3.8. Corollary. *If X is an arcwise connected continuum, then the set of all points at which X is colocally connected spans X.*

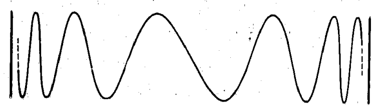
3.9. Remark. The above corollary fails for continua with two arc-components. The well-known $(\sin 1/x)$-curve is such an example.

3.10. Corollary. *Let X be a continuum with two arc-components lying in a strongly locally connected space M. Then there is a point $p \in \mathrm{Fr}_M X$ at which X is colocally connected.*

Proof. Let A and B be the arc-components of X. There is a point $x \in \bar{A} \cap \bar{B}$. Let $E = \{x\}$. Clearly, $\mathrm{Fr}_M X \neq \emptyset$ (otherwise X would be a locally connected continuum). By 2.1 we have $(\mathrm{Fr}_M X) \setminus E \neq \emptyset$. Since there is no surjection from X onto an indecomposable continuum, by 3.1 we get a point in $\mathrm{Fr}_M X$ with the desired properties.

3.11. Corollary. *Every continuum with two arc-components contains a point at which it is colocally connected.*

3.12. Remark. The above corollary fails for continua with three arc-components. The continuum pictured below is such an example.



In [7] some results similar to the above ones are established for continua with countable number of arc-components.

References

[1] D. Bellamy, *Composants of Hausdorff indecomposable continua: A mapping approach*, Pacific J. Math. 47 (1973), pp. 303–309.

[2] D. E. Bennett and J. B. Fugate, *Continua and their non-separating subcontinua*, Dissertationes Math. (in press).

[3] K. Borsuk, *Über sphäroidale und H-sphäroidale Räume*, Матем. Сборник, T. 1 (43) (1936).

[4] K. Borsuk, *Theory of Retracts*, Warszawa 1967.

[5] F. B. Jones, *Concerning aposyndetic and non-aposyndetic continua*, Bull. Amer. Math. Soc. 58 (1952), pp. 137–151.

[6] J. Krasinkiewicz and P. Minc, *Dendroids and their endpoints*, Fund. Math. 99 (1978), pp. 227–244.

[7] — — *Continua with countable number of arc-components*, Fund. Math. (this volume), pp. 119–127.

[8] K. Kuratowski, *Topology*, vol. II, New York–London–Warszawa 1968.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
and
WARSAW UNIVERSITY

Accepté par la Rédaction 16. 8. 1976

A hereditarily normal strongly zero-dimensional space containing subspaces of arbitrarily large dimension

by

Elżbieta Pol and **Roman Pol** (Warszawa)

Abstract. We construct a hereditarily normal space X with $\dim X = 0$ containing for every $n = 1, 2, \ldots$ a perfectly normal subspace X_n such that $\dim X_n = \mathrm{Ind}\, X_n = n$ and $\mathrm{locdim}\, X_n = 0$.

There was an old problem raised by E. Čech [3] whether the covering dimension dim is monotone in the class of hereditarily normal spaces; the analogous problem for the large inductive dimension Ind was raised by C. H. Dowker [4] (see also [1; Ch. VII] and [14; Problem 11-14]).

Under the assumption of an existence of Souslin's continuum V. V. Filippov [10] solved these problems in the negative exhibiting a hereditarily normal space X with $\dim X = 0$ containing for $n = 1, 2, \ldots$ a subspace X_n with $\dim X_n = \mathrm{Ind}\, X_n = n$, and later on the authors [18] constructed (using only the usual set theory) a hereditarily normal space X with $\dim X = 0$ containing a subspace Y with $\dim Y = \mathrm{Ind}\, Y = 1$ (¹).

In this paper we improve our previous result [18] by a construction of a hereditarily normal space X with $\dim X = 0$ containing for $n = 1, 2, \ldots$ a subspace X_n with $\dim X_n = \mathrm{Ind}\, X_n = n$. This construction is in fact very similar to our former construction [18; Sec. 3]. However, to obtain the stronger result we needed another approach to the dimensional properties of this construction (exhibited in [17]) and some special results on the structure of complete metrizable spaces (proved in [20]) to apply this idea.

