Continua and their open subsets with connected complements

by

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Abstract. In this paper it is proved that some continua contain nowhere dense subcontinua (or points) having arbitrarily small neighborhoods with connected complements.

1. Introduction. All spaces considered in this paper are metric. By a neighborhood is always meant an open set. The term continuum stands for a nonvoid compact connected space.

It is evident that not all continua contain nonvoid open and non-dense subsets whose complements are connected. For example indecomposable continua do not have this property and, in fact, this is the characteristic property of indecomposable continua. Observe also that there are decomposable continua such that no proper subcontinuum has arbitrarily small neighborhoods with connected complements. Proper subcontinua of a continuum X having arbitrarily small neighborhoods with connected complements will be called subcontinua of colocal connectedness of X. For the concept of colocalizations of topological properties in general see the paper of Borsuk [3] (comp. also [4, p. 227]). Precisely, let us agree to use the following terminology. Let $F \neq \emptyset$ be a subset of a space X and let C be another subset of X. Then the pair (X, F) is said to be *colocally connected at* C provided that for each neighborhood U of C in X there is a neighborhood $V \subset U$ of C in X such that $F \setminus V$ is contained in a single component of $X \setminus V$. In case $F = X$ we will simply say that X is *colocally connected at* C instead of saying that (X, X) is *colocally connected at* C. If $C = \{p\}$, then we say that (X, F) is *colocally connected at* p. In [6] it was proved that every dendroid X contains some points at which it is colocally connected. In this paper we generalize this result for all arcwise connected continua (see 3.7). This fact is closely related to arcwise connectivity (see 3.11 and 3.12). However one can prove an analogous result for a larger class of continua. Namely, we shall show that every continuum which cannot be mapped onto an indecomposable continuum contains nowhere dense subcontinua at which it is colocally connected (see 3.6). We will consider the following situation. Continua under consideration will be treated as subsets of "nice" spaces. In such a case we shall show that the continuum always contains subcontinua of colocal connectednes in its boundary. By a "nice" space we shall mean a topologically complete space M such that each point $p \in M$ has arbitrarily small neighborhoods U such that $U\setminus\{p\}$ is connected. Such spaces will be called *strongly locally connected*. Observe that manifolds of dimension $\geqslant 2$ and the Hilbert cube belong to this class. Note also that each connected open subset of a strongly locally connected space is arcwise connected.

It is convenient for us to use the notion of aposyndesis. A continuum X is said to be *aposyndetic* at a set $C\subset X$ with respect to a point $p\in X$ provided there exists a continuum $D\subset X$ such that $C\subset \operatorname{Int} D\subset D\subset X\setminus\{p\}$ (see [5]).

Corollary 3.5 provides an affirmative answer to Question 5.14 and a negative answer to Question 5.16 from [2].

We thank Professor D. Bellamy for his remarks concerning this paper.

2. Auxiliary lemmas. In this section we establish several lemmas needed in the sequal.

2.1. LEMMA. *Let X be a nondegenerate continuum lying in a strongly locally connected space M. Then no point of $\operatorname{Fr}_M X$ is isolated in $\operatorname{Fr}_M X$.*

Proof. Let $p\in \operatorname{Fr}_M X$ and let U be a neighborhood of p in M. We can assume that $U\setminus\{p\}$ is connected. Since $U\setminus\{p\}$ meets both X and $M\setminus X$ and $U\setminus\{p\}$ is connected we conclude that $(U\setminus\{p\})\cap \operatorname{Fr}_M X\neq\emptyset$, which completes the proof.

2.2. LEMMA. *Let X be a continuum lying in a strongly locally connected space M. If $(X, \operatorname{Fr}_M X)$ is colocally connected at a point $p\in X$, then X is colocally connected at p.*

Proof. Let U be an arbitrary neighborhood of p in X. It suffices to show that there is a neighborhood V of p in X contained in U such that $X\setminus U$ is a subset of a single component of $X\setminus V$.

Let G be a neighborhood of p in M such that $G\cap X\subset U$ and $G\setminus\{p\}$ is arcwise connected. Let H be a neighborhood of p in X such that $\overline{H}\subset G$, $(\operatorname{Fr}_M X)\setminus H\neq\emptyset$ and $(\operatorname{Fr}_M X)\setminus H$ is contained in a component C_0 of $X\setminus H$. Now we shall show that only a finite number of components of $X\setminus H$ meets $X\setminus U$. If not, then there is a sequence of points $q_1, q_2, \ldots$ from $X\setminus U$ converging to a point q such that the elements of this sequence all belong to different components of $X\setminus H$. Since $q\in X\setminus U$ and $\overline{H}\subset U$ there is an arcwise connected neighborhood W of q in M contained in $M\setminus\overline{H}$. There are two different components $D_1\ni q_i$ and $D_2\ni q_j$ of $X\setminus H$ such that $q_i, q_j\in W$. Assume $D_1\neq C_0$. Let q_iq_j be an arc in W joining q_i and q_j. Clearly, $q_iq_j\not\subset X$. Let r be the first point on q_iq_j belonging to $\operatorname{Fr}_M X$. Clearly, $r\in D_1$ which is a contradiction.

