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On the fixed point index and the Nielsen fixed point theorem of symmetric product mappings

by

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Abstract. In this paper we study essential fixed point sets of symmetric product maps. We define fixed point index of a symmetric product map of a finite polyhedron. In the special case when $G = S_n$, the symmetric group, we define fixed point classes and the Nielsen number of a symmetric product map and prove the Nielsen fixed point theorem for symmetric product maps of finite polyhedra.

1. Introduction. Let X be a topological space and X^n be the Cartesian product with usual topology. A group G of permutations of the numbers $[1, 2, \ldots, n]$ can be considered as a group of homeomorphisms on X^n by defining, for $\alpha \in G$ and $(x_1, x_2, \ldots, x_n) \in X^n$, $\alpha(x_1, x_2, \ldots, x_n) = (x_{\alpha(1)}, x_{\alpha(2)}, \ldots, x_{\alpha(n)})$. The orbit space with identification topology is denoted by X^n/G. A map $f\colon X \to X^n/G$ is called a *symmetric product map*. A point $x \in X$ is said to be a *fixed point of* f if $\eta(z) = f(x)$ implies that x is a coordinate of z, where $z \in X^n$ and $\eta\colon X^n \to X^n/G$ is the identification map. C. N. Maxwell defined the Lefschetz number $L(f)$ of a symmetric product map and proved the Lefschetz fixed point theorem for symmetric product maps in the case when X is a compact polyhedron [6]. The Lefschetz fixed point theorem for symmetric product mappings also hold in the case when X is a metric absolute neighborhood retract and f is a compact map [5].

A fixed point x of the map $f\colon X \to X^n/G$ is called an *essential fixed point* if each map sufficiently close to f has a fixed point arbitrary close to x. Essential fixed points and essential fixed point sets for a single valued maps have been investigated by Fort [3] and O'Neill [7] respectively.

In this paper we study essential fixed point sets of symmetric product maps. We define fixed point index of a symmetric product map of a finite polyhedron. In the special case when $G = S_n$, the symmetric group, we define fixed point classes and the Nielsen number of a symmetric product map and prove the Nielsen fixed point theorem for symmetric product maps of finite polyhedra.

2. Preliminaries. Let $\pi_i\colon X^n \to X$ be the ith projection and $\alpha \in G$, the for $z \in X^n$ $\pi_i \alpha(z) = \pi_{\alpha(i)} z$, where $i = 1, 2, \ldots, n$.

Let d be a metric on X and $\bar{d}$ be the usual euclidean metric on X^n. A metric $\tilde{d}$ may be defined on X^n/G by defining

$$\tilde{d}(\eta(z), \eta(z')) = \inf\{\bar{d}(z, \alpha z')|\ \alpha \in G\}$$

where $z, z' \in X^n$. There is a metric like real valued continuous function ω on $X \times X^n/G$ defined by

$$\omega(x, \eta(z)) = \inf\{\bar{d}(x, \pi_i(z))|\ i = 1, 2, \ldots, n\}$$

where $x \in X$ and $z \in X^n$. The map ω satisfies the following inequality

$$\omega(x, y) \leqslant \omega(x, y') + \tilde{d}(y, y')$$

for any $x \in X$ and $y, y' \in X^n/G$.

Let X be a finite polyhedron with a fixed basic triangulation T. Let K be a subdivision of the basic triangulation of X. Assume that K is an ordered complex, that is to say that a partial ordering $\leqslant$ is defined on the set V, the set of vertices of K, which is a linear ordering on any subset of V in S, where S denotes the set of simplexes of K. A triangulation K^n of X^n may be obtained as follows ([2], p. 67). The set of vertices of K^n is V^n. Let π_i: $V^n \to V$ be the ith projection. For $w, w' \in V^n$, define $w \leqslant w'$ if $\pi_i w \leqslant \pi_i w'$ for all $i = 1, 2, \ldots, n$. A subset $t = (w_0, \ldots, w_p)$ of V^n is a simplex of K^n if it is linearly ordered and the vertices $\pi_i w_0, \ldots, \pi_i w_p$ span a simplex of K for all $i = 1, \ldots, n$. It follows from the definition of K^n that the projections are simplicial and that G is a group of order preserving functions on K^n. Since each $\alpha \in G$ is order preserving on V^n, we have that if $(w, \alpha w)$ is a simplex of K^n, then $w \leqslant \alpha w \leqslant \alpha^2 w \leqslant \ldots \leqslant \alpha^k w = w$ for some integer k, hence $\alpha w = w$.

Let $\mathrm{Sd}(K^n)$ denote the first barycentric subdivision of K^n (the set B of vertices of $\mathrm{Sd}(K^n)$ consists in all barycenters b_t of simplexes t of K^n). The group G operates on $\mathrm{Sd}(K^n)$ by $\alpha b_t = b_{\alpha t}$ and the simplicial map φ: $\mathrm{Sd}(K^n) \to K^n$ (which associates to b_t the least vertex of t) commutes with each $\alpha \in G$. Furthermore, if $(b_t, b_{t'})$ and $(b_t, \alpha b_{t'})$ are both simplexes of $\mathrm{Sd}(K^n)$, then $\alpha b_t = b_t$ [6].

A triangulation $K(n, G)$ for X^n/G can now be defined as follows [6]. The set A of vertices of $K(n, G)$ is the set of equivalence classes of elements of B under G. A subset $(a_0, \ldots, a_p)$ of A form a simplex of $K(n, G)$ if there exists $b_i \in a_i$ so that $(b_0, \ldots, b_p)$ is a simplex of $\mathrm{Sd}(K^n)$. If another choice is made, say $b'_i \in a_i$ so that $(b'_0, \ldots, b'_p)$ is a simplex of $\mathrm{Sd}(K^n)$, then $b'_i = \alpha_i b_i$ for some $\alpha_i \in G$, $i = 0, 1, \ldots, p$. For any i we have (b_i, b_p) and $(\alpha_i b_i, \alpha_p b_p)$ simplexes of $\mathrm{Sd}(K^n)$, and therefore (b_i, b_p) and $(b_i, \alpha_i^{-1}\alpha_p b_p)$ are simplexes of $\mathrm{Sd}(K^n)$. By a previous argument, we have $\alpha_i^{-1}\alpha_p b_i = b_i$, and hence $\alpha_p b_i = \alpha_i b_i$. Therefore

$$(b'_0, \ldots, b'_p) = (\alpha_0 b_0, \ldots, \alpha_p b_p) = (\alpha_p b_0, \ldots, \alpha_p b_p) = \alpha_p(b_0, \ldots, b_p).$$

The p-simplexes of $K(n, G)$ are therefore in one to one correspondence with the equivalence classes of p-simplexes of $\mathrm{Sd}(K^n)$.

Let $C(K)$, $C(K^n)$, $C(\mathrm{Sd}(K^n))$ and $C(K(n, G))$ be the integral chain groups defined on oriented simplexes of the respective complexes. There exists a chain

map μ_K: $C(K(n, G)) \to C(K)$ which is obtained by the following commutative diagram

$$\begin{array}{ccc} C(\mathrm{Sd}(K^n)) & \xrightarrow{\varphi_\#} & C(K^n) \\ {\scriptstyle \eta_\#}\downarrow & & \downarrow{\scriptstyle \sum_{i=1}^{n} \pi_{i\#}} \\ C(K(n, G)) & \xrightarrow[\mu_K]{} & C(K) \end{array}$$

For an oriented simplex $t = (a_0, \ldots, a_p)$, choose any oriented simplex $\sigma = (b_0, \ldots, b_p)$ such that $\eta(\sigma) = t$. Then

$$\mu_K(\sigma) = \sum_{i=1}^{n} \pi_{i\#}\varphi_\#(\sigma).$$

This definition is independent of the choice of σ [6]. If f: $K \to L$ is an order preserving simplicial map between two order complexes, then ([6], Lemma 1)

$$f_\# \mu_K = \mu_L \tilde{f}_\#$$

where $\tilde{f}$: $K(n, G) \to L(n, G)$ is the simplicial map induced by the map f.