1. Terminology, notation and auxiliary results.

1.1. Our terminology follows [5]. We shall denote by N the set of natural numbers, by I the real unit interval $[0, 1]$ and by I^n the unit n-dimensional cube; ∂I^n stands for the boundary of the cube I^n, i.e., the points in I^n at least one of whose

(¹) It is worth while to notice that compact hereditarily normal spaces missing the monotonicity of the dimensions dim and Ind were constructed, under some set-theoretical hypothesis stronger than the continuum hypothesis, by V. V. Fedorčuk [6], [7] and A. Ostaszewski [15], and, more recently, under the continuum hypothesis, by V. V. Fedorčuk [8] and E. Pol [16]; these examples, especially those in [6] and [15] have many further very interesting properties.

coordinates is 0 or 1. If $f\colon X\to Y$ is a mapping and $Z\subset X$ then $f|Z$ denotes the restriction of f to the set Z.

The word "dimension" stands for the covering dimension dim and Ind denotes the large inductive dimension. A space X is said to be of *local dimension at most n* (abbreviated $\operatorname{locdim} X\leqslant n$) if each point $x\in X$ has an open neighbourhood U with $\dim \overline{U}\leqslant n$ (see [1] or [14]).

1.2. Given an ordinal ξ we denote by $D(\xi)$ the space of all ordinals less than ξ endowed with the discrete topology. A set $S\subset\omega_1$ (we identify ω_1 with the set of all countable ordinals) is said to be *stationary* if it intersects every cofinal and closed with respect to the order topology subset in ω_1 (see [12; Appendix 1.5]). An immediate consequence of non-measurability of $\aleph_1$ (see [13; Ch. IX, § 3]) is that every stationary set in ω_1 can be split into two disjoint stationary sets (note that all not stationary sets in ω_1 form a σ-ideal containing all singletons).

1.3. For each ordinal $\xi\leqslant\omega_1$ we put $B(\xi) = D(\xi)^N$, i.e., $B(\xi)$ is the space of all sequences of ordinals less than ξ endowed with the pointwise topology; in particular $B(\omega_1) = B(\aleph_1)$ is the so called Baire's space of weight $\aleph_1$ (see [5]). We put also

$$B_\xi = B(\xi)\setminus\bigcup_{\alpha<\xi} B(\alpha)\quad\text{and}\quad B(S) = \bigcup\{B_\xi\colon \xi\in S\}\quad\text{for}\quad S\subset\omega_1 .$$

In the sequel we use the following

LEMMA (cf. [20; Sec. 3]). *Let T be a stationary set in ω_1 and let U be an open set in $B(\aleph_1)$ containing the set $B(T)$. Then the set $\{\xi\colon B_\xi\not\subset U\}$ is not stationary.*

The lemma can be justified shortly as follows: the closed set $F = B(\aleph_1)\setminus U$ does not contain topologically the space $B(\aleph_1)$ ([20; Lemma 3.2]), whence F being a completely metrizable space is the union of countably many locally separable subspaces (A. H. Stone's theorem [21; Theorem 2]) and therefore F intersects only "not stationary many" sets B_ξ ([20; Theorem 2.2]); see also the proof of Proposition 3.5 in [20] (2).

1.4. We shall describe a perfectly normal space B which will be a base for our construction (this space was also exploited in [18]).

We give the set $B = B(\omega_1)$ a new topology, finer than the metrizable topology of the Baire's space, by taking as a base the sets $U\cap B(\xi)$, where U is an open set in the Baire's space $B(\aleph_1)$ and $\xi<\omega_1$. In other words we enrich the topology of $B(\aleph_1)$ by new open sets $B(\xi)$; see also [18; Sec. 3] and [19; Example] for another description of the space B.