Let $C_0, C_1, C_2, \ldots, C_n$ be the component of $X\setminus H$ covering $X\setminus U$. Since $C_k\cap \operatorname{Fr}_X H\neq\emptyset$ for $0\leqslant k\leqslant n$ and $\operatorname{Fr}_X H\subset G$, then for $k\geqslant 1$ there is an arc $a_kb_k\subset G\setminus\{p\}$ such that $a_k\in C_k$ and $b_k\in C_0$. If $a_kb_k\subset X$ let $c_k=b_k$. If $a_kb_k\not\subset X$ let c_k be the first point on a_kb_k belonging to $\operatorname{Fr}_M X$. Let a_kc_k be the subarc of a_kb_k between a_k and c_k.

Let $V\subset H\setminus \bigcup_{k=1}^{n} a_kb_k$ be a neighborhood of p in X such that $(\operatorname{Fr}_M X)\setminus V$ is contained in a single component of $X\setminus V$. Observe that $X\setminus U$ is contained in the same component of $X\setminus V$ because

$$X\setminus U\subset \bigcup_{k=0}^{n} C_k\subset C_0\cup \bigcup_{k=1}^{n}(C_k\cup a_kc_k)\subset X\setminus V$$

and each component of $C_0\cup \bigcup_{k=1}^{n}(C_k\cup a_kc_k)$ meets $\operatorname{Fr}_M X$. This completes the proof.

The proof of the following lemma is similar to the proof of the Theorem in [1].

2.3. LEMMA. *Let X be a continuum lying in a locally connected complete space M and let C be a subcontinuum of X such that $(\operatorname{Fr}_M X)\setminus C\neq\emptyset$. Assume that there is no surjection from X onto an indecomposable continuum sending C to a single point. Then there is a point $p\in(\operatorname{Fr}_M X)\setminus C$ such that X is aposyndetic at C with respect to p.*

Proof. Suppose the lemma fails.

We are going to construct a sequence of functions $f_n\colon X\to[0,1]$ satisfying the following conditions:

(1) $C\subset \operatorname{Int}_X f_n^{-1}(0)$,

(2) $\operatorname{Fr}_M X\cap \operatorname{Int}_X f_n^{-1}(1)\neq\emptyset$,

(3) $f_n=\varphi\circ f_{n+1}$,

where $\varphi\colon[0,1]\to[0,1]$ is defined by

$$\varphi(t)=\begin{cases}2t & \text{for}\quad 0\leqslant t\leqslant\frac12,\\ 2-2t & \text{for}\quad \frac12\leqslant t\leqslant 1.\end{cases}$$

Let $U\supset C$ and V be open subsets of X such that $\overline{U}\cap\overline{V}=\emptyset$ and $V\cap \operatorname{Fr}_M X\neq\emptyset$. Let f_1 be the Uryshon function sending $\overline{U}$ to 0 and $\overline{V}$ to 1, and assume the functions $f_1, \ldots, f_n$ have been constructed. It remains to construct f_{n+1}.

Let C^* be the component of $X\setminus \operatorname{Int}_X f_n^{-1}(1)$ containing C. By our supposition and (2) there is a point $c\in C\cap\overline{X\setminus C^*}$. Let G be a connected neighborhood of c in M such that $G\cap X\subset \operatorname{Int}_X f_n^{-1}(0)$ (see (1)). Let d be a point from $G\cap X\setminus C^*$. There is an arc $dc\subset G$ joining d and c. Clearly, $dc\not\subset X$. Let e be the first point on dc belonging to $\operatorname{Fr}_M X$. Since $d\notin C^*$, then $e\notin C^*$. There are two closed disjoint sets A and B such that $X\setminus \operatorname{Int}_X f_n^{-1}(1)=A\cup B$, $C^*\subset A$ and $e\in B$. Now define

$$f_{n+1}(x)=\begin{cases}\frac12 f_n(x) & \text{for}\quad x\in A\cup f_n^{-1}(1),\\ 1-\frac12 f_n(x) & \text{for}\quad x\in B.\end{cases}$$

The function f_{n+1} is well-defined continuous and satisfies (3). Observe that

$$C\subset \operatorname{Int}_X f_n^{-1}(0)\setminus B=\operatorname{Int}_X f_{n+1}^{-1}(0)$$

and

$$e\in \operatorname{Int}_X f_n^{-1}(0)\setminus A=\operatorname{Int}_X f_{n+1}^{-1}(1),$$

which proves (1) and (2) because $e\in \operatorname{Fr}_M X$.