3. The fixed point index. Let X be a finite polyhedra and f: $X \to X^n/G$ be a map. Let K be a subdivision of the basic triangulation of X, regarded as an ordered complex. Let f': $K' \to K(n, G)$ be a simplicial approximation of f, where K' is a subdivision of K. Let $\Psi_\#$: $C(K) \to C(K')$ be the usual subdivision chain map. The composition $\mu_K f'_\# \Psi_\#$: $C(K) \to C(K)$ is called the chain map induced by the map f and it will be denoted by $f_\#$. Let $\hat{\sigma}^i$ denote the elementary cochain dual to the positively oriented simplex σ^i such that $\langle \sigma^i, \hat{\sigma}^j \rangle = \hat{\sigma}^j(\sigma^i) = \delta_{ij}$.

Let U be an open subset of X and f: $X \to X^n/G$ be a map which has no fixed point on the boundary of U. Let $f_\#$: $C(K) \to C(K)$ be a chain map induced by f. We define the index of f on U to be the number $I(X, f, U)$ given by

$$I(X, f, U) = \sum_{i=1}^{n} \sum_{\sigma_p \in \mathrm{St}_K U} (-1)^p \langle f_\#(\sigma_p), \hat{\sigma}_p \rangle$$

where p denotes the dimension of the simplex σ_p.

(3.1) Lemma. *Let X be a finite polyhedron and C be a closed subset of X which contains no fixed point of the map f: $X \to X^n/G$. Then there exists a positive number $\varepsilon = \varepsilon(C)$ such that if K is a triangulation of X and* mesh $|K| < \frac{1}{2}\varepsilon\sqrt{n}$ *and if $f_\#$ is a chain map induced by the map f, and if σ_p is an oriented simplex such that $d(|\sigma_p|, C) < \frac{1}{2}\varepsilon$, then $\langle f_\#(\sigma_p), \hat{\sigma}_p \rangle = 0$.*

Proof. From definition of the function ω, it follows that $x \in X$ is a fixed point of f iff $\omega(x, f(x)) = 0$. Since C contains no fixed point of f, it follows that $\omega(x, f(x)) > 0$ for all $x \in C$. Since C compact, there exists a number $\xi > 0$ such that $\omega(x, f(x)) > \xi$ for all $x \in C$. Since f is uniformly continuous, there exists a number $\delta > 0$ such that if $x, y \in X$ and $d(x, y) < \delta$, then $\tilde{d}(f(x), f(x)) < \frac{1}{8}\xi$. Let

$\varepsilon = \varepsilon(C) = \min(\delta, \frac{1}{8}\xi)$. Let mesh $|K| < \frac{1}{2}\varepsilon\sqrt{n}$ and $f'\colon K' \to K(n, G)$ be a simplicial approximation to f. We claim that if $y \in X$ and $d(y, C) < \varepsilon$, then $\omega(y, f'(y)) > \varepsilon$.

There exists an $x \in C$ such that $d(x, y) = d(y, C) < \varepsilon < \delta$. It follows that $\bar{d}(f(x), f(y)) < \frac{1}{8}\xi$. Also since mesh$|K| < \frac{1}{2}\varepsilon\sqrt{n}$ and f' is a simplicial approximation of f, we have $\bar{d}(f(y, f'(y)) < \text{mesh}|K(n, G)| < \frac{1}{2}\varepsilon < \frac{1}{8}\xi$. It follows that

$$\begin{aligned}\xi < \omega(x, f(x)) &\leqslant \omega(x, f'(y)) + \bar{d}(f'(y), f(x))\\ &\leqslant \omega(x, f'(y)) + \bar{d}(f'(y), f(y)) + \bar{d}(f(y), f(x))\\ &\leqslant \omega(x, f'(y)) + \tfrac{1}{8}\xi + \tfrac{1}{8}\xi .\end{aligned}$$

Hence $\omega(x, f'(y)) > \xi - \frac{1}{4}\xi = \frac{3}{4}\xi$.

Suppose $\omega(y, f'(y)) < \varepsilon < \frac{1}{8}\xi$. Let $\eta(z) = f'(y)$, where $z \in X^n$, then for some i, $1 \leqslant i \leqslant n$, $\omega(y, f'(y)) = d(y, \pi_i(z)) < \varepsilon < \xi$. It follows that

$$\omega(x, f'(y)) \leqslant d(x, \pi_i(z)) \leqslant d(x, y) + d(y, \pi_i(z)) < \varepsilon + \varepsilon = 2\varepsilon < \tfrac{1}{4}\xi$$

which is a contradiction because $\omega(x, f'(y)) > \frac{3}{4}\xi$. Hence if $d(y, C) < \varepsilon$, then $\omega(y, f'(y)) > \varepsilon$.

Let $\Psi_{\#}\colon C(K) \to C(K')$ be the standard subdivision chain map. Let σ_p be an oriented p-simplex of K and $\Psi_{\#}(\sigma_p) = \sum_j \lambda_j \sigma_{pj}$, where $\lambda_j = \pm 1$ and σ_{pj} is anoriented p-simplex of K' such that $|\sigma_{pj}| \subset |\sigma_p|$. Let $f'(\sigma_{pj}) = \eta(\tau_{pj})$, where τ_{pj} is a p-simplex of $\mathrm{Sd}(K^n)$. Let $\varphi\colon \mathrm{Sd}(K^n) \to K^n$ be the standard order preserving simplicial map. Let $\varphi(\tau_{pj}) = \gamma_{pj}$, where γ_{pj} is a p-simplex of K^n such that $|\tau_{pj}| \subset |\gamma_{pj}|$. It follows that

$$f_{\#}(\sigma_p) = \mu_K f'_{\#} \Psi_{\#}(\sigma_p) = \mu_K f'_{\#}\Big(\sum_j \lambda_j \sigma_{pj}\Big) = \sum_j \lambda_j \mu_K f'_{\#}(\sigma_{pj}) .$$

Since $f'(\sigma_{pj}) = \eta(\tau_{pj})$, it follows by definition of μ_K that

$$f_{\#}(\sigma_p) = \sum_j \sum_{i=1}^{n} \lambda_j \pi_{i\#} \varphi_{\#}(\tau_{pj}) = \sum_{i=1}^{n} \sum_j \lambda_j \pi_{i\#}(\gamma_{pj}) .$$

If $\langle f_{\#}(\sigma_p), \hat{\sigma}_p \rangle \neq 0$, then there exist i and j such that $\pi_i(\gamma_{pj}) = \pm\sigma_p$. If $y \in |\sigma_{pj}| \subset |\sigma_p|$, then $f'(y) \in |f'(\sigma_{pj})| = \eta(|\tau_{pj}|)$ and there exists $z \in |\tau_{pj}|$ such that $\eta(z) = f'(y)$. Since $|\tau_{pj}| \subset |\gamma_{pj}|$ and $|\pi_i(\gamma_{pj})| = |\sigma_p|$, it follows that $\pi_i(z) \in |\sigma_p|$ and

$$\omega(y, f'(y)) \leqslant d(y, \pi_i(z)) < \tfrac{1}{2}\varepsilon\sqrt{n} < \varepsilon .$$

However, if $d(|\sigma_p|, C) < \frac{1}{2}\varepsilon$, then for every $y \in |\sigma_{pj}| \subset |\sigma_p|$, $d(y, C) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon\sqrt{n} < \varepsilon$ and hence $\omega(y, f'(y)) > \varepsilon$. It follows that $\langle f_{\#}(\sigma_p), \hat{\sigma}_p \rangle = 0$.

