Between Martin's Axiom and Souslin's Hypothesis

by

Kenneth Kunen * (Madison) and **Franklin D. Tall** ** (Toronto)

Abstract. The consequences of Martin's Axiom plus the negation of the continuum hypothesis are analyzed. In particular the "Souslin type" and the "combinatorial" are distinguished. This analysis leads to a study of various kinds of countable chain condition partial orders.

The consequences of Martin's Axiom plus the negation of the continuum hypothesis in practice seem to fall into two categories: those that straightforwardly imply Souslin's Hypothesis, and those that do not. Among the former is

H: *In a topological space in which every collection of disjoint open sets is countable, every uncountable collection of open sets has an uncountable subcollection such that each finite subset of it has non-empty intersection.*

Among the latter are various combinatorial propositions concerning sets of natural numbers. A typical one is

P: *Let* $\{A_\alpha\}_{\alpha<\omega_1}$ *be subsets of* ω *such that each finite intersection of the* A_α*'s is infinite. Then there is an infinite* $A \subseteq \omega$ *such that for every* α, $A - A_\alpha$ *is finite.*

A number of mathematicians have wondered whether these combinatorial consequences of Martin's Axiom are equivalent to it. We shall show that they are not, by establishing that they do not imply Souslin's Hypothesis.

The combinatorial consequences of Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ (when stated in the "$\aleph_1$" form rather than the "$<2^{\aleph_0}$" form) readily imply $2^{\aleph_0} > \aleph_1$, while the Souslin type consequences do not. Jensen's proof [DJ] of the consistency of Souslin's Hypothesis with the continuum hypothesis raised the hope of proving a similar result for H. The conjunction of H with the continuum hypothesis has a number of attractive consequences, e.g. that compact countable chain condition spaces of cardinality $\leqslant 2^{\aleph_0}$ are separable [T_2]. Unfortunately, it also implies $0 = 1$ (Theorem 6 below). This answers a question apparently first raised at the U.C.L.A. set theory conference in 1967, and later discussed at the Prague [A] and Keszthely [T_1] topology conferences. In fact, H implies $2^{\aleph_0} > \aleph_1$. We shall give three proofs, all quite elementary.

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We shall also consider various other propositions intermediate between Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ and Souslin's Hypothesis.

Before proceeding further we need a large number of definitions.

TOPOLOGICAL DEFINITIONS. A collection of sets is *linked* (*centered*) if each pair (finite subcollection) has non-empty intersection. A topological space satisfies the *countable chain condition* or is CCC if every collection of disjoint open sets is countable. A space is CCC-*productive* if its product with every CCC space is CCC. A space has *property* (*K*) (for Knaster [K]) (has *precaliber* $\aleph_1$) (has *caliber* $\aleph_1$) if each uncountable collection of open sets includes an uncountable subcollection which is linked (is centered) (has non-empty intersection). A completely regular Hausdorff space is *absolute* G_δ if it is a G_δ in some compactification.

NOTE. It is easy to show that separable implies caliber $\aleph_1$ implies precaliber $\aleph_1$ implies property (*K*) implies CCC-productive implies CCC. Every complete metric space and every locally compact Hausdorff space is absolute G_δ (see e.g. [E]).

PARTIAL ORDER DEFINITIONS. Let $\mathscr{P} = (P, \leqslant)$ be a partial order. $p, q \in P$ are *compatible* if there is an $r \in P$ such that $r \leqslant p$ and $r \leqslant q$. $p, q \in P$ are *incompatible* if they are not compatible. $S \subseteq P$ is compatible (incompatible) if it is pairwise compatible (incompatible). $\mathscr{P}$ is CCC if every incompatible collection is countable. $\mathscr{P}$ is CCC-productive if its product with every CCC partial order is CCC. $\mathscr{P}$ has property (*K*) (has precaliber $\aleph_1$) if each uncountable $S \subseteq P$ has an uncountable subset which is compatible (such that for each collection of finitely many elements of it, there is something below all of them). $D \subseteq P$ is *dense* if for each $p \in P$, there is a $d \in D$ such that $d \leqslant p$. Let $\mathscr{D}$ be a collection of dense subsets of P. $G \subseteq P$ is $\mathscr{D}$-*generic* if

1) if $p \geqslant q \in G$, then $p \in G$,

2) if $p, q \in G$, there is an $r \in G$ such that $r \leqslant p$ and $r \leqslant q$.

