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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

On the pseudoachromatic number of a graph

by

V. N. Bhave * (Kalamazoo, Mich.)

Abstract. The pseudoachromatic number $\psi_s(G)$ of a graph G is the maximum number of colors which may be assigned to the points of G so that for every two colors, there exist adjacent points assigned these colors (adjacent points may have the same color). We investigate (1) the effect on the pseudoachromatic number by removing points or lines and (2) general bounds for the pseudoachromatic number. A graph G is pseudominimal if $\psi_s(G-x) < \psi_s(G)$ for every line x in G. The structure of pseudominimal graphs and a technique for constructing pseudominimal graphs is discussed.

By a graph we mean a finite undirected graph without multiple lines and loops. For a graph G, let $V(G)$ and $X(G)$ denote respectively its point set and line set. In general we follow the notations in [3].

A collection $P = \{V_1, V_2, \ldots, V_n\}$ of nonempty subsets of a nonempty set V is a *partition* of V if (i) $V = \bigcup_{i=1}^{n} V_i$ and (ii) $V_i \cap V_j = \emptyset$ for $i \neq j$.

Let P be a partition of $V(G)$ of a graph G. The *partition graph* $P(G)$ of G is the graph with point set P where V_i and V_j are adjacent if and only if there exist $v_i \in V_i$ and $v_j \in V_j$ such that v_iv_j is a line in G. A partition P of $V(G)$ is *complete* if $P(G)$ is a complete graph.

A set S of points in G is *independent* if no two points of S are adjacent.

A homomorphism of a graph G onto a graph G' is a function φ from $V(G)$ onto $V(G')$ such that whenever uv is a line in G, $\varphi(u)\varphi(v)$ is a line in G'; G' is called a homomorphic *image* of G.

If every set in a partition P of $V(G)$ is independent, we say that P is an *independent* partition of G. It is easy to see that a partition graph $P(G)$ of G is a homomorphic image if P is an independent partition. If P is an independent partition and $P(G)$ is a homomorphic image, we call P itself a homomorphism of G.

By a *k-coloring* of a graph G we mean a mapping $f\colon V(G) \to \{1, 2, \ldots, k\}$ satisfying one or both of the following conditions:

C_1: For every line uv in G, $f(u) \neq f(v)$.

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C_2: For every $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$, there exists a line uv in G such that $f(u) = i$ and $f(v) = j$.

The *chromatic number* $\chi(G)$ of G is the smallest number n such that G has an n-coloring satisfying C_1 and C_2. The *achromatic number* $\psi(G)$ of G is the largest n such G has an n-coloring satisfying C_1 and C_2. The *pseudoachromatic number* $\psi_s(G)$ of G is the largest n such that G has an n-coloring satisfying C_2. The achromatic number was introduced in [4] and further studied in [1]. The pseudoachromatic number was introduced in [2].

Let $\mathfrak{p}_G$ denote the class of all partition graphs of G, and $\overline{\mathfrak{p}}_G$ denote the class of all partition graphs of G which are homomorphic images of G. It is easy to see that

LEMMA 1. *For a graph G,*

1) $\chi(G) = \min\{n\colon K_n \in \overline{\mathfrak{p}}_G\}$,

2) $\psi(G) = \max\{n\colon K_n \in \overline{\mathfrak{p}}_G\}$ *and*

3) $\psi_s(G) = \max\{n\colon K_n \in \mathfrak{p}_G\}$.

THEOREM 1. *For any graph G and a point $u \in V(G)$,*

$$\psi_s(G) \geqslant \psi_s(G-u) \geqslant \psi_s(G)-1 .$$

Proof. Let $\psi_s(G-u) = n$. Then there exists a partition $P' = \{V_1', V_2', \ldots, V_n'\}$ of $V(G)-\{u\}$ such that $P'(G-u) = K_n$. Now, since $P = \{V_1' \cup \{u\}, V_2', \ldots, V_n'\}$ is a partition of $V(G)$ such that $P(G) = K_n$, it follows that $\psi_s(G) \geqslant n = \psi_s(G-u)$. To prove the other inequality, suppose $\psi_s(G) = n$. Then, there exists a partition P of $V(G)$ such that $P(G) = K_n$. Let $u \in V_1 \in P$. Clearly the points of $P-\{V_1\}$ induce K_{n-1} in $P(G)$ and hence

$$\psi_s(G-u) \geqslant n-1 = \psi_s(G)-1 .$$

In [1] Geller and Kronk have proved the following: If x is a line of a graph G and $\psi(G)$ the achromatic number of G, then

$$\psi(G)+1 \geqslant \psi(G-x) \geqslant \psi(G)-1 .$$

The following theorem gives a corresponding result for $\psi_s(G)$.