The properties of the space B were exactly investigated in [19]; let us recall that B is perfectly normal (but not paracompact) and that the sets $B(\xi)$ are open — and — closed subspaces with a countable base which cover B. Note also that every set B_ξ is closed in B.

(2) The reader is referred to Fleissner [9; Corollary 3.5] for a straightforward combinatorial proof of this lemma.

In the sequel the following statement parallel of Lemma 1.3 will play the key role.

LEMMA. *Let S be a stationary set in ω_1 and let H be a G_δ-set in B containing the set $B(S)$. Then the set $\{\xi\colon B_\xi\not\subset H\}$ is not stationary, i.e., there exists a not stationary set $K\subset\omega_1$ such that $B\setminus B(K)\subset H$.*

It is easy to see that we can restrict ourselves to the case of open H. By [19; Lemma 2] there exists an open in $B(\aleph_1)$ set $U\subset H$ such that the set

$$L = \{\xi\colon B_\xi\cap(H\setminus U)\neq\emptyset\}$$

is not stationary (observe, that for the function $\varkappa$ defined in [19; Example] we have $\varkappa^{-1}(\xi) = B_\xi$). Thus $U\supset B(T)$, where the set $T = S\setminus L$ is stationary in ω_1, and therefore by Lemma 1.3 the set $\{\xi\colon B_\xi\not\subset H\}\subset\{\xi\colon B_\xi\not\subset U\}$ is not stationary.

2. THEOREM. *There exists for every $n = 1, 2, \ldots$ a perfectly normal space X_n with $\operatorname{locdim} X_n = 0$ and $\dim X_n = \operatorname{Ind} X_n = n$. Moreover each X_n is locally second-countable.*

2.1. We begin with some necessary notation. For every $0\leqslant m\leqslant n$ let us denote by R_n^m the set of points in the cube I^n exactly m of whose coordinates are rational and let us denote by L_n^m the set of points in I^n at least m of whose coordinates are rational, i.e., $L_n^m = \bigcup_{j=m}^{n} R_n^j$ (see [11; Example II 12 and III 6]).

Recal ([11]) that $\dim R_n^m = 0$ and that each set L_n^m is the union of countably many compact subsets of I^n.

2.2. We pass to the definition of the spaces X_n.

Let us split ω_1 into $n+1$ disjoint stationary sets $S_0, \ldots, S_n$ (see 1.2) and let us define (see 1.3 and 1.4)

$$X_n = \bigcup_{m=0}^{n} B(S_m)\times R_n^m\subset B\times I^n ,$$

where X_n is endowed with the subspace topology of the product of the perfectly normal space B defined in 1.4 and the n-dimensional cube I^n. Thus X_n is perfectly normal [5; Problem 4.5.16] and locally second-countable.

2.3. Let us verify that $\operatorname{locdim} X_n = 0$.

The sets $B(\xi)$ form an open — and — closed cover of B (see 1.4) and hence the sets $V_\xi = B(\xi)\times I^n$ form an open — and — closed cover of $B\times I^n$. Since

$$V_\xi\cap X_n = \bigcup_{m=0}^{n}\bigcup\{B_\alpha\times R_n^m\colon \alpha\leqslant\xi\ \text{and}\ \alpha\in S_m\}$$

and since each set $B_\alpha\times R_n^m$ is closed in X_n (see 1.4) we infer from the sum theorem ([11; Theorem III 2]; notice that V_ξ is second-countable) that

$$\dim(V_\xi\cap X_n)\leqslant\sup\{\dim(B_\alpha\times R_n^m)\colon \alpha\leqslant\xi, 0\leqslant m\leqslant n\} = 0\quad\text{(cf. 2.1)} .$$

2.4. The following lemma is crucial for our further reasonings:

LEMMA. *Let G be a G_δ-set in the space $B\times I^n$ containing the space X_n. Then there exists a point $x\in B$ such that the set $\{x\}\times I^n$ is contained in G.*

Proof. We shall define inductively a sequence $K_n, K_{n-1}, \ldots, K_1$ of not stationary sets in ω_1 such that the following condition (m) is satisfied for $1\leqslant m\leqslant n$

(m) $$(B\setminus B(K_m))\times L_n^m\subset G.$$

For convenience let us put $K_{n+1} = \emptyset = L_n^{n+1}$, so condition (n+1) holds, and let us assume that we have defined the set K_{m+1} satisfying (m+1); we shall define the set K_m.