3) G meets each $D \in \mathscr{D}$.

$G \subseteq P$ is *weakly* $\mathscr{D}$-*generic* if it satisfies 1) and 3) and for each finite subset S of G, there is a $p \in P \leqslant$ all members of S. $A_\varkappa$ [MS] is the assertion that *for each* CCC *partial order* $\mathscr{P}$ *and for each collection* $\mathscr{D}$ *of* $\varkappa$ *dense subsets of* P, *there is a* $\mathscr{D}$-*generic* G. *Martin's Axiom is* $(\forall \varkappa < 2^{\aleph_0}) A_\varkappa$.

As foreshadowed by our choice of terminology, there are obvious connections between the two sets of concepts, e.g.

THEOREM 1. a) *Consider the four properties*: CCC, CCC-*productive, property* (*K*), *precaliber* $\aleph_1$. *The assertion that every topological space with one of the properties has another of them, is equivalent to the corresponding statement for partial orders.*

b) *Martin's Axiom holds if and only if in every compact Hausdorff* CCC *space, the intersection of fewer than* $2^{\aleph_0}$ *dense open sets is dense.*

Some other equivalents:

THEOREM 2. *The following are equivalent*:

a) H (*i.e. every* CCC *space has precaliber* $\aleph_1$).

b) *Every compact Hausdorff* CCC *space has caliber* $\aleph_1$.

c) *For every* CCC *partial order and collection* $\mathscr{D}$ *of* $\aleph_1$ *dense sets, there is an uncountable* $\mathscr{E} \subseteq \mathscr{D}$ *and a weakly* $\mathscr{E}$-*generic* G.

d) *In every compact Hausdorff* CCC *space, the intersection of a descending collection of* $\aleph_1$ *dense open sets is dense.*

THEOREM 3. *The following are equivalent*:

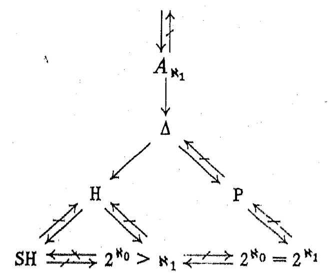
a) *Every* CCC *absolute* G_δ *space has caliber* $\aleph_1$.

b) *For every compact Hausdorff* CCC *space* X *and collections of dense open sets* $\mathscr{D} = \{D_n\}_{n<\omega}$, $\mathscr{E} = \{E_\alpha\}_{\alpha<\omega_1}$, *there is an uncountable* $\mathscr{F} \subseteq \mathscr{E}$ *such that* $\bigcap \mathscr{F} \cap \bigcap \mathscr{D} \neq \emptyset$.

c) *For every* CCC *partial order* $\mathscr{P}$ *and collections* $\mathscr{D} = \{D_n\}_{n<\omega}$ *and* $\mathscr{E} = \{E_\alpha\}_{\alpha<\omega_1}$ *of dense sets, there is an uncountable* $\mathscr{F} \subseteq \mathscr{E}$ *and a weakly* $(\mathscr{D} \cup \mathscr{F})$-*generic* G.

There is a third set of equivalents involving boolean algebras which we omit. There is an essentially uniform proof for Theorems 1 and 2, consisting of chasing through regular open algebras and Stone spaces. It can be dug out of [MS] and [Ju] where 1b) is done. Portions of Theorem 2 are done a different way in [T$_2$]. We shall give a proof of Theorem 3, using these two ideas, just to illustrate the technique. Let any of the equivalents in the theorem be symbolized by Δ. Δ is of interest topologically [T$_2$] and because as we shall later show, Δ implies P. Before proceeding with the proofs, we illustrate with the following diagram the relationships among the propositions we have been considering.

THEOREM 4. *Martin's Axiom plus* $2^{\aleph_0} > \aleph_1$



Reading from the top down, the non-trivial one can be gleaned from [MS]. The implication to Δ is in [T$_2$] or our Theorem 3. $\Delta \to$H is obvious. As mentioned, we shall prove $\Delta \to$P. We shall construct a model for P plus not SH, thus establishing P$\not\to$H or Δ, since H$\to$SH (e.g. [Ju] or [T$_1$]). We shall prove H$\to 2^{\aleph_0} > \aleph_1$, hence SH$\not\to$H, since by [DJ], SH is consistent with $2^{\aleph_0} = \aleph_1$. The other relations between SH and $2^{\aleph_0} = \aleph_1$ can be found in [Je$_1$], [Te], and [ST]. P$\to 2^{\aleph_0} = 2^{\aleph_1}$ is in [R]. The non-converse is well-known. The missing arrows all represent interesting problems. A recent addition to the picture is due to C. Herink who has proved that SH plus $2^{\aleph_0} = \aleph_2 \not\leftrightarrow$Martin's Axiom.