(3.2) DEFINITION. A triple (X, f, U) will be called an *admissible triple* if X is a finite polyhedron, $f\colon X \to X^n/G$ is a map and U is an open subset of X which has no fixed points of f on its boundary.

Let $F(f)$ denote the set of fixed points of the map $f\colon X \to X^n/G$.

(3.3) LEMMA. *Let (X, f, U) be an admissible triple and $U_1, \ldots, U_k$ be mutually disjoint open subsets of U such that $\{\mathrm{cl}\, U - \bigcup_{i=1}^{k} U_i\} \cap F(f) = \emptyset$ (empty set). Then there exists $\varepsilon > 0$ such that if* mesh$|K| < \frac{1}{2}\varepsilon\sqrt{n}$, *then*

$$I(X, f, U) = \sum_{j=1}^{k} I(X, f, U_j) .$$

Proof. Let $C = \mathrm{cl}\, U - \bigcup_{i=1}^{k} U_i$, then by Lemma (3.1) there exists $\varepsilon = \varepsilon(C)$ such that if mesh$|K| < \frac{1}{2}\varepsilon\sqrt{n}$ and $d(|\sigma_p|, C) < \frac{1}{2}\varepsilon$, then $\langle f_{\#}(\sigma_p), \hat{\sigma}_p \rangle = 0$, where $f_{\#}$ is the chain map induced by f. It follows that

$$\begin{aligned} I(X, f, U) &= \sum_{p=0}^{\infty} \sum_{\sigma_p \in \mathrm{St}_K U} (-1)^p \langle f_{\#}(\sigma_p), \hat{\sigma}_p \rangle\\ &= \sum_{p=0}^{\infty} \sum_{j=1}^{k} \sum_{\sigma_p \in \mathrm{St}_K U_j} (-1)^p \langle f_{\#}(\sigma_p), \hat{\sigma}_p \rangle\\ &= \sum_{j=1}^{k} \sum_{p=0}^{\infty} \sum_{\sigma_p \in \mathrm{St}_K U_j} (-1)^p \langle f_{\#}(\sigma_p), \hat{\sigma}_p \rangle\\ &= \sum_{j=1}^{k} I(X, f, U_j) . \end{aligned}$$

If $\{U_1, \ldots, U_k\}$ is a collection of open sets of X such that $\mathrm{cl}\, U_i \cap \mathrm{cl}\, U_j = \emptyset$ for $i \neq j$, and $F(f) \subset \bigcup_{i=1}^{k} U_i$, then we say that $\{U_1, \ldots, U_k\}$ partitions $F(f)$. We have

(3.4) LEMMA. *Let $\{U_1, \ldots, U_k\}$ partition the set $F(f)$, the set of fixed points of the map $f\colon X \to X^n/G$. If the triangulation K of X is sufficiently fine, then*

$$L(f) = \sum I(X, f, U_j)$$

where $L(f)$ is the Lefschetz number of the map f.

We shall show that the index $I(X, f, U)$ defined above is independent of the choice of the subdivision K of the basic triangulation of X and the simplicial map $f'\colon K' \to K(n, G)$ provided that K is sufficiently fine. This is accomplished by the following lemma the proof of which is similar to Lemma 2.2 in [7].

(3.5) LEMMA. *Let X be a finite polyhedron and $f\colon X \to X^n/G$ be a map. Let f_i $(i = 1, 2)$ be a simplicial approximation from a triangulation K_i' to $K_i(n, G)$. Let B_i be a K_i-subpolyhedron of X and A_i a K_i'-subpolyhedron such that*

(1) $\mathrm{cl}\,\mathrm{St}_{K_1}(B_1) \cap \mathrm{cl}\,\mathrm{St}_{K_2}(B_2) = \emptyset$ *and*

(2) $f_i(A_i) \subset B_i^n/G$,

then $f_1|A_1$ and $f_2|A_2$ have a common extension f' which is a simplicial approximation of f from a triangulation K' to $K(n, G)$.

Proof. Let $C_i = \mathrm{clSt}_{K_i}B_i$ $(i = 1, 2)$. Since all triangulations are subdivisions of a common triangulation, we may extend $K_1|C_1$ and $K_2|C_2$ to a triangulation K. Simmilarly, extend $K_1'|\mathrm{clSt}_{K_1'}A_1$ and $K_2'|\mathrm{clSt}_{K_2'}A_2$ to a triangulation K' and let $K^{(j)}$ denotes the jth barycentric subdivision of K' modulo $A_1 \cup A_2$ (that is modify the usual subdivision by adding no new vertices to $A_1 \cup A_2$). $K^{(j)}$ is also an extension of $K_1'|A_1$ and $K_2'|A_2$. If $K^{(j)}$ is sufficiently fine and the vertex v is not in the $\mathrm{St}_{K_i'}A_i$, then $\mathrm{St}(v)$ has arbitrarily small diameter. Choose $K^{(j)}$ so fine that if v is one of its vertices in $X-(\mathrm{St}_{K_1'}A_1 \cup \mathrm{St}_{K_2'}A_2)$, then $\mathrm{St}(v) \subset f^{-1}(\mathrm{St}(w))$ for some $w \in K(n, G)$. Let $f'(v) = w$. If $v \in A_i$, let $f'(v) = f_i(v)$. Finally if $v \in \mathrm{St}_{K_i'}A_i - A_i$, then there exists a vertex v' of K_i such that $\mathrm{St}(v) \subset \mathrm{St}(v')$. Let $f'(v) = f_i(v')$. The vertex assignment above determines the desired simplicial approximation of f.

(3.6) Lemma. *If (X, f, U) is admissible triple, then the index $I(X, f, U)$ is independent of the choice of the simplicial approximation and the subdivision K of the basic triangulation, provided K is sufficiently fine.*

Proof. For $A \subset X$, let $\mathrm{St}^n(A) = \mathrm{Stcl}(\mathrm{St}^{n-1}(A))$, $n \geqslant 2$. Let $\varepsilon = \frac{1}{5}\varepsilon(\partial U)$, where ∂U denotes the boundary of U and $\varepsilon(\partial U) > 0$ is the number obtained by Lemma (3.1) for $C = \partial U$. Let K_1 and K_2 be the triangulations of X such that $\mathrm{mesh}|K_i| < \frac{1}{2}\varepsilon\sqrt{n}$, $i = 1, 2$. Let f_i: $K_i' \to K_i(n, G)$ be a simplicial approximation of f. We wish to show that if $f_{1\#}$ and $f_{2\#}$ are the chain maps induced by f_1 and f_2 respectively, then $I(X, f_{1\#}, U) = I(X, f_{2\#}, U)$.