Let us put $S = S_m\setminus K_{m+1}$ and let us observe that

(*) $$B(S)\times L_n^m\subset G.$$

Indeed, we have $B(S_m)\times R_n^m\subset X_n\subset G$ and, by (m+1), $(B\setminus B(K_{m+1}))\times L_n^{m+1}\subset G$; thus $B(S)\times(R_n^m\cup L_n^{m+1})\subset G$, but $R_n^m\cup L_n^{m+1} = L_n^m$ (see 2.1).

Let $L_n^m = \bigcup_k Z_k$, where Z_k is a compact set (see 2.1). Since, by (*), we have $B(S)\times Z_k\subset G$ for every k, there exist G_δ-sets G_k in B such that $B(S)\subset G_k$ and $G_k\times Z_k\subset G$ (3). Let $H = \bigcap_k G_k$, then

$$H\times L_n^m\subset \bigcup_k G_k\times Z_k\subset G.$$

Since H is a G_δ-set in B containing the set $B(S)$ with S stationary, there exists, by Lemma 1.4, a not stationary set K_m such that $B\setminus B(K_m)\subset H$ which implies that $(B\setminus B(K_m))\times L_n^m\subset G$, i.e., the condition (m) holds. The inductive step is done.

Now, let us look at the set K_1. By condition (1) we have $(B\setminus B(K_1))\times L_n^1\subset G$ and, because K_1 is not stationary, there exists a point $x\in B(S_0\setminus K_1)$. We obtain thus $\{x\}\times R_n^0\subset X_n\subset G$ and $\{x\}\times L_n^1\subset G$ and hence $\{x\}\times(R_n^0\cup L_n^1)\subset G$; but

$$R_n^0\cup L_n^1 = L_n^0 = I^n$$

and this completes the proof.

2.5. LEMMA. *Let E be a topological space and let $f\colon F\to Z$ be a continuous mapping of a closed subset F of E into a compact metrizable space Z. Let A be a dense subset of E such that the restriction $f\,|\,F\cap A$ has a continuous extension over A with values in Z. Then there exist a G_δ-subset G of E containing A and a continuous extension $g\colon G\to Z$ of the mapping $f\,|\,F\cap G$ over G.*

Proof. There exist a G_δ-subset H of E containing A and a continuous extension $h\colon H\to Z$ of the mapping $f\,|\,F\cap A$ (first extend $f\,|\,F\cap A$ over A and then choose H and h; see [2; Ch. IX, § 2, 3 and Ch. I, § 8, 5]). The set

$$W = \{x\in F\cap H\colon f(x)\neq h(x)\}$$

is an F_σ-set in H disjoint from A. One can take now $G = H\setminus W$ and $g = h\,|\,G$.

(3) This follows easily, for example, from the closedness of the projection $B\times Z_k\to B$ parallel of the compact space Z_k.

2.6. We are ready for the proof that $\dim X_n = \operatorname{Ind} X_n = n$.

The inequality $\operatorname{Ind} X_n\leqslant \operatorname{Ind}(B\times I^n)\leqslant n$ follows from the fact that $\dim B = 0$ (which can be verified easily on the ground of the results of [19]) by some well-known theorems of the dimension theory (see for example [14; Corollary 11.11 and Theorem 25.6]). Because $\dim\leqslant\operatorname{Ind}$ for normal spaces it remains to prove that $\dim X_n\geqslant n$.