Now for the proof of Theorem 3.

We first prove the equivalence of the two topological versions. Suppose first that every CCC absolute G_δ space has caliber $\aleph_1$. Let X be a compact Hausdorff CCC space, $\mathscr{D} = \{D_n\}_{n<\omega}$, $\mathscr{E} = \{E_\alpha\}_{\alpha<\omega_1}$, collections of dense open subsets of X. $\bigcap \mathscr{D}$ is dense in X by the Baire category theorem, and is therefore absolute G_δ. It is CCC since in general, Y is CCC if and only if $\overline{Y}$ is. For each α, $E_\alpha \cap \bigcap \mathscr{D}$ is open in $\bigcap \mathscr{D}$, so by hypothesis there is an uncountable $\mathscr{F} \subseteq \mathscr{E}$ such that $\bigcap \mathscr{F} \cap \bigcap \mathscr{D} \neq \emptyset$. Conversely, let Y be CCC and absolute G_δ. Let $\{U_\alpha\}_{\alpha<\omega_1}$ be open in Y. Since Y is CCC, an open U can be found such that every open subset of U intersects uncountably many (not necessarily distinct) U_α's. U is also CCC and absolute G_δ, and for each β, $V_\beta = \bigcup_{\beta \leqslant \alpha < \omega_1} U_\alpha$ is dense open in U. There is a compact Hausdorff CCC space X and dense open subsets $\{D_n\}_{n<\omega}$ of X, such that $U = \bigcap_{n<\omega} D_n$ and $\overline{U} = X$. There are dense open W_β in X such that $W_\beta \cap U = V_\beta$. By 3b) there is an uncountable $F \subseteq \omega_1$ such that $\bigcap_{n<\omega} D_n \cap \bigcap_{\beta \in F} W_\beta \neq \emptyset$. But then $\bigcap_{\beta \in F} V_\beta \neq \emptyset$. Since the V_β's are decreasing, it follows that $\bigcap_{\alpha<\omega_1} V_\alpha \neq \emptyset$, and so, finally, some uncountable collection of U_α's must have non-empty intersection.

To prove b) and c) equivalent, recall [Je$_2$, p. 50] that for each partial order $\mathscr{P}$ there is a unique complete boolean algebra $\mathrm{RO}(\mathscr{P})$ and a homomorphism e such that $e``P$ is dense in $\mathrm{RO}(\mathscr{P})$ (i.e. in the partial order of non-zero elements) and $p, q \in P$ are compatible if and only if $e(p)$ and $e(q)$ are. Also recall that the sets $W_b = \{\mathscr{U}: \mathscr{U}$ is an ultrafilter on $\mathrm{RO}(\mathscr{P})$ and $b \in \mathscr{U}\}$ form a clopen base for the compact Hausdorff space $\mathrm{St}(\mathrm{RO}(\mathscr{P}))$, and that $\mathrm{RO}(\mathscr{P})$ is isomorphic to the clopen algebra of $\mathrm{St}(\mathrm{RO}(\mathscr{P}))$. Using these facts, it is easy to prove and is well-known that $\mathscr{P}$ is CCC if and only if $\mathrm{RO}(\mathscr{P})$ is, if and only if $\mathrm{St}(\mathrm{RO}(\mathscr{P}))$ is. Furthermore, D is dense in $\mathscr{P}$ if and only if $e``D$ is dense in $\mathrm{RO}(\mathscr{P})$, if and only if $\{W_b: b \in e``D\}$ is dense in the inclusion order on the non-empty open subsets of $\mathrm{St}(\mathrm{RO}(\mathscr{P}))$. To go from b) to c) then, if $\{D_n\}_{n<\omega}$ and $\{E_\alpha\}_{\alpha<\omega_1}$ are dense in $\mathscr{P}$, there is an uncountable $F \subseteq \omega_1$ and an

$$x \in \bigcap_{n<\omega} \left(\bigcup \{W_b: b \in e``D_n\}\right) \cap \bigcap_{\alpha \in F} \left(\bigcup \{W_b: b \in e``E_\alpha\}\right).$$