Let $U = U_1$ and $U_2 = X - \mathrm{clSt}^5(U)$. Let $B_i = \mathrm{clSt}_{K_i}(U_i)$. Let $A_i = f_i^{-1}(B/G)$. It is easy to see that U_2 has no fixed point of f on its boundary and

$$\mathrm{clSt}_{K_1}(B_1) \cap \mathrm{clSt}_{K_2}(B_2) = \emptyset .$$

It follows from Lemma (3.5) that $f_1|A_1$ and $f_2|A_2$ have a common extension f' which is a simplicial approximation from a triangulation K' to $K(n, G)$. It follows that

$$I(X, f_{1\#}, U_1) = I(X, f'_{\#}, U_1) \quad \text{and} \quad I(X, f_{2\#}, U_2) = I(X, f'_{\#}, U_2) .$$

Since $\{U_1, U_2\}$ partition $F(f)$, it follows from Lemma (3.4) that

$$L(f) = I(X, f'_{\#}, U_1) + I(X, f'_{\#}, U_2) = I(X, f_{1\#}, U_1) + I(X, f_{2\#}, U_2) .$$

Simmilarly

$$L(f) = I(X, f_{2\#}, U_1) + I(X, f_{2\#}, U_2) .$$

It follows that

$$I(X, f_{1\#}, U_1) + I(X, f_{2\#}, U_2) = I(X, f_{2\#}, U_1) + I(X, f_{2\#}, U_2) .$$

Hence

$$I(X, f_{1\#}, U_1) = I(X, f_{2\#}, U_1) .$$

(3.7) Lemma. *Let H: $X \times I \to X^n/G$ be a homotopy such that (X, H_t, U) is admissible for every $t \in I$, where $H_t(x) = H(x, t)$ for $x \in X$ and $t \in I$. Then $I(X, H_0, U) = I(X, H_1, U)$.*

Proof. Let K be sufficiently fine triangulation of X. Since X is a compact polyhedron there exists a number $\delta > 0$ such that if f, g: $X \to X^n/G$ are two maps such that $\tilde{d}(f(x), g(x)) < \delta$ for all $x \in X$, then f and g have a common simplicial approximation. It follows that if f and g have no fixed points on the boundary of U, then $I(X, f, U) = I(X, g, U)$.

By compactness of I and uniform continuity of H we can find a finite subdivision, $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_m = 1$, of the interval I such that $\tilde{d}(H_{t_{i-1}}(x), H_{t_i}(x)) < \delta$, for all $x \in X$ and all i, $i = 1, \ldots, m$. It follows that

$$I(X, H_0, U) = I(X, H_{t_0}, U) = \ldots = I(X, H_{t_m}, U) = I(X, H_1, U) .$$

In the proof of Lemma (3.5) we assumed that all triangulations of X are subdivisions of the basic triangulation of X. Now we shall show that the index is actually independent of the choice of the basic triangulation of X.

Let us denote the index $I(X, f, U)$ by $I(T, f, U)$, if T is the basic triangulation of X used in the definition of the index.

Let (X, f, U) be an admissible triple. As in the proof of Lemma (3.1) we can find numbers $\xi > 0$ and $\delta > 0$ such that $\omega(x, f(x)) > \xi$ for all $x \in \partial U$ and if $x, y \in X$ such that $d(x, y) < \delta$, then $\tilde{d}(f(x), f(y)) < \frac{1}{8}\xi$. Let $\varepsilon = \min(\delta, \frac{1}{8}\xi)$.

Let T_1 and T_2 be two triangulations of X (not necessarily the subdivisions of the same basic triangulation of X). Let K and L be the subdivisions of T_1 and T_2 respectively such that $\mathrm{mesh}|K|$ and $\mathrm{mesh}|L| < \frac{1}{4}\varepsilon\sqrt{n}$ and g: $K \to L$ be a simplicial approximation to the identity map I_X: $X \to X$. Let q: $K^{(1)} \to L^{(1)}$ be the barycentric map defined by g, that is $q(b_t) = b_{g(t)}$, where b_t is the barycenter of the simplex t of K. The first barycentric subdivisions $K^{(1)}$ and $L^{(1)}$ are ordered complexes and q is an order preserving simplicial map. Let f': $L^{(r)} \to K^{(1)}(n, G)$ be a simplicial approximation of f. Let e: $K^{(r)} \to L^{(r)}$ be the rth barycentric map induced by the map g, where $K^{(r)}$ and $L^{(r)}$ are the rth barycentric subdivisions of K and L respectively. Let $\tilde{q}$: $K^{(1)}(n, G) \to L^{(1)}(n, G)$ be the simplicial map induced by the map q [6]. Let $|q|$, $|e|$: $X \to X$ and $|\tilde{q}|$: $X^n/G \to X^n/G$ be the topological maps defined by the simplicial maps q, e and $\tilde{q}$ respectively. We have the following lemmas concerning the maps defined above.

(3.8) Lemma. *If (X, f, U) is an admissible triple then $(X, f|e|, U)$ and $(X, |\tilde{q}|f, U)$ are admissible and*

$$I(T_1, f, U) = I(T_1, f|e|, U) \quad \text{and} \quad I(T_2, f, U) = I(T_2, |\tilde{q}|f, U) .$$

Proof. Since g is a simplicial approximation of the identity map and e is defined by g, it follows that for each $x \in X$, $|e|(x)$ and x belong to the same closed simplex of L. Let h: $X \times I \to X$ be the homotopy defined by, for $x \in X$ and $t \in I$, $h(x, t) = t|e|(x) + (1-t)x$. Then $H = fh$: $X \times I \to X^n/G$ is a homotopy between the maps f and $f|e|$. Also, for $x \in X$ and $t \in I$, $d(h_t(x), x) < \operatorname{mesh}|L| < \frac{1}{4}\varepsilon\sqrt{n}$. Which implies that $\tilde{d}(fh_t(x), f(x)) < \frac{1}{8}\xi$. We have

$$\omega(x, f(x)) \leqslant \omega(x, fh_t(x)) + \tilde{d}(fh_t(x), f(x)) \leqslant \omega(x, H_t(x)) + \tfrac{1}{8}\xi .$$

Hence $\omega(x, H_t(x)) > \frac{7}{8}\xi$. It follows that (X, H_t, U) is admissible for all $t \in I$.

Similarly, there exists a homotopy $S\colon X \times I \to X^n/G$ between the maps $|\tilde{q}|f$ and f such that (X, S_t, U) is admissible for all $t \in I$. It now follows from Lemma (3.7) that

$$I(T_1, f|e|, U) = I(T_1, f, U) \quad \text{and} \quad I(T_2, f, U) = I(T_2, |\tilde{q}|f, U) .$$

(3.9) LEMMA. (i) *The following diagram comutes.*

$$\begin{array}{ccccccc}
C(K^{(1)}) & \xrightarrow{q_\#} & C(L^{(1)}) & & C(K^{(1)}) & \xrightarrow{q_\#} & C(L^{(1)}) \\
\downarrow{\scriptstyle \chi_K} & & \downarrow{\scriptstyle \chi_L} & & \uparrow{\scriptstyle \mu_{K^{(1)}}} & & \uparrow{\scriptstyle \mu_{L^{(1)}}} \\
C(K^{(r)}) & \xrightarrow[e_\#]{} & C(L^{(r)}) & \xrightarrow{f'_\#} & C(K^{(1)}(n, G)) & \xrightarrow[\tilde{q}_\#]{} & C(L(n, G))
\end{array}$$

where χ_K and χ_L are the standard barycentric chain maps.

(ii) *If σ is a p-simplex of $\mathrm{St}_{K^{(1)}} U$ and $\langle \mu_{K^{(1)}} f'_\# e_\# \chi_K(\sigma), \hat{\sigma} \rangle \neq 0$, then $q(\sigma) \in \mathrm{St}_{L^{(1)}} U$.*

(iii) *If τ is a p-simplex in $\mathrm{St}_{L^{(1)}} U$ and $\langle \mu_{L^{(1)}} q_\# f'_\# \chi_L(\tau), \hat{\tau} \rangle \neq 0$, then $\emptyset \neq q^{-1}(\sigma) \in \mathrm{St}_{K^{(1)}} U$.*

Proof of (i). Commutativity of the right square follows from the definition of μ and $\tilde{q}$ ([6], Lemma 1). Commutativity of left square is easy to show by using induction on the order of barycentric subdivision. For $r = 2$, the result is well known ([4], Prop. 3.59).