THEOREM 2. *For any graph G and a line x of G*

$$\psi_s(G) \geqslant \psi_s(G-x) \geqslant \psi_s(G)-1 .$$

Proof. Suppose $\psi_s(G) = n$. Then there exists a partition P of $V(G)$ with $P(G) = K_n$. If the line x joins a point of V_i and a point of V_j where $V_i, V_j \in P$, $i \neq j$, then the partition $P' = (P-\{V_i, V_j\}) \cup \{V_i \cup V_j\}$ of $V(G)$ will be such that $P'(G-x) = K_{n-1}$. Therefore, $\psi_s(G-x) \geqslant n-1 = \psi_s(G)-1$. On the other hand, if P is a partition of $V(G)$ such that $P(G-x) = K_m$, where $m = \psi_s(G-x)$, then for the same partition P, $P(G) = K_m$ and hence $\psi_s(G) \geqslant m = \psi_s(G-x)$.

We observe that for the cycle C_4 of length four, $\psi_s(C_4) = 3$ and $\psi(C_4) = 2$ and for any line x of C_4, $\psi_s(C_4-x) = 3 = \psi_s(C_4-x)$. The following theorem shows that this result is true in general.

THEOREM 3. *For any graph G with $\psi_s(G) > \psi(G)$, there exists a line x such that $\psi_s(G-x) = \psi_s(G)$.*

Proof. Let $\psi_s(G) > \psi(G)$ and $n = \psi_s(G)$. Then there is a partition P of $V(G)$ such that $P(G) = K_n$. Clearly, the partition is not a homomorphism, for otherwise $\psi(G) \geqslant n$. Hence there is a line joining points of the same set $V_i \in P$. This line x will be such that $P(G-x) = K_n$. Therefore, $\psi_s(G-x) \geqslant n$, but $\psi_s(G-x) \leqslant n$ by Theorem 2. Thus, $\psi_s(G-x) = n$.

The converse of the above theorem is not true. For example the graph G in Figure 1 has a line x such that $\psi_s(G-x) = \psi_s(G)$. But, $\psi_s(G) = \psi(G) = 4$.

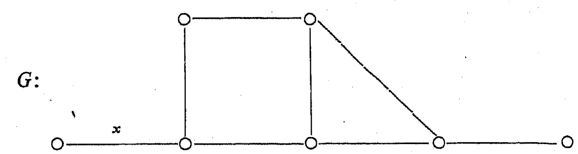

Fig. 1

THEOREM 4. *For any graph G with p points*

$$\psi_s(G) \leqslant \tfrac{1}{2}(p+\chi(G)) .$$

Proof. Let $\chi(G) = n$. Then there exists a complete partition

$$P = \{V_1, V_2, \ldots, V_n\}$$

of $V(G)$ where each V_i is independent in G. Let $\psi_s(G) = r$ and $P' = \{W_1, W_2, \ldots, W_r\}$ be a complete partition of $V(G)$. It is clear that for $i \neq j$, $W_i \cup W_j$ is not contained in any V_k for all $i, j \in \{1, 2, \ldots, r\}$ and $1 \leqslant k \leqslant n$. Hence, at least $r-n$ sets in P' contain points from different sets in P. Thus,

$$2(r-n) \leqslant p-n \quad \text{or} \quad r \leqslant \tfrac{1}{2}(p+n) .$$

In particular if G is a spanning subgraph of K_{mn} with $m \leqslant n$ then $\chi(G) = 2$, $p \leqslant 2n$ and $\psi_s(G) \leqslant n+1$. Therefore, $\psi(G) \leqslant n+1$, an upperbound obtained in [1].

The graph G in Figure 2 has $p = 6$, $\chi(G) = 2$ and $\psi_s(G) = 4$, and thus the upper bound is attained.

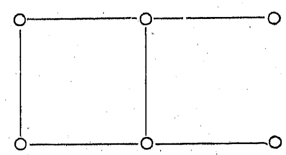

Fig. 2

As usual, let $\{r\}$ denote the least integer not less than the real number r.

THEOREM 5. *For a graph G with p points and maximum degree $\Delta(G)>0$.*

$$p\geqslant n\left\{\frac{n-1}{\Delta(G)}\right\}, \quad \text{where} \quad n=\psi_s(G).$$

Proof. Let $P=\{V_1, V_2, \dots, V_n\}$ be a partition of $V(G)$ such that $P(G)=K_n$. Then the degree of every $V_i\in P$ in $P(G)$ is $n-1$. Now,

$$n-1=\deg_{P(G)}V_i\leqslant\sum_{v\in V_i}\deg_G v\leqslant|V_i|\Delta(G).$$

Hence, $\frac{n-1}{\Delta(G)}\leqslant|V_i|$ and as $|V_i|$ is an integer

$$\left\{\frac{n-1}{\Delta(G)}\right\}\leqslant|V_i| \quad \text{for each } V_i\in P.$$

Hence

$$n\left\{\frac{n-1}{\Delta(G)}\right\}\leqslant\sum|V_i|=p.$$

THEOREM 6. *If G is a graph with q lines then $\psi(G)\leqslant\psi_s(G)\leqslant r$, where r is the maximum integer with $\binom{r}{2}\leqslant q$.*

Proof. Let $P(G)=K_n$, where $n=\psi_s(G)$. Then it is easy to see that G has at least $\binom{n}{2}$ lines, i.e. $\binom{n}{2}\leqslant q$. Hence the result follows.