This completes the construction.

Let K denote the limit of an inverse sequence $\{I_n, \varphi_n\}$, where $I_n = [0,1]$ and $\varphi_n = \varphi$ for each $n \geqslant 1$. It is well-known that K is homeomorphic to the Knaster indecomposable continuum (see [8], p. 204, Ex. 1, comp. also [1]). But conditions (1), (2) and (3) imply that the functions f_n induce a map from X onto K sending C to the point $(0, 0, \ldots) \in K$, contrary to our assumption. This completes the proof.

2.4. Lemma. *Let $C_1, C_2, \ldots$ be a strictly increasing sequence of continua in a continuum X, i.e. $C_n \subset \operatorname{Int} C_{n+1}$ for each $n \geqslant 1$. Then X is colocally connected at each component of $X \setminus \bigcup_{n=1}^{\infty} C_n$.*

Proof. Let K be a component of $X \setminus \bigcup_{n=1}^{\infty} C_n$ and let U be a neighborhood of K in X. We have to construct a neighborhood V of K contained in U such that $X \setminus V$ is connected. Let G be a neighborhood of K contained in U such that $\operatorname{Fr} G \subset \bigcup_{n=1}^{\infty} C_n$. Since $\operatorname{Fr} G$ is compact there exists an index m such that $\operatorname{Fr} G \subset C_m$. Let $V = G \setminus C_m$. Then $X \setminus V = C_m \cup (X \setminus G)$ is connected because every component of $X \setminus G$ meets $\operatorname{Fr} G \subset C_m$. This proves the lemma.

The following lemma is not needed here in the general form in which it is stated; however it will be used in this general form in a subsequent paper [7]. (For our purposes herein, the set A could be omitted from the statement.)

2.5. Lemma. *Let X be a continuum lying in a strongly locally connected space M. Let E be a subcontinuum of X such that $(\operatorname{Fr}_M X) \setminus E \neq \emptyset$, and let A be a nonvoid countable subset of X. Assume that for every point $a \in A$ and for every continuum $C \subset X$ such that $E \subset C$ and $(\operatorname{Fr}_M X) \setminus C \neq \emptyset$ there is a point $x \in \operatorname{Fr}_M X$ such that X is aposyndetic at $C \cup \{a\}$ with respect to x. Then there is a proper subset G of X such that*

(i) *G is connected and open in X,*

(ii) *$A \cup E \subset G$,*

(iii) *$\operatorname{Fr}_X G$ is a continuum at which $\bar{G}$ is colocally connected,*

(iv) *$\operatorname{Fr}_X G \subset \operatorname{Fr}_M X \subset \bar{G}$,*

(v) *each subcontinuum of X meeting both G and $\operatorname{Fr}_X G$ contains $\operatorname{Fr}_X G$.*

2.6. Remark. The set $\operatorname{Fr}_X G$ is nowhere dense in $\operatorname{Fr}_M X$.

In fact, suppose there is a point $q \in \operatorname{Fr}_X G$ and a connected neighborhood U of q in M such that $U \cap \operatorname{Fr}_M X \subset \operatorname{Fr}_X G$. Let p be a point from $G \cap U$. Let pq be an arc in U joining p and q. Let r be the first point of pq belonging to $\operatorname{Fr}_X G$. Let pr be the subarc of pq. Observe that $pr \cap \operatorname{Fr}_M X = \{r\}$. Thus $pr \subset X$. By (v) we infer that $\operatorname{Fr}_X G = \{r\}$. Hence 2.6 follows from 2.1.

Proof of 2.5. Arrange A into a sequence $a_1, a_2, \ldots$ Let $H_1, H_2, \ldots$ be a sequence of open subsets of X such that $\operatorname{diam} H_n \xrightarrow[n\to\infty]{} 0$ and the sets $H_n \cap \operatorname{Fr}_M X$ are nonvoid and form a base for open sets in $\operatorname{Fr}_M X$. Now we construct a sequence $W_0, W_1, \ldots$ of open subsets of X and a sequence of continua $E_0, E_1, \ldots$ such that for each $n \geqslant 1$ the following conditions are fulfilled:

$(1)_n$ $W_{n-1} \cap \operatorname{Fr}_M X \neq \emptyset$,

$(2)_n$ $W_{n-1} \cap E = \emptyset$ and E_{n-1} is the component of $X - W_{n-1}$ containing E,

$(3)_n$ $\operatorname{diam} W_n < 1/n$,

$(4)_n$ $E_{n-1} \cup \{a_n\} \subset \operatorname{Int}_X E_n$,

$(5)_n$ $E_n \cap H_n \neq \emptyset$,

$(6)_n$ if there is apoint $x \in H_n \cap \operatorname{Fr}_M X$ such that X is aposyndetic at $E_{n-1} \cup \{a_n\}$ with respect to x, then $W_n \subset H_n$.