(ii) Let $\sigma \in \mathrm{St}_{K^{(1)}} U$. It follows that $\sigma \cap U \neq \emptyset$. It follows from the hypothesis in (ii) and Lemma (3.1) that $d(|\sigma|, \partial U) > \frac{1}{2}\varepsilon$. Since σ is connected, we have $|\sigma| \subset U$. Since q is the barycentric map defined by the simplicial approximation to the identity map, there exists a simplex γ of L such that $|\mathrm{cl}\gamma| \cap |\sigma| \neq \emptyset \neq |q(\sigma)| \cap |\mathrm{cl}\gamma|$. It follows that $d(|\gamma|, \partial U) > \frac{1}{4}\varepsilon\sqrt{n}$ and that $|\gamma| \cap U \neq \emptyset$. As before this implies that $|\gamma| \subset U$. Hence $q(\sigma) \cap U \neq \emptyset$.

(iii) Let τ be a p-simplex of $\mathrm{St}_{L^{(1)}} U$. Let $\chi_L(\tau) = \sum_j \lambda_j \tau_j$, where τ_j is a p-simplex of $L^{(1)}$ such that $|\tau_j| \subset |\tau|$ and $\lambda_j = \pm 1$, for all j. It follows from the commutativity of diagram in (i) that

$$\mu' q_\# f'_\# \chi_L(\tau) = \tilde{q}_\# \mu f'_\# \chi_L(\tau) = \sum_j \lambda_j \tilde{q}_\# \mu f'_\#(\tau_j)$$

where $\mu' = \mu_{L^{(1)}}$ and $\mu = \mu_{K^{(1)}}$ for simplicity. If γ_j is a p-simplex of $\mathrm{Sd}((K^{(1)})^n)$ such that $\eta(\gamma_j) = f'(\tau_j)$, then by definition of μ, it follows that

$$\mu' \tilde{q}_\# f'_\# \chi_L(\tau) = \sum_j \lambda_j q_\# \sum_{i=1}^{n} \pi_{i\#} \varphi_\#(\gamma_j) = \sum_{i=1}^{n} \sum_j \lambda_j q_\# \pi_{i\#} \varphi(\gamma_j) .$$

Since $\langle \mu' \tilde{q}_\# f'_\# \chi_L(\tau), \hat{\tau} \rangle \neq 0$, it follows that there exists i and j, $1 \leqslant i \leqslant n$, such that $q_\# \pi_{i\#} \varphi(\gamma_j) = \pm\tau$. Let $\pi_i \varphi(\gamma_j) = \sigma$, then $|q(\sigma)| = |\tau|$. As before $|\tau| \cap U \neq \emptyset$ and $d(|\tau|, \partial U) > \frac{1}{2}\varepsilon$ implies that $|\sigma| \cap U \neq \emptyset$. Hence $\sigma \in q^{-1}(\tau) \subset \mathrm{St}_{K^{(1)}} U$.

(3.10) *If T_1 and T_2 are two triangulation of X (not necessarily the subdivisions of the same basic triangulation of X) and (X, f, U) is an admissible triple, then*

$$I(T_1, f, U) = I(T_2, f, U) .$$

Proof. Let K and L be the subdivisions of T_1 and T_2 respectively, as in the proof of Lemmas (3.8) and (3.9). Since index is independent of the subdivision of the basic triangulation (for sufficiently fine subdivisions) and simplicial approximation of f, it follows that

$$I(T_1, f, U) = I(K^{(1)}, f, U) \quad \text{and} \quad I(T_2, f, U) = I(L^{(1)}, f, U) .$$

From Lemma (3.8) we have

$$I(T_1, f, U) = I(T_1, f|e|, U) = I(K^{(1)}, f|e|, U)$$

and

$$I(T_2, f, U) = I(T_2, |\tilde{q}|f, U) = I(L^{(1)}, |\tilde{q}|f, U) .$$

Hence it is sufficient to show that $I(K^{(1)}, f|e|, U) = I(L^{(1)}, |\tilde{q}|f, U)$.

Let $q^\#$ denotes the cochain map dual to the chain map $q_\#$. It is easy to see that for any positively oriented simplex τ of $L^{(1)}$, $q^\#\hat{\tau} = \sum \hat{\sigma}$, where summation is over all oriented simplexes σ of $K^{(1)}$ for which $q(\sigma) = \tau$. It now follows from Lemma (3.9) that

$$\begin{aligned}
I(K^{(1)}, f|e|, U) &= \sum_{p=0}^{\infty} \sum_{\sigma_p \in \mathrm{St}_{K^{(1)}} U} (-1)^p \langle \mu f'_\# e_\# \chi_K(\sigma_p), \hat{\sigma}_p \rangle \\
&= \sum_{p=0}^{\infty} \sum_{\tau_p \in \mathrm{St}_{L^{(1)}} U} (-1)^p \sum_{q(\sigma_p) = \tau} \langle \mu f'_\# e_\# \chi_K(\sigma_p), \hat{\sigma}_p \rangle \\
&= \sum_{p=0}^{\infty} \sum_{\tau_p \in \mathrm{St}_{L^{(1)}} U} (-1)^p \langle \mu f'_\# \chi_L q_\#(\sigma_p), \sum_{q(\sigma_p) = \tau_p} \hat{\sigma}_p \rangle \\
&= \sum_{p=0}^{\infty} \sum_{\tau_p \in \mathrm{St}_{L^{(1)}} U} (-1)^p \langle \mu f'_\# \chi_L(\tau_p), q^\# \hat{\tau}_p \rangle \\
&= \sum_{p=0}^{\infty} \sum_{\tau_p \in \mathrm{St}_{L^{(1)}} U} (-1)^p \langle q_\# \mu f'_\# \chi_L(\tau_p), \hat{\tau}_p \rangle \\
&= \sum_{p=0}^{\infty} \sum_{\tau_p \in \mathrm{St}_{L^{(1)}} U} (-1)^p \langle \mu' \tilde{q}_\# f'_\# \chi_L(\tau_p), \hat{\tau}_p \rangle \\
&= I(L^{(1)}, |\tilde{q}|f, U) .
\end{aligned}$$

This completes the proof of Lemma (3.10).

Some more properties of the index that can be derived from the results in this section are listed in the following theorem.

(3.11) Theorem. *Let U and V be open sets of X whose boundaries contain no fixed points of $f\colon X\to X^n/G$, then the index has the following properties.*

I1. *If U contains no fixed point of f, then $I(X,f,U)=0$.*

I2. $I(X,f,U)+I(X,f,V)=I(X,f,U\cup V)+I(X,f,U\cap V)$.

I3. *There exists a number $\varepsilon>0$ such that if $g\colon X\to X^n/G$ and $\tilde d(f(x),g(x))<\varepsilon$ for all $x\in X$, then (X,g,U) is an admissible triple and $I(X,f,U)=I(X,g,U)$.*

Proof. If U contains no fixed point of f and W is an open subset of U and $\mathrm{cl}W\subset U$, then $\{U, X-\mathrm{cl}W\}$ partition $F(f)$. It follows from Lemma (3.4) that

$$L(f)=I(X,f,U)+I(X,f,X-\mathrm{cl}W).$$

Since $F(f)\subset X-\mathrm{cl}W$, it follows from Lemma (3.3) that

$$L(f)=I(X,f,X-\mathrm{cl}W).$$

Hence we have $I(X,f,U)=0$.