Then $\{p: e(p) \in x\}$ is the desired subset of P. Conversely, if $\{D_n\}_{n<\omega}$, $\{E_\alpha\}_{\alpha<\omega_1}$ are dense open subsets of the compact Hausdorff CCC space X, let $\mathscr{P}$ be the inclusion order on the non-empty open subsets of X. Let $h(D_n) = \{U \in P: \overline{U} \subseteq D_n\}$. Similarly define $h(E_\alpha)$. Then for all n and all α, $h(D_n)$ and $h(E_\alpha)$ are dense in $\mathscr{P}$. Hence there is an uncountable $F \subseteq \omega_1$ and a $G \subseteq P$ which is weakly $(\{h(D_n)\}_{n<\omega} \cup \{h(E_\alpha)\}_{\alpha \in F})$-generic. Then

$$\emptyset \neq \bigcap \{\overline{U}: U \in G\} \subseteq \bigcap_{n<\omega} D_n \cap \bigcap_{\alpha \in F} E_\alpha .$$

We do not know whether the "weakly" can be removed from Theorems 2 or 3. On the other hand, no strength is lost if in the statement of Martin's Axiom, "generic" is replaced by "weakly generic", or even if mere compatibility is required in addition to 1) and 3). This can be verified by a standard density argument.

We next prove

THEOREM 5. $\Delta \to \mathrm{P}$.

First we need some results of Rothberger [R].

Consider the quasi-order E on subsets of ω defined by $a \preccurlyeq b$ if $a-b$ is finite. Say $a \prec b$ if $a \preccurlyeq b$ and not $b \preccurlyeq a$. E has an *Ω-limit* if there exist $\{a_\alpha\}_{\alpha<\omega_1}$ in E such that

$$a_1 \prec a_2 \prec \ldots \prec a_\alpha \prec \ldots \prec a_{\omega_1}$$

and for any $d \in E$, if $a_\alpha \preccurlyeq d$ for all $\alpha < \omega_1$, and $d \preccurlyeq a_{\omega_1}$, then $a_{\omega_1} \preccurlyeq d$.

LEMMA [R]. *P holds if and only if E has no Ω-limits.*

Rothberger further notes (p. 38) that if there is an Ω-limit, there exist $\{Z_\alpha\}_{\alpha<\omega_1}$ such that $\alpha < \beta$ implies $Z_\alpha \succcurlyeq Z_\beta$ and if $Z_\alpha \succcurlyeq Z$ for all α, then $Z \preccurlyeq \emptyset$. Thus, to establish P, we need only consider the special case of a descending collection. So let $\{A_\alpha\}_{\alpha<\omega_1}$ be infinite subsets of ω such that $\alpha < \beta \to A_\alpha \succcurlyeq A_\beta$. The standard Martin's Axiom proof [B] of P uses the partial order $\mathscr{P} = \langle P, \leqslant \rangle$ defined by

$$P = \{\langle h, H\rangle: h \subseteq \omega, H \subseteq \omega_1;\ h, H \text{ finite}\}.$$

$$\langle h', H'\rangle \leqslant \langle h, H\rangle \quad \text{if } h' \supseteq h,\ H' \supseteq H \text{ and for each } \alpha \in H,\ h'-h \subseteq A_\alpha,$$

and the dense sets

$$D_n = \{\langle h, H\rangle: h \text{ has cardinality } \geqslant n\},$$

$$E_\alpha = \{\langle h, H\rangle: \alpha \in H\}.$$

Let G be weakly generic for all the D_n's and uncountably many E_α's. Then $\bigcup \{h: \text{for some } H, \langle h, H\rangle \in G\}$ is infinite, and $\preccurlyeq$ uncountably many A_α's, hence $\preccurlyeq$ all of them.

Consider the reduced measure algebra $\mathscr{B}$ of Lebesgue measurable subsets of the unit interval modulo sets of measure zero. Every measure algebra has property (K) [HT] (i.e. the natural partial order on non-zero elements of the algebra has property (K)). Thus, if Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ holds, $\mathscr{B}$ has precaliber $\aleph_1$. On the other hand we shall prove that the continuum hypothesis implies $\mathscr{B}$ does not have precaliber $\aleph_1$, and thus

THEOREM 6. H *implies* $2^{\aleph_0} > \aleph_1$.

