Y is a confluent image of $(-\infty, +\infty)$ if and only if Y is arcwise connected and each closed connected proper subset of Y is a confluent image of $[0, +\infty)$.*

Proof. Assume $f\colon (-\infty, +\infty) \xrightarrow{\text{onto}} Y$ is a confluent mapping onto a metric space Y. Clearly, Y is arcwise connected. Let K be a closed connected proper subset of Y. Let C be a component of $f^{-1}[K]$. Since C is a closed subset of $(-\infty, +\infty)$,

(i) $C = [a, b]$, some $a, b \in (-\infty, +\infty)$ or

(ii) $C = (-\infty, s]$ or $C = [s, +\infty)$, some $s \in (-\infty, +\infty)$.

Since f is confluent,

$$f[C] = K.$$

Let $f_C\colon C \xrightarrow{\text{onto}} K$ denote the restriction of f to C. It is easy to see that f_C is confluent. Assume C is as in (i). Then, since the confluent image of an arc is an arc [2, Corollary 20, p. 32] and $f_C[C] = K$, we have that K is an arc. Hence, by (3.1), K is a confluent image of $[0, +\infty)$. Next, assume C is as in (ii). Then, since f_C is confluent, K is a confluent image of $[0, +\infty)$. This proves half of (6.1). Conversely, assume Y is an arcwise connected metric space such that each closed connected proper subset of Y is a confluent image of $[0, +\infty)$. Then, clearly,

(*) each closed connected proper subset of Y is arcwise connected

and, by (2.1),

(**) Y contains no simple triod.

By the arcwise connectivity of Y, (**), and (3.2) of [6], there exists a one-to-one continuous function g from a connected subset L of $(-\infty, +\infty)$ onto Y. Since a circle is a confluent image of $(-\infty, +\infty)$ by (3.2), we assume for the purpose of proof that

(***) Y is not a circle.

Hence, by (*) through (***), (a) through (c) of (4.8) hold. So, by (4.8), g is confluent. Therefore, since L is a confluent image of $(-\infty, +\infty)$ (see (3.2)) and since the composition of confluent mappings is confluent, it now follows that Y is a confluent image of $(-\infty, +\infty)$. This completes the proof of (6.1).

The following result shows how two closed connected subsets of a confluent image of $(-\infty, +\infty)$ must intersect. A proof of it can be based on (2.1), (5.1), and (6.1); we omit the details.

(6.2) THEOREM. *If a metric space Y is a confluent image of $(-\infty, +\infty)$ and if A and B are each closed connected subsets of Y such that $A \not\subset B$, $B \not\subset A$, and $A \cap B \neq \emptyset$, then $A \cap B$ satisfies one of the following:*

(1) *$A \cap B$ consists of only one point;*

(2) *$A \cap B$ is an arc;*

(3) *$A \cap B$ consists of exactly two points, in which case Y must be a circle.*

7. Concluding comments.

(1) Observe that by (2.2) every metric confluent image of a half-line, or of a line, is one-dimensional, hence embeddable in 3-space. Some are not embeddable in the plane. For example, let A denote a composant of the dyadic solenoid. As is well-known, A is an arcwise connected metric space such that each closed connected

proper subset of Λ is an arc. Hence, by (6.1), Λ is a confluent image of $(-\infty, +\infty)$. However, as is known, Λ is not embeddable in the plane. It would be interesting to know exactly which confluent images of a half-line, or of a line, are embeddable in the plane. However, this is not known for one-to-one images of $[0, +\infty)$ as in (4) of (5.1).

(2) By (4) of (5.1), each nondegenerate closed connected proper subset of an indecomposable confluent image of $[0, +\infty)$ is an arc. The corresponding statement, for confluent images of $(-\infty, +\infty)$, is false. For example, let Γ denote the composant "you see" of the continuum in Example 1 of [3, pp. 204–205]. Then, Γ satisfies (4) of (5.1). Now, delete the origin from Γ and denote the new space by Γ_0,

$$\Gamma_0 = \Gamma \setminus \{(0, 0)\}.$$

Then, it can be seen that each closed connected proper subset of Γ_0 is an arc or a half-line. Hence, since Γ_0 is arcwise connected, Γ_0 is a confluent image of $(-\infty, +\infty)$ by (6.1). Also, Γ_0 has a closed subset which is a half-line.

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Capacitability and determinacy

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Abstract. We establish a conjecture of Mycielski and show that the Axiom of Determinacy implies that every set in three-dimensional Euclidean space is capacitable. The capacitability of projective sets follows from projective determinacy, but the case of Σ_2^1 sets requires no determinacy at all, but only some such weaker assumption as the existence of a measurable cardinal.

Introduction. This paper explores the analogy between (Newtonian) *capacity* and Lebesgue measure. Mycielski and Świerczkowski proved that the Axiom of Determinacy (AD) implies that every set of real numbers is Lebesgue measurable [11] (1). Of course this has to be in the absence of the Axiom of Choice (AC), in view of the classical derivation due to Vitali of a non-measurable set from AC. Also, Solovay showed that even without AD, it is at least consistent that every set of reals be Lebesgue measurable [17]. The model that he obtains by forcing which satisfies this property, also satisfies the principle of Dependent Choice (DC):

$$(\forall \alpha \in X)(\exists \beta)\langle \alpha, \beta\rangle \in A \Rightarrow (\exists f)(\forall n)\langle f(n), f(n+1)\rangle \in A .$$

Now the notion of (Newtonian) *capacitability* (of sets in three-dimensional Euclidean space R^3) has certain analogies with Lebesgue measurability. For example, Choquet [1] showed that analytic sets are capacitable. Accordingly, Mycielski conjectured that in Solovay's model, all sets in R^3 might be capacitable ([17], p. 2, Remark 5), and also indicated how this might be proved, via the proposition BC (2) (for "Borel Choice"):

Let X and T be complete separable metric spaces, μ a non-negative, non-atomic Borel measure on X. If $U \subseteq X \times T$, then there is a Borel set $B \subseteq X$ and a Borel measur-

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(1) We have not given a statement of AD in the paper, since we do not use it directly. [3] is a recent comprehensive survey article.

(2) This formulation of BC was conveyed to the author privately by Professor Mycielski. The version stated in [17] occurs as Proposition 5(a) of Theorem 1 on page 1. This is the special case for $X = T =$ the real numbers, and $\mu =$ Lebesgue measure. However, as Mycielski has pointed out, this case and the general case are both equivalent to the special case when X is the Baire space, in view of the reduction mentioned just before Theorem 1 in this paper.