To prove I2, we notice that the case when $U\cap V=\emptyset$ is a direct consequence of Lemma (3.2). To prove I2 in the case when $U\cap V\neq\emptyset$, we consider the following sets, namely, $U-\mathrm{cl}\,V$, $V-\mathrm{cl}\,U$ and $U\cap V$. Applying Lemma (3.2) to the collections $\{U, U-\mathrm{cl}\,V, U\cap V\}$, $\{V, V-\mathrm{cl}\,U, U\cap V\}$ and $\{U\cup V, U-\mathrm{cl}\,V, V-\mathrm{cl}\,U, U\cap V\}$ we obtain

$$I(X,f,U)=I(X,f,U-\mathrm{cl}\,V)+I(X,f,U\cap V),$$

$$I(X,f,V)=I(X,f,V-\mathrm{cl}\,U)+I(X,f,V\cap U),$$

$$I(X,f,U\cup V)=I(X,f,V-\mathrm{cl}\,U)+I(X,f,U-\mathrm{cl}\,V)+I(X,f,U\cap V).$$

It follows that

$$I(X,f,U\cup V)+I(X,f,U\cap V)=I(X,f,U)+I(X,f,V).$$

Proof of I3 is the same as a part of the proof of Lemma (3.7). However, δ has to be chosen more carefully, in order to make sure that (X,g,U) is admissible.

4. Fixed point classes and the Nielsen theorem. Let $f\colon X\to X^n/S_n$ be a map, where S_n is the group of all permutations of the numbers $[1,\dots,n]$. A point $z\in X^n$ is called *admissible with respect to* $x\in F(f)$ if $\eta(z)=f(x)$ and $\pi_1(z)=x$. Since S_n is the group of all permutations the numbers $[1,\dots,n]$, it is easy to see that for every $x\in F(f)$ there exists a point $z\in X^n$ which is admissible with respect to x.

Two points x and $x'\in F(f)$ are said to be *f-equivalent* if there exists points z and $z'\in X^n$, which are admissible with respect to x and x' respectively, and a path $C\colon I\to X^n$ from z to z' such that $\eta C\simeq f\pi_1 C$. It is easy to show that the relation of f-equivalence is an equivalence relation if $F(f)$. The equivalence classes are called the *fixed point classes of f.*

For $y\in X^n/S_n$ and $\varepsilon>0$, let $N(y,\varepsilon)=\{y'\in X^n/S_n|\ \tilde d(y,y')<\varepsilon\}$.

(4.1) Lemma. *If X is a compact polyhedron, then there exists a number $\delta>0$ such that if $C, D\colon I\to N(y,\delta)\subset X^n/S_n$ are paths, where y is any point of X^n/S_n, and $C(0)=D(0)$, $C(1)=D(1)$, then $C\simeq D$.*

Proof. Since a compact polyhedron is an absolute neighborhood retract, it follows from a known result ([1], Lemma 4) that fo revery $y\in X^n/S_n$, there exists a number $\delta>0$ such that if $C, D\colon I\to N(y,\delta)\subset X^n/S_n$ are two paths and $C(0)=D(0)$, $C(1)=D(1)$, then $C\simeq D$. Cover X^n/S_n by open sets $\{N(y,\delta(y))\}$. Let the Lebesgue number of this cover be 2δ.

(4.2) Lemma. *If X is a compact polyhedron and $f\colon X\to X^n/S_n$ is a map, then there exists a number $\varepsilon>0$ such that if $x, x'\in F(f)$ and $d(x,x')<\varepsilon$, then x is f-equivalent to x'.*

Proof. For $A\subset X$, let $D(A)=\mathrm{Sup}\{d(x,x')|\ x,x'\in A\}$. Since X is compact and locally path connected, it is easy to find a number $\lambda>0$ such that if $x, x'\in X$ and $d(x,x')<\lambda$, then there exists a path connected open set U such that $x,x'\in U$ and $D(U)<\frac{1}{2}\delta\sqrt{n}$, where δ is the number in Lemma (4.1). Since f is uniformly continuous, there exists a number $\theta>0$ such that if $x, x'\in X$ and $d(x,x')<\theta$, then $\tilde d(f(x),f(x'))<\min(\frac{1}{3}\lambda, \frac{1}{2}\delta\sqrt{n})$. Let $\xi=\min(\theta,\frac{1}{3}\lambda,\frac{1}{2}\delta\sqrt{n})$. Once again we can find a number $\varepsilon>0$ such that if $x, x'\in X$ and $d(x,x')<\varepsilon$, then there exists an open path connected set V such that $D(V)<\xi$ and $x,x'\in V$.

Let $x,x'\in F(f)$ and $d(x,x')<\varepsilon$. Let $z=(x_1,\dots,x_n)$ and $z'=(x_1',\dots,x_n')$ be elements of X^n such that $\eta(z)=f(x)$, $\eta(z')=f(x')$, $\pi_1(z)=x_1=x$ and $\pi_1(z')=x_1'=x'$. Since $d(x,x')<\varepsilon<\theta$ we have $\tilde d(f(x),f(x'))<\frac{1}{3}\lambda$. There exists $\alpha\in S_n$ such that $\tilde d(f(x),f(x'))=\bar d(z,\alpha z')<\frac{1}{3}\lambda$. It follows that $d(x_i,x_{\alpha(i)})<\frac{1}{3}\lambda$, for all i, $i=1,\dots,n$. Let $\beta\in S_n$ be such that $\beta\alpha(1)=1$, $\beta(1)=\alpha(1)$ and $\beta(i)=i$ for all i, $i=2,\dots,n$ and $i\neq\alpha(1), 1$. Let $\alpha(j)=1$, for some j, $1\leqslant j\leqslant n$. It follows that

$$d(x_j,x_1')=d(x_j,x'_{\alpha(j)})\leqslant\bar d(z,\alpha z')<\tfrac{1}{3}\lambda.$$

Since $\beta\alpha(1)=1$, we have

$$d(x_1,x'_{\beta\alpha(1)})=d(x_1,x_1')=d(x,x')<\varepsilon<\tfrac{1}{3}\lambda.$$

It follows that

$$d(x_j,x_{\beta\alpha(j)})\leqslant d(x_j,x_1')+d(x_1',x_1)+d(x_1,x'_{\beta\alpha(1)})<\tfrac{1}{3}\lambda+\tfrac{1}{3}\lambda+\tfrac{1}{3}\lambda=\lambda.$$

Hence $d(x_i,x'_{(i)})<\lambda$ for all i, $i=1,\dots,n$.

By the choice of the numbers ε and λ, there exist open sets $O_1,\dots,O_n$ of X such that $D(O_1)<\xi$, $D(O_i)<\frac{1}{2}\delta\sqrt{n}$ for $i=2,\dots,n$, and z, $\beta\alpha z'\in O_1\times O_2\times\dots\times O_n=O$. Since O is path connected, there exists a path $C\colon I\to O$ such that $C(0)=z$ and $C(1)=\beta\alpha z'$. Clearly $\pi_1 C\colon I\to O_1\subset X$ is a path joining x and x'. Since $D(O_1)<\varepsilon<\theta$, it follows that $f\pi_1 C(I)\subset N(\eta(z),\delta)$. Also $\eta C(I)\subset N(\eta(z),\delta)$. Since

$$\eta C(0)=\eta(z)=f(x)=f\pi_1 C(0)$$

and

$$\eta C(1) = \eta(\beta\alpha z') = \eta(z') = f(x') = f\pi_1 C(1)$$

it follows from Lemma (4.1) that $\eta C \simeq f\pi_1 C$. Hence x is f-equivalent to x'.

It follows from Lemma (4.2) that fixed point classes of a map $f\colon X \to X^n/S_n$ are open and closed subsets of $F(f)$. Since $F(f)$ is compact, it follows that f has only finite number of fixed point classes. Hence we have the following theorem.

(4.3) THEOREM. *If X is compact polyhedron, then the map $f\colon X \to X^n/S_n$ has only finite number of fixed point classes* (compare [1], Theorem 6).