If $G=qK_2$, where $q=\binom{r}{2}$ then $\psi_s(G)=r$. This shows that the bound is attained.

DEFINITION. A graph H is *partition realizable* from a graph G, if $P(G)=H$ for some partition P of $V(G)$.

LEMMA 2. *If G is a graph with q lines and no isolated points, then G is partition realizable from qK_2. a graph with q copies of K_2.*

LEMMA 3. *If H is a subgraph of G and q, q_1 are the number of lines in G and H respectively, then G is partition realizable from $H\cup rK_2$, where $r=q-q_1$.*

In Lemmas 2 and 3 in forming the partition graphs, we observe that no lines of G are destroyed and hence the partitions are homomorphisms of G.

THEOREM 7. *If a and b $(a>b>1)$ are any two integers, then there exists a graph G with $\psi(G)=\psi_s(G)=a$ and $\chi(G)=b$.*

Proof. Let $G=K_b\cup rK_2$ where $r=\binom{a}{2}-\binom{b}{2}$. Then by Lemma 3 there exists a partition P of $V(G)$ with $P(G)=K_a$. Hence $\psi_s(G)\geqslant a$. Also as G has $\frac{1}{2}a(a-1)$ lines, $\psi_s(G)\leqslant a$. Hence $\psi_s(G)=a$. Since, P is a homomorphism of G we have $\psi(G)=a$. $\chi(G)=b$ follows, since K_b is a component of G and every other component of G is K_2. This completes the proof.

DEFINITION. A graph G is *n-pseudominimal* if $\psi_s(G-x)<\psi_s(G)=n$ for every line x in G.

DEFINITION. A graph G is *n-achrominimal* if $\psi(G-x)<\psi(G)=n$ for every line x in G.

THEOREM 8. *If G is n-achrominimal, then G has exactly $\binom{n}{2}$ lines and conversely, if $\psi(G)=n$ and G has $\binom{n}{2}$ lines then G is n-achrominimal.*

Proof. Let G be n-achrominimal. Let $P(G)=K_n$, where $P=\{V_1, V_2, \dots, V_n\}$ is a homomorphism. Then each set V_i is independent and there is only one line joining V_i and V_j in G for $i,j\in\{1, 2, \dots, n\}$, $i\neq j$. This implies that G has $\binom{n}{2}$ lines. Conversely if $\psi(G)=n$ and G has $\binom{n}{2}$ lines, then for any line x in G we have, $\psi(G-x)<n$ by Theorem 6. Hence G is achrominimal.

THEOREM 9. *A graph G is n-achrominimal if and only if it is n-pseudominimal.*

Proof. Let G be an n-pseudominimal graph. Then there exists a complete partition $P=\{V_1, V_2, \dots, V_n\}$ of $V(G)$. We claim that P is a homomorphism of G. For, suppose $V_i\in P$ is not independent and let x be a line of G with both end points in V_i. Then for some partition P of $V(G)$, we get $P(G-x)=K_n$. This implies that $\psi_s(G-x)\geqslant n$, which is a contradiction. Thus P is a homomorphism of G. Hence $\psi(G)\geqslant n$, but by Lemma 1, $\psi(G)\leqslant n$. Hence, $\psi(G)=n$. Also for every line x of G, $\psi(G-x)\leqslant\psi_s(G-x)<\psi_s(G)=\psi(G)$. This implies that G is n-achrominimal.

Conversely, suppose G is n-achrominimal. Then by Theorem 8, G has $\binom{n}{2}$ lines. Hence by Theorem 6, $\psi_s(G)\leqslant n=\psi(G)$. But by Lemma 1, $\psi(G)\leqslant\psi_s(G)$, hence $\psi_s(G)=n$. Again by Theorem 6, $\psi_s(G-x)<n$ for any line x. Hence, G is n-pseudominimal.

DEFINITION. An *n-minimal* graph is one which is n-pseudominimal (and hence n-achrominimal).

Let H be a subgraph of G. Then we shall denote the subgraph of G obtained by deleting all lines of H and the resulting isolated points in G by $G-H$. It is clear that.