Let $F = B\times\partial I^n$ and let $f\colon F\to\partial I^n$ be the projection parallel of the space B. We shall verify that the mapping $f\,|\,F\cap X_n$ cannot be extended continuously over X_n to a mapping with values in ∂I^n. Indeed, in the opposite case, by Lemma 2.5 (where $E = B\times I^n$, $Z = \partial I^n$ and $A = X_n$), there would exist G, a G_δ-set in E containing X_n, and a continuous mapping $g\colon G\to\partial I^n$ which extends $f\,|\,F\cap G$. However, by Lemma 2.4, there exists a point $x\in B$ such that the set $\{x\}\times I^n$ is contained in G. Since $f\,|\,\{x\}\times\partial I^n$ is in fact the identity of ∂I^n, the mapping $r\colon I^n\to\partial I^n$ defined by $r(t) = g(x, t)$ provides a retraction of the n-dimensional cube onto its boundary, a contradiction.

Thus X_n admits a continuous mapping of a closed subset into ∂I^n which is not extendable over X_n and hence, by a classical theorem of P. S. Aleksandroff (see [1] or [14]), $\dim X_n\geqslant n$.

3. THEOREM. *There exists a hereditarily normal space X such that $\dim X = 0$ and X contains for every $n = 1, 2, \ldots$ a subspace X_n with $\dim X_n = \operatorname{Ind} X_n = n$. Moreover, X is a Lindelöf space and there exists a point $p\in X$ such that the space $X\setminus\{p\}$ is perfectly normal and locally second-countable.*

Proof (cf. [14, Theorem 11.17] and [18; Example 2]). Let us add to the free union $U = \bigoplus_{n=1}^{\infty} X_n$ of the spaces X_n constructed in the precede section a point p which does not belong to U, i.e., $X = \{p\}\cup\bigoplus_{n=1}^{\infty} X_n$. We give X a topology letting U to be an open subspace in X and taking as a base of neighbourhoods of the point p the sets $\{p\}\cup V$, where V is an open — and — closed set in U and the space $U\setminus V$ has a countable base. All the properties of the space X stated in Theorem are easily verified (cf. [18; Example 2]).

Added in Proof. The reader is referred for a brief exposition of the main idea of this paper to a note of the authors in Proceedings of the Fourth Prague Top. Symp., 1976, Part B, Contributed Paper, pp. 357–359.

References

[1] P. S. Aleksandroff and B. A. Pasynkow, *Introduction to Dimension Theory* (in Russian), Moscow 1973.

[2] N. Bourbaki, *Topologie Générale*, ch. I and II, Paris 1940 and ch. IX, Paris 1949.

[3] E. Čech, *Problem* 53, Colloq. Math. 1 (1948), p. 332.
[4] C. H. Dowker, *Local dimension of normal spaces*, Quart. J. Math. Oxford 6 (1955), pp. 101–120.
[5] R. Engelking, *General Topology*, Warszawa 1977.
[6] V. V. Fedorčuk, *Compatibility of some theorems of general topology with the axioms of set theory* (in Russian), Dokl. Akad. Nauk SSSR 220 (1975), pp. 786–788.
[7] — *Fully closed mappings and a compatibility of some theorems of general topology with the axioms of set theory* (in Russian), Mat. Sbornik 99 (1976), pp. 3–33.
[8] — *On the dimension of hereditarily normal spaces*, Proc. London. Math. Soc. 36 (1978), pp. 163–175.
[9] W. G. Fleissner, *Separation properties in Moore spaces*, Fund. Math. 98 (1978), pp. 279–286.
[10] V. V. Filippow, *On the dimension of normal spaces* (in Russian), Dokl. Akad. Nauk SSSR 209 (1973), pp. 805–807.
[11] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton 1941.
[12] I. Juhász, *Cardinal Functions in Topology*, Amsterdam 1971.
[13] K. Kuratowski and A. Mostowski, *Set Theory*, Amsterdam 1971.
[14] K. Nagami, *Dimension Theory*, New York 1970.
[15] A. Ostaszewski, *A perfectly normal countably compact scattered space which is not strongly zero-dimensional*, J. London Math. Soc. 14 (1976), pp. 167–177.
[16] E. Pol, *Remark about Juhász–Kunen–Rudin construction of a hereditarily separable non-Lindelöf space*, Bull. Acad. Polon. Sci. 24 (1976), pp. 749–752.
[17] — *Strongly metrizable spaces of large dimension all separable subspaces of which are zero-dimensional*, Colloq. Math. 39 (1978), pp. 25–27.
[18] — and R. Pol, *A hereditarily normal strongly zero-dimensional space with a subspace of positive dimension and an N-compact space of positive dimension*, Fund. Math. 97 (1977), pp. 43–50.
[19] R. Pol, *A perfectly normal locally metrizable not paracompact space*, Fund. Math. 97 (1977), pp. 37–42.
[20] — *Note on decompositions of metrizable spaces II*, Fund. Math. 100 (1978), pp. 129–143.
[21] A. H. Stone, *Non-separable Borel sets II*, Gen. Topology Appl. 2 (1972), pp. 249–270.