We give three proofs. The first is due to P. Erdös and is included with his kind permission. It establishes that if every set of reals of power less than continuum has measure zero, then there is a collection of $2^{\aleph_0}$ elements of $\mathscr{B}$ such that no subcollection of power $2^{\aleph_0}$ has the property that every finite subset has non-zero meet. The second gets an analogous collection of $\aleph_1$ elements by using the existence of a family $\{f_n\}_{n<\omega}$ of functions from ω_1 to ω such that for each uncountable subset X of ω_1, all but finitely many f_n map X onto all of ω. This is a weak variant of Proposition P_3 of [S], which is equivalent to the continuum hypothesis, and asserts the existence of a countable family of real-valued functions of a real variable, such that for each uncountable set X of reals, all but finitely many members of the family map X onto all the reals. The kind of family we are considering can exist in models where $2^{\aleph_0}>\aleph_1$, e.g. a model obtained by adjoining $\aleph_2$ Cohen reals to a model of the continuum hypothesis. In such a model there are non-measurable sets of power less than continuum. The third proof is more topological, involving $\mathrm{St}(\mathscr{B})$.

First proof. Let $[0,1]=\{r_\alpha\}_{\alpha<2^{\aleph_0}}$. Let $R_\alpha=\{r_\beta\colon \beta\geqslant\alpha\}$. By hypothesis, $\mu(R_\alpha)=1$. Therefore there is an F_σ of measure 1 included in R_α, and hence an increasing sequence of closed sets included in R_α, with measures approaching 1. In fact, for any r, $0<r<1$, there is a closed set of measure r included in R_α. To see this, take a closed set $F\subseteq R_\alpha$ such that $\mu(F)\geqslant r$, and consider the continuous function on the unit interval defined by $f(x)=\mu([0,x]\cap F)$. We can therefore define closed sets $F_\alpha\subseteq R_\alpha$ such that $0\neq\mu(F_\alpha)\neq\mu(F_\beta)$ for all $\alpha,\beta<2^{\aleph_0}$. It follows that, letting $[F_\alpha]$ be the equivalence class of F_α modulo null sets, that if $\alpha\neq\beta$, $[F_\alpha]\neq[F_\beta]$. Suppose there were an $A\subseteq 2^{\aleph_0}$ of power $2^{\aleph_0}$ such that for every finite $C\subseteq A$, $0\neq\bigwedge_{\alpha\in C}[F_\alpha]$ (where $\bigwedge$ is the meet in $\mathscr{B}$). Then $0\neq\mu(\bigcap_{\alpha\in C}F_\alpha)$ and so $\emptyset\neq\bigcap_{\alpha\in C}F_\alpha$, i.e. A is centered. By compactness, $\bigcap_{\alpha\in A}F_\alpha\neq\emptyset$. Therefore $\bigcap_{\alpha\in A}R_\alpha\neq\emptyset$. But $\alpha<\beta$ implies $R_\alpha\supseteq R_\beta$, and A is cofinal in $2^{\aleph_0}$, so $\bigcap_{\alpha<2^{\aleph_0}}R_\alpha\neq\emptyset$. But clearly $\bigcap_{\alpha<2^{\aleph_0}}R_\alpha=\emptyset$.

Second proof. For each $n\in\omega$ and each i, $1\leqslant i\leqslant 2^{n+2}$, let $J(n,i)=$ the interval $[(i-1)2^{-n-2}, i\cdot 2^{-n-2}]$. Let $F_\alpha=[0,1]-\bigcup\{J(n,f_n(\alpha))\colon n\in\omega\}$, where $J(n,f_n(\alpha))$ is interpreted as empty if $f_n(\alpha)>2^{n+2}$. Let X be an uncountable subset of ω_1. Then $\bigcap_{\alpha\in X}F_\alpha=\emptyset$, for if $y\in\bigcap_{\alpha\in X}F_\alpha$, then $(\forall\alpha\in X)(\forall n\in\omega)(y\notin J(n,f_n(\alpha)))$. But for n sufficiently large, f_n maps X onto ω, so the $J(n,f_n(\alpha))$'s cover $[0,1]$, a contradiction. To complete the proof, as before we need closed sets representing distinct elements of the measure algebra. Since each F_α has measure $\geqslant\frac{1}{2}$, this can be done.