LEMMA 4. *Let a graph H be partition realizable from G and H_1 be an induced subgraph of H, then*

(i) *H_1 is partition realizable from an induced subgraph G_1 of G and*

(ii) *$H-H_1$ is partition realizable from $G-G_1$.*

We observe that $K_n-K_{n-1}=K_{1,n-1}$.

COROLLARY 4.1. *If K_n is partition realizable from G, then there exists an induced subgraph G_1 of G such that*

(i) *K_{n-1} is partition realizable from G_1 and*

(ii) *$K_{1,n-1}$ is partition realizable from $G-G_1$.*

Lemma 4 suggests a method of constructing the set of all n-minimal graphs from the set of all $(n-1)$-minimal graphs. We consider the graphs with no isolated points. The method is as follows.

Let $\{G_i\}$ be the collection of all $(n-1)$-minimal graphs and $\{H_j\}$ be the collection of all graphs with $n-1$ lines, such that $K_{1,n-1}$ is partition realizable from H_j. Since K_{n-1} is partition realizable from G_i and $K_{1,n-1}$ is partition realizable from H_j, K_n is partition realizable from each of the graphs formed below. Further, each of the following graphs has $\binom{n}{2}$ lines and hence each is n-minimal.

Consider a graph G_i. Let $P = \{V_1, V_2, \ldots, V_{n-1}\}$ be a complete partition of $V(G_i)$. It is easy to see that each H_j has at least $n-1$ points of degree one, say u_r, for $r = 1, 2, \ldots, n-1$.

Let G be a graph obtained from G_i and H_j by identifying some, all or none of the points u_r with the points of G_i such that, no two points u_r are identified with the points of the same set $V_i \in P$. We claim that any n-minimal graph is isomorphic to a graph obtained above. For, let G be n-minimal and $P(G) = K_n$. Then as K_{n-1} is an induced subgraph of K_n, there exists an induced subgraph of G_i', of G such that K_{n-1} is partition realizable from G_i' and $K_{1,n-1}$ is partition realizable from $G-G_i'$, by Corollary 4.1. Therefore, G_i' has $\binom{n-1}{2}$ lines and hence it is $(n-1)$-minimal and $G-G_i'$ has $n-1$ lines. Therefore G is isomorphic to one of the graphs obtained above.

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WESTERN MICHIGAN UNIVERSITY

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Continuous extenders in normal and collectionwise normal spaces

by

David J. Lutzer (Pittsburgh, Pa.) and **Teodor C. Przymusiński** (Warszawa) *

Abstract. For a Banach space B and a space Z denote by $C^*(Z, B)$ the space of continuous, bounded mappings $\varphi\colon Z \to B$ with the sup-norm topology and by $P^*(Z)$ (resp. $P^*_\lambda(Z)$) the space of continuous, bounded (resp. bounded and λ-separable) pseudometrics $\varrho\colon Z\times Z \to \boldsymbol{R}$ on Z with the topology of the subspace of $C^*(Z\times Z) = C^*(Z\times Z, \boldsymbol{R})$.

THEOREM 1. *Let F be a closed subset of a λ-collectionwise normal space X and B a Banach space of weight $\leqslant \lambda$. There exist continuous extenders*:

$$e\colon C^*(F, B) \to C^*(X, B) \quad \text{and} \quad E\colon P^*_\lambda(F) \to P^*_\lambda(X).$$

COROLLARY 1. *Let F be a closed subset of a collectionwise normal space X and let B be a Banach space. There exist continuous extenders*:

$$e\colon C^*(F, B) \to C^*(X, B) \quad \text{and} \quad E\colon P^*(F) \to P^*(X).$$

COROLLARY 2. *Let F be a closed subset of a normal space X. There exist continuous extenders*:

$$e\colon C^*(F) \to C^*(X) \quad \text{and} \quad E\colon P^*_{\omega_0}(F) \to P^*_{\omega_0}(X).$$

The above extenders are homeomorphic (but, in general, neither linear nor isometric) embeddings.

§ 1. Introduction. The symbol λ will always denote infinite cardinal number and $\boldsymbol{R}$ stands for the real line. A T_1-space X is *λ-collectionwise normal* if each discrete collection of cardinality $\leqslant\lambda$ of subsets of X can be separated by disjoint open sets. A space is *normal* if and only if it is ω_0-collectionwise normal (cf. [E]; Theorem 2.1.14).

For a Banach space B and a topological space Z, $C^*(Z, B)$ will denote the Banach space of all continuous, bounded functions $\varphi\colon Z\to B$ with the sup-norm $\|\varphi\| = \sup_{z\in Z} \|\varphi(z)\|$. If $B = \boldsymbol{R}$, then we write $C^*(Z)$ instead of $C^*(Z, \boldsymbol{R})$. A pseudo-

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