Let Γ be a fixed point class of $f\colon X \to X^n/S_n$. Let U be an open subset of X such that $U \cap F(f) = \Gamma$ and $F(f) \cap \partial U = \emptyset$. Existence of such an open set U follows from Lemma (4.2). Let $I(\Gamma) = I(X, f, U)$. Then $I(\Gamma)$ is called the *index of the fixed point class* Γ. It follows from Lemma (3.2) that $I(\Gamma)$ is independent of the choice of the open set U. The fixed point class Γ is called *essential* if $I(\Gamma) \neq 0$, otherwise it is called *inessential.*

(4.4) DEFINITION. The Nielsen number of a symmetric product map $f\colon X \to X^n/S_n$ of a finite polyhedron X is the *number of essential fixed point classes of f*. The Nielsen number of the map f is denoted by $N(f)$.

It is easy to see that if the Lefschetz number $L(f) \neq 0$, then $N(f) \neq 0$ and f has at least $N(f)$ fixed points.

5. H-related classes and the Nielsen fixed point theorem. Let $h\colon X \times I \to X^n/S_n$ be a homotopy. Let $C\colon I \to X$ be a path in X. Then $\langle h; C\rangle\colon I \to X^n/S_n$ is the path defined by, for $t \in I$, $\langle h; C\rangle(t) = h(C(t), t)$. A point $x_0 \in F(h_0)$ is said to be *h-related to a point* $x_1 \in F(h_1)$ if there exists points $z_0, z_1 \in X^n$, which are admissible with respect to x_0 and x_1 respectively, and a path $C\colon I \to X^n$ from z_0 to z_1 such that $\eta C \simeq \langle h; \pi_1 C\rangle$.

(5.1) THEOREM. *Let X be a compact polyhedron and $h\colon X \times I \to X^n/S_n$ be a homotopy. If $x_0, x_0' \in F(h_0)$ and $x_1, x_1' \in F(h_1)$ such that x_0 is h-related to x_1, x_0' is h-related to x_1' and x_0 is h_0-equivalent to x_0', then x_1 is h_1-equivalent to x_1'.*

Proof. Since x_0 is h-related to x_1 and x_0' is h-related to x_1', there exists points z_0, z_1, z_0' and z_1' which are admissible with respect to x_0, x_1, x_0' and x_1' respectively, and paths A and B from z_0 to z_1 and from z_0' to z_1' respectively, such that $\eta A \simeq \langle h; \pi_1 A\rangle$ and $\eta B \simeq \langle h; \pi_1 B\rangle$. Since x_0 is h_0-equivalent to x_0', there exists points $\bar{z}_0$ and $\bar{z}_0'$ which are admissible with respect to x_0 and x_0' respectively, and path E from $\bar{z}_0$ to $\bar{z}_0'$ such that $\eta E \simeq h_0 \pi_1 E$.

Since $\eta(z_0) = \eta(\bar{z}_0)$ and $\eta(z_0') = \eta(\bar{z}_0')$, there exist $\alpha, \beta \in S_n$ such that $\alpha z_0 = \bar{z}_0$ and $\beta z'_0 = \bar{z}_0'$ and $\alpha(1) = 1 = \beta(1)$. Then

$$C(t) = \begin{cases} \alpha A(1-3t) & \text{for } 0 \leqslant t \leqslant \frac{1}{3}, \\ E(3t-1) & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ \beta B(3t-2) & \text{for } \frac{2}{3} \leqslant t \leqslant 1 \end{cases}$$

is a path from αz_1 to $\beta z_1'$. Let $C'\colon I \to X^n/S_n$ be a path from $h_1(x_1)$ to $h_1(x_0)$ defined as follows

$$C'(t) = \begin{cases} h(\pi_1 \alpha A(1-3t), 1-3t) & \text{for } 0 \leqslant t \leqslant \frac{1}{3}, \\ h(\pi_1 E(3t-1), 0) & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ h(\pi_1 \beta B(3t-2), 3t-2) & \text{for } \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

Consider the homotopy

$$g(t, s) = \begin{cases} h(\pi_1 \alpha A(1-3t), 1-3t(1-s)) & \text{for } 0 \leqslant t \leqslant \frac{1}{3}, \\ h(\pi_1 E(3t-1), s) & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ h(\pi_1 \beta B(3t-2), 3(t-1)(1-s)+1) & \text{for } \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

It is easy to see that g is a homotopy between C' and $h_1 \pi_1 C$. Since $\pi_1 \alpha = \pi_{\alpha(1)} = \pi_1$ and $\pi_1 \beta = \pi_1$, it follows that

$$\eta\alpha A = \eta A \simeq \langle h; \pi_1 A\rangle = \langle h; \pi_1 \alpha A\rangle$$

and

$$\eta\beta B = \eta B \simeq \langle h; \pi_1 B\rangle = h; \pi_1 \beta B\rangle.$$

Let P, Q and R be the homotopies between $\eta\alpha A$ and $\langle h; \pi_1 \alpha A\rangle$, $\eta\beta B$ and $\langle h; \pi_1 \beta B\rangle$ and ηE and $h_0 \pi_1 E$, respectively. Then it is easy to see that the homotopy H, defined as follows

$$H(t, s) = \begin{cases} P((1-3t), s) & \text{for } 0 \leqslant t \leqslant \frac{1}{3}, \\ R((3t-1), s) & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ Q((3t-2), s) & \text{for } \frac{2}{3} \leqslant t \leqslant 1 \end{cases}$$

is a homotopy between ηC and C'. It follows that

$$\eta C \simeq C' \simeq h_1 \pi_1 C.$$

Since C is a path between αz_1 and $\beta z_1'$, which are admissible with respect to x_1 and x_1' respectively, it follows that x_1 is h_1-equivalent to x_1'.

(5.2) THEOREM. *Let X and h be as in Theorem (5.1). If $x_0, x_0' \in F(f)$ and $x_1, x_1' \in F(h_1)$ such that x_0 is h_0-equivalent to x_0', x_1 is h_1-equivalent to x_1' and x_0 is h-related to x_1, then x_0' is h-related to x_1'.*

The proof of this theorem is similar to the proof of Theorem (5.1), hence it is omitted.

Let $h\colon X \times I \to X^n/S_n$ be a homotopy. A fixed point class Γ of h_0 is said to be *h-related to a fixed point class Γ' of h_1*, if a point of Γ is h-related to a point Γ'. It is clear from Theorem (5.2) that this definition is independent of the choice of points in Γ and Γ'.

(5.3) LEMMA. *If X is a finite polyhedron and $\Gamma_1, \ldots, \Gamma_k$ are fixed point classes of the map $f\colon X \to X^n/S_n$, then there exists a number $\varepsilon(f) > 0$ such that if $h\colon X \times I \to X^n/S_n$*

is a homotopy such that $h_0 = f$ *and* $\operatorname{Sup}_{t\in I} \varrho(f, h_t) < \varepsilon(f)$, *where* ϱ *is the metric in* $(X^n/S_n)^X$, *then*

(i) *Every fixed point class* Γ' *of* h_1 *is h-related to a fixed point class* Γ_j *of f, for some* j, $1 \leqslant j \leqslant k$.

(ii) *If a class* Γ_j *of f is h-related to a class* Γ' *of* h_1, *then* $I(\Gamma_j) = I(\Gamma')$.

(iii) *If a class* Γ_j *of f is not h-related to any class of* h_1, *then* $I(\Gamma_j) = 0$.

(iv) $N(f) = N(h_1)$.