Third proof. Consider $\mathrm{St}(\mathscr{B})$. It is not difficult to prove it is the union of $\leqslant 2^{\aleph_0}$ nowhere dense sets [We]. First category sets are nowhere dense [H], so, assuming the continuum hypothesis, the version of H in Theorem 2d) is contradicted.

We have actually shown that under CH, there is a space with property (K) that fails to have precaliber $\aleph_1$. After completion of the first version of this paper we learned that R. Laver and F. Galvin had independently used CH to construct a CCC space with non-CCC square. The first author has recently used CH to construct a CCC-productive space which does not have property (K) [Wa].

It remains open whether $2^{\aleph_0}<2^{\aleph_1}$ can replace CH in any of these applications. It would be interesting if H were consistent with $2^{\aleph_0}<2^{\aleph_1}$:

THEOREM 7. H *plus* $2^{\aleph_0}<2^{\aleph_1}$ *implies every compact Hausdorff* CCC *hereditarily normal space is hereditarily separable.*

Proof. By [$\check{\mathrm{S}}_1$], assuming $2^{\aleph_0}<2^{\aleph_1}$, such a space is hereditarily CCC. By [T_2], H then suffices to make the space hereditarily separable.

At present, all that is known is the Martin's Axiom plus $2^{\aleph_0}>\aleph_1$ ensures that every compact Hausdorff perfectly normal space is hereditarily separable [Ju].

As mentioned earlier, a number of people, e.g. [He], [M], [T_3], [vD], [vR], have raised the question of whether various combinatorial consequences of Martin's Axiom are sufficient to yield the full strength of the axiom. The answer is no for all of those we have checked, by the following reasoning. All of the partial orders used for establishing these consequences — such as P — have property (K). The reader may check for his favorite, that the usual proof that the partial order is CCC, actually shows it has property (K). It is tedious but routine to check that in every step of the consistency proof (see [ST] or [Je_2]) of Martin's Axiom plus $2^{\aleph_0}>\aleph_1$, "CCC" can be replaced by "property (K)". The only non-trivial point is to check that a property (K) partial order in the final model had property (K) when it first appeared. This can be proved by an application of Theorem 11 bellow, since the final model is a property (K) extension of the intermediate model. Thus, letting "MAK" stand for "Martin's Axiom restricted to partial orders with property (K)", we get

THEOREM 8. *If* $\mathscr{M}$ *is a countable transitive model of* ZFC$+2^{\aleph_1}=\aleph_2$, *there is a generic extension (via a property* (K) *partial order)* $\mathscr{M}[G]$ *which satisfies* $2^{\aleph_0}=\aleph_2$ *and* MAK.

On the other hand, we shall prove

THEOREM 9. *Let* $\mathscr{T}=(T,\prec)$ *be a Souslin tree in* $\mathscr{M}$, *a countable transitive model of* ZFC. *Let* $\mathscr{P}$ *be a partial order with property* (K) *in* $\mathscr{M}$. *Let* G *be* $\mathscr{P}$*-generic over* $\mathscr{M}$. *Then* $\mathscr{T}$ *is a Souslin tree in* $\mathscr{M}[G]$.

Thus, starting with a model in which there is a Souslin tree and $2^{\aleph_1}=\aleph_2$, e.g. L, and then iterating over partial orders with property (K), we get

COROLLARY 10. $2^{\aleph_0}=\aleph_2$ *is consistent with not* SH (*and hence not* H) *and* MAK.

Theorem 9 will follow as an easy corollary to the next result, which we shall state in greater generality than necessary.

THEOREM 11. *Let* $\mathscr{M}$ *be a countable transitive model of* ZFC. *Let* $\mathscr{P}$ *be a partial order with property* (K) *in* $\mathscr{M}$. *Let* G *be* $\mathscr{P}$*-generic over* $\mathscr{M}$. *Let* $f\in\mathscr{M}[G]$, $A\in\mathscr{M}$, $f\colon\omega_1\to A$. *Let* $\varphi(\alpha,\beta,x,y)$ *be downward absolute for transitive* ε*-models. Suppose* $\mathscr{M}[G]\models(\forall\alpha,\beta\in\omega_1)(\varphi(\alpha,\beta,f(\alpha),f(\beta)))$. *Then in* $\mathscr{M}$ *there is an uncountable* $S\subseteq\omega_1$ *and* $\{x_\alpha\}_{\alpha\in S}$, $x_\alpha\in A$, *such that* $(\forall\alpha,\beta\in S)\varphi(\alpha,\beta,x_\alpha,x_\beta)$.