DEPARTMENT OF MATHEMATICS AND MECHANICS, WARSAW UNIVERSITY
WYDZIAŁ MATEMATYKI I MECHANIKI, UNIWERSYTET WARSZAWSKI

Accepté par la Rédaction le 16. 8. 1976

On the fixed point index and the Nielsen fixed point theorem of symmetric product mappings

by

Samuel Masih (Albany, Ga.)

Abstract. In this paper we study essential fixed point sets of symmetric product maps. We define fixed point index of a symmetric product map of a finite polyhedron. In the special case when $G = S_n$, the symmetric group, we define fixed point classes and the Nielsen number of a symmetric product map and prove the Nielsen fixed point theorem for symmetric product maps of finite polyhedra.

1. Introduction. Let X be a topological space and X^n be the Cartesian product with usual topology. A group G of permutations of the numbers $[1, 2, \ldots, n]$ can be considered as a group of homeomorphisms on X^n by defining, for $\alpha \in G$ and $(x_1, x_2, \ldots, x_n) \in X^n$, $\alpha(x_1, x_2, \ldots, x_n) = (x_{\alpha(1)}, x_{\alpha(2)}, \ldots, x_{\alpha(n)})$. The orbit space with identification topology is denoted by X^n/G. A map $f\colon X \to X^n/G$ is called a *symmetric product map*. A point $x \in X$ is said to be a *fixed point of f* if $\eta(z) = f(x)$ implies that x is a coordinate of z, where $z \in X^n$ and $\eta\colon X^n \to X^n/G$ is the identification map. C. N. Maxwell defined the Lefschetz number $L(f)$ of a symmetric product map and proved the Lefschetz fixed point theorem for symmetric product maps in the case when X is a compact polyhedron [6]. The Lefschetz fixed point theorem for symmetric product mappings also hold in the case when X is a metric absolute neighborhood retract and f is a compact map [5].

A fixed point x of the map $f\colon X \to X^n/G$ is called an *essential fixed point* if each map sufficiently close to f has a fixed point arbitrary close to x. Essential fixed points and essential fixed point sets for a single valued maps have been investigated by Fort [3] and O'Neill [7] respectively.

In this paper we study essential fixed point sets of symmetric product maps. We define fixed point index of a symmetric product map of a finite polyhedron. In the special case when $G = S_n$, the symmetric group, we define fixed point classes and the Nielsen number of a symmetric product map and prove the Nielsen fixed point theorem for symmetric product maps of finite polyhedra.

2. Preliminaries. Let $\pi_i\colon X^n \to X$ be the ith projection and $\alpha \in G$, the for $z \in X^n$ $\pi_i\alpha(z) = \pi_{\alpha(i)}z$, where $i = 1, 2, \ldots, n$.