Proof. Let $\lambda > 0$ and $\delta > 0$ are the numbers as in the proof of Lemmas (4.2) and (4.1) respectively. Let $\theta > 0$ be a number such that if $x, x' \in X$ and $d(x, x') < \theta$, then $\tilde{d}(f(x), f(x')) < \min(\frac{1}{6}\lambda, \frac{1}{2}\delta)$. Let $\xi = \min(\frac{1}{6}\lambda, \frac{1}{2}\delta, \theta)$. Since $\Gamma_1, \ldots, \Gamma_k$ are mutually disjoint compact subset of X, there exist open sets $U_1, \ldots, U_k$ such that $\Gamma_1 \subset U_i$ for all i, $i = 1, \ldots, k$, and $U_i \cap U_j = \emptyset$, $i \neq j$. Let m be a large positive integer such that $d(\Gamma_i, X - U_i) > \xi/m$ for all i. For $x \in \Gamma_j$, let $U(x)$ be an open connected set such that $D(U(x)) < \xi/m$. Cover Γ_j with a finite number of sets $U(x_1), \ldots, U(x_p)$, where $x_1, \ldots, x_p \in \Gamma_j$. Let $G_j = \bigcup_{i=1}^{p} U(x_i)$. Then $G_1, \ldots, G_k$ are mutually disjoint open sets such that $\Gamma_i \subset G_i$, for all i, $i = 1, \ldots, k$, and $F(f) \subset \bigcup_{i=1}^{k} G_i = V$. Let $\varepsilon(X - V) > 0$ be a number such that $\omega(x, f(x)) > \varepsilon(X - V)$ for all $x \in X - V$. Let $\varepsilon(f) = \min(\xi, \varepsilon(X - V))$. We claim that $F(h_t) \subset V$ for all $t \in I$. If not, there exists $x \in F(h_t) \cap (X - V)$ such that

$$\omega(x, f(x)) \leqslant \omega(x, h_t(x)) + \tilde{d}(f(x), h_t(x)) < 0 + \varepsilon(f) < \varepsilon(X - V)$$

which is a contradiction. Hence $F(h_t) \subset V$ for all $t \in I$.

Let $x \in G_j \cap F(h_1)$, then there exists $x_j \in \Gamma_j$ and an open connected set $U(x_j)$ such that $D(U(x_j)) < \xi/m$ and $x \in U(x_j)$. Let $C\colon I \to U(x_j)$ be path such that $C(0) = x_j$ and $C(1) = x$. If $z, z_j \in X^n$ are points which are admissible with respect to x and x_j respectively, then for $t \in I$, we have

$$\tilde{d}(\eta(z_j), \langle h; C\rangle(t)) = \tilde{d}(f(x_j), h_t(t)) = \tilde{d}(f(x_j), f(C(t))) + \tilde{d}(f(C(t)), h_t(t))$$
$$< \tfrac{1}{2}\delta + \varepsilon(f) < \tfrac{1}{2}\delta + \tfrac{1}{2}\delta = \delta.$$

Hence $\langle h; C\rangle(I) \subset N(\eta(z_j), \delta)$. Also

$$\tilde{d}(\eta(z_j), \eta(z)) = \tilde{d}(f(x_j), h_1(x)) \leqslant \tilde{d}(f(x_j), f(x)) + \tilde{d}(f(x), h_1(x))$$
$$< \tfrac{1}{6}\lambda + \varepsilon(f) < \tfrac{1}{6}\lambda + \tfrac{1}{6}\lambda = \tfrac{1}{3}\lambda.$$

It follows, as in the proof of Lemma (4.2), that there exist open path connected sets $O_1, \ldots, O_n$ of X and an element $\alpha \in S_n$ such that $D(O_i) < \frac{1}{2}\delta\sqrt{n}$, for all i, $i = 1, \ldots, n$, $\pi_1 \alpha = \pi_1$ and $z, \alpha z \in O_1 \times O_2 \ldots \times O_n = O$. Since O is path connected, it easy to construct a path $C'\colon I \to O \subset X^n$ from z_j to αz such that $\pi_1 C' = C$. It follows that $\eta(C') \subset N(\eta(z_j), \delta)$. Since

$$\eta C'(0) = \eta(z_j) = f(x_j) = h_0 C(0) = \langle h; C\rangle(0)$$

and

$$\eta C'(1) = \eta(\alpha z) = h_1(x) = h_1 C(1) = \langle h; C\rangle(1).$$

It follows from Lemma (4.1) that $\eta C' \simeq \langle h; C\rangle = \langle h; \pi_1 C'\rangle$. Hence x is h-related to x_j, that is, Γ' is h-related to Γ_j, where Γ' is the fixed point class of h_1 containing x. We claim that $\Gamma' = G_j \cap F(h_1)$. If $x' \in \Gamma'$ such that $x' \notin G_j \cap F(h_1)$, then there exists i, $i \neq j$, $1 \leqslant i \leqslant k$, such that $x' \in G_i \cap F(h_1)$. As before this implies that x is h-related to some point x'' of Γ_i. It follows from Theorem (5.1) that x'' is f-equivalent to x_j, but this is a contradiction, for, $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$. Hence $\Gamma' \subseteq G_j \cap F(h_1)$. Similarly $G_j \cap F(h_1) \subseteq \Gamma'$. Thus every fixed point class Γ' of h_1 is h-related to one and only one fixed point class of f. It follows from Lemma (3.7) that

$$I(\Gamma_j) = I(X, f, G_j) = I(X, h_1, G_j) = I(\Gamma').$$

If, however, a fixed class Γ_j of f is not h-related to any fixed point class of h_1, then it follows that $F(h_1) \cap G_j = \emptyset$. By property I1 of Theorem (4.11) we have

$$I(\Gamma_j) = I(X, f, G_j) = I(X, h_1, G_j) = 0.$$

We see that there is a one to one correspondence between the set of essential fixed point classes of f and h_1. Hence $N(f) = N(h_1)$.

(3.4) Theorem. *If X is a finite polyhedron and f, g:* $X \to X^n/S_n$ *are maps such that* $f \simeq g$, *then* $N(f) = N(g)$.

Proof. Let $h\colon X \times I \to X^n/S_n$ be a homotopy between f and g. We shall show that for $r \in I$, the Nielsen number of the map $h_r\colon X \to X^n/S_n$, where $h_r(x) = h(x, r)$, is a constant in a neighborhood of r in I. Let $C\colon I \to (X^n/S_n)^X$ be the map defined by, for $s \in I$, $C(s) = h_s$. Let $r \in I$ and $H_r\colon I \to R$ (R is the field of real numbers) be the map defined by, $H_r(s) = \varrho(h_r, h_s)$, where ϱ is the metric of $(X^n/S_n)^X$ and $s \in I$. Let $g\colon X \times I \to X^n/S_n$ be a homotopy between h_r and h_s defined by

$$g(x, t) = h(x, (1-t)r + ts) \quad \text{for} \quad t \in I \text{ and } x \in X.$$

Let $\varepsilon(h_r) > 0$ be the number of Lemma (5.3) for the map h_r. Since H_r is continuous and $H_r(r) = 0$ it follows that there exists a number $\delta(r) > 0$ such that if $s \in I$ and $|r - s| < \delta(r)$, then $H_r(s) < \varepsilon(h_r)$. Since $|r - ((1-t)r + ts)| = |t|\,|r - s| < |r - s| < \delta(r)$ for all $t \in I$, it follows that $\operatorname{Sup}\varrho(h_r, g_t) < \varepsilon(h_r)$. Hence from Lemma (5.3), we have $N(h_r) = N(h_s)$ for all $s \in I$ and $|r - s| < \delta(r)$. This proves that $N(h_r)$ is a constant in a neighborhood of r in I, for all $r \in I$. Since I is connected, it follows that $N(h_r)$ is constant for all $r \in I$. Hence $N(f) = N(h_1) = N(g)$.

We state the Nielsen fixed point theorem for symmetric product map of a finite polyhedron as follows.

(5.5) Theorem. *If X is a finite polyhedron and f:* $X \to X^n/S_n$ *is a map, then one can associate a number* $N(f)$ *with the map f such that if the Lefschetz number* $L(f) \neq 0$, *then* $N(f) \neq 0$ *and if g:* $X \to X^n/S_n$ *is a map homotopic to f, then g has at least* $N(f)$ *fixed points.*

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