Proof. Note that since $\mathscr{P}$ is CCC, $\omega_1=\omega_1^{\mathscr{M}[G]}$. Let $p\in G$,

$$p\Vdash(\forall\alpha\in\check{\omega}_1)(\exists x)(x\in\check{A}\wedge \underset{\sim}{f}(\alpha)=x)\wedge(\forall\alpha,\beta\in\check{\omega}_1)\varphi(\alpha,\beta,\underset{\sim}{f}(\alpha),\underset{\sim}{f}(\beta)).$$

Then for every $\alpha\in\omega_1$, there is a $p_\alpha\leqslant p$ and an $x_\alpha\in A$ such that $p_\alpha\Vdash\underset{\sim}{f}(\check{\alpha})=\check{x}_\alpha$. By property ($K$) there is an uncountable $S\subseteq\omega_1$ such that $\{p_\alpha\}_{\alpha\in S}$ is compatible. Let $\alpha,\beta\in S$. Let $q\leqslant p_\alpha$ and $q\leqslant p_\beta$. Then $q\Vdash\varphi(\check{\alpha},\check{\beta},\check{x}_\alpha,\check{x}_\beta)$. By absoluteness, $\varphi(\alpha,\beta,x_\alpha,x_\beta)$. Thus $(\forall\alpha,\beta\in S)\varphi(\alpha,\beta,x_\alpha,x_\beta)$.

To prove Theorem 9, first observe that "$\mathcal{T}$ is a tree" is absolute. Since cardinals are preserved, $|\mathcal{T}| = \aleph_1$ in $\mathcal{M}[G]$. It therefore suffices to show $\mathcal{T}$ has no uncountable chains or antichains in $\mathcal{M}[G]$. In the first case, apply the theorem to

$$\text{"}f(\alpha) \in T \wedge f(\beta) \in T \wedge (\alpha < \beta \to (f(\alpha), f(\beta)) \in \prec)\text{"}.$$

In the second case, to

$$\text{"}f(\alpha) \in T \wedge f(\beta) \in T \wedge (\alpha \neq \beta \to (f(\alpha), f(\beta)) \notin \preccurlyeq \wedge (f(\beta), f(\alpha)) \notin \preccurlyeq)\text{"}.$$

Theorem 11 has a number of curious applications in infinitary combinatorics as well as in topology. Consider for example the question of whether $\omega_1 \to [\omega_1]^2_{\omega_1}$ (see e.g. [EH] for the definition). It is unknown whether this is consistent with ZFC, but it fails if the continuum hypothesis is assumed. Theorem 11 easily shows that a counterexample to $\omega_1 \to [\omega_1]^2_{\omega_1}$ is preserved by a partial order with property (K), so that MAK plus $2^{\aleph_0} > \aleph_1$ plus $\omega_1 \nrightarrow [\omega_1]^2_{\omega_1}$ is consistent with ZFC. These same remarks hold with "$\omega_1 \to [\omega_1]^2_{\omega_1}$" replaced everywhere by "there does not exist a regular, first countable, hereditarily separable, non-Lindelöf space".

A closer examination of the partial orders used to establish P and many other combinatorial consequences shows that not only do they have property (K) and indeed precaliber $\aleph_1$, but even a stronger property — they are *σ-centered*, i.e. they are the union of countably many collections, each of which is finitely compatible. The topological counterpart is having a σ-centered topology. Call a space with such a topology σ-centered. Clearly separable implies σ-centered implies precaliber $\aleph_1$. It is easy to see that compact Hausdorff spaces are σ-centered if and only if they are separable. Thus there exist compact spaces with precaliber $\aleph_1$ which are not σ-centered, e.g. the product of $2^{\aleph_0}$ copies of the 2-point discrete space. On the other hand, Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ implies that compact Hausdorff CCC spaces with a π-base (a dense subset of the inclusion ordering on the non-empty open sets) of cardinality $< 2^{\aleph_0}$ are separable [HJ]. It follows that CCC partial orders of cardinality $< 2^{\aleph_0}$ are σ-centered.

The fact that σ-centered compact Hausdorff spaces are separable yields that Martin's Axiom restricted to the class of compact separable spaces implies, if $2^{\aleph_0} > \aleph_1$, that $2^{\aleph_0} = 2^{\aleph_1}$, since it implies P. This is not so for compact hereditarily separable spaces. By [Š$_2$] such spaces have countable π-bases. It can be shown that Martin's Axiom for the class of compact spaces with countable π-bases is equivalent to the real line not being the union of fewer than $2^{\aleph_0}$ nowhere dense sets, which is consistent with $\aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$.

References

[A] A. V. Arhangel'skiĭ, *On cardinal invariants*, in: J. Novak, ed., *General Topology and its Relations to Modern Analysis and Algebra* III, Proceedings of the third Prague topological symposium, 1971 (Academia, Prague, 1972), pp. 37–46.

[B] D. D. Booth, *Countably indexed ultrafilters*, Ph. D. Thesis, University of Wisconsin, Madison, 1969.
[DJ] K. Devlin and H. Johnsbråten, *The Souslin Problem*, Lecture Notes Math. 405, Berlin 1975.
[vD] E. K. van Douwen, *Martin's Axiom and pathological points in $\beta X \setminus X$*, preprint.
[E] R. Engelking, *General Topology*, Warszawa 1977.
[EH] P. Erdös and A. Hajnal, *Unsolved problems in set theory*, Proc. Symp. Pure Math. 13 (I) (1971), pp. 17–48.
[H] P. R. Halmos, *Lectures on boolean algebras*, Princeton 1963.
[He] S. H. Hechler, *On some weakly compact spaces and their products*, Gen. Top. Appl. 5 (1975), pp. 83–93.
[HJ] A. Hajnal and I. Juhász, *A consequence of Martin's Axiom*, Res. Paper No. 110, Dept. of Mathematics, Univ. of Calgary, Calgary, Alberta.
[HT] A. Horn and A. Tarski, *Measures in boolean algebras*, Trans. Amer. Math. Soc. 64 (1948), pp. 467–497.
[Je$_1$] T. Jech, *Nonprovability of Souslin's hypothesis*, Comment. Math. Univ. Carolinae 8 (1967), pp. 293–296.
[Je$_2$] — *Lectures in set theory with particular emphasis on the method of forcing*, Lecture Notes Math. 217, Berlin 1971.
[Ju] I. Juhász, *Cardinal Functions in Topology*, Amsterdam 1971.
[K] B. Knaster, *Sur une propriété caractéristique de l'ensemble des nombres réels*, Mat. Sb. 16 (1945), pp. 281–288.
[M] R. D. Mauldin, *On generalized rectangles and countably generated families*, preprint.
[MS] D. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic 2 (1970), pp. 143–178.
[R] F. Rothberger, *On some problems of Hausdorff and of Sierpiński*, Fund. Math. 35 (1948), pp. 29–46.
[vR] R. v. B. Rucker, *Cantor's Continuum Problem*, preprint.
[Š$_1$] B. Šapirovskiĭ, *On separability and metrizability of spaces with Souslin's condition*, Soviet Math. Dokl. 13 (1972), pp. 1633–1638.
[Š$_2$] — *Canonical sets and character. Density and weight in compact spaces*, Soviet Math. Dokl. 15 (1974), pp. 1282–1287.
[S] W. Sierpiński, *Hypothèse du Continu*, New York 1956.
[ST] R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. of Math. 94 (1971), pp. 201–245.
[T$_1$] F. D. Tall, *Souslin's conjecture revisited*, in: Colloquia Mathematica Societatis János Bolyai 8. Topics in Topology, Amsterdam 1974, pp. 609–615.
[T$_2$] — *The countable chain condition vs. separability — applications of Martin's Axiom*, Gen. Top. Appl. 4 (1974), pp. 315–340.
[T$_3$] — *An alternative to the continuum hypothesis and its uses in general topology*, preprint.
[Te] S. Tennenbaum, *Suslin's problem*, Proc. Nat. Acad. Sci. U.S.A. 59 (1968), pp. 60–63.
[Wa] M. L. Wage, *Almost disjoint sets*, preprint.
[We] W. A. R. Weiss, *Some applications of set theory to topology*, Thesis, University of Toronto, Toronto 1975.

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