Let $W_0 = X \setminus E$ and assume the sets $W_0, \ldots, W_{n-1}$ have been constructed. It remains to construct the set W_n. This will be done using just $(1)_n$ and $(2)_n$.

First assume that there is a point $x \in H_n \cap \operatorname{Fr}_M X$ such that X is aposyndetic at $E_{n-1} \cup \{a_n\}$ with respect to x. Hence there is a continuum $B \subset X$ such that $E_{n-1} \cup \{a_n\} \subset \operatorname{Int}_X B \subset B \subset X \setminus \{x\}$. In this case let W_n be a neighborhood of x satisfying $(3)_n$ such that $\overline{W}_n \subset H_n \setminus B$. Clearly, all the appropriate conditions are fulfilled.

For the other case take a point $x \in \operatorname{Fr}_M X$ such that X is aposyndetic at $E_{n-1} \cup \{a_n\}$ with respect to x. Let $B \subset X$ be a continuum such that

$$E_{n-1} \cup \{a_n\} \subset \operatorname{Int}_X B \subset B \subset X \setminus \{x\}.$$

Let W_n be a neighborhood of x in X disjoint from B and satisfying $(3)_n$. It remains to prove $(5)_n$. We shall prove more: $H_n \cap \operatorname{Fr}_M X \subset E_n$. In fact, otherwise X is aposyndetic at $E_{n-1} \cup \{a_n\}$ $(\subset \operatorname{Int}_X E_n)$ with respect to every point from $H_n \cap \operatorname{Fr}_M X \setminus E_n$, and the first case of the construction applies, contrary to our assumption.

This completes the construction.

Set

$$G = \bigcup_{n=0}^{\infty} E_n.$$

Clearly, by $(4)_n$ the set G is connected and open, and by $(2)_n$ and $(4)_n$ condition (ii) holds. Now we prove that $\operatorname{Fr}_X G$ is a continuum. Suppose, to the contrary, that there is a separation $\operatorname{Fr}_X G = P \cup Q$ into two closed disjoint nonvoid sets. Let U be a neighborhood of P in $\bar{G}$ and let V be a neighborhood of Q in $\bar{G}$ such that $d(U, V) > 0$, where $d(U, V) = \inf\{\varrho(u, v)\colon u \in U\ v \in V\}$. Since $\operatorname{Fr}_{\bar{G}} U \cup \operatorname{Fr}_{\bar{G}} V$ is a compact subset of G there is an index n such that E_n contains this set. Let $p \in P$ and $q \in Q$ be arbitrary points. Let D_p be the component of $\bar{U}$ containing p and let D_q be the component of $\bar{V}$ containing q. Clearly,

$$D_p \cap \operatorname{Fr}_{\bar{G}} U \neq \emptyset \neq D_q \cap \operatorname{Fr}_{\bar{G}} V.$$

Hence $D_p \cap E_n \neq \emptyset \neq D_q \cap E_n$. Let $m > n$ be an index such that $1/m < d(U, V)$. Without loss of generality one can assume that $W_m \cap \bar{U} = \emptyset$ (see $(3)_m$). It follows that $D_p \cup E_n$ is a continuum containing E and missing W_m. Hence $p \in E_m \subset G$, contrary to the choice of p. Thus $\operatorname{Fr}_X G$ is a continuum.

The continuum $\bar{G}$ is colocally connected at $\mathrm{Fr}_X G$ by $(4)_n$ and 2.4, which proves (iii).

Since $\operatorname{diam} H_n \xrightarrow[n\to\infty]{} 0$ and $\{H_n \cap \mathrm{Fr}_M X\}$ is a base for $\mathrm{Fr}_M X$, by $(5)_n$ we get $\mathrm{Fr}_M X \subset \bar{G}$. To complete the proof of (iv) it remains to show that $\mathrm{Fr}_X G \subset \mathrm{Fr}_M X$. We shall prove this showing that if T is a component of $\mathrm{Int}_M X$ meeting G, then $T \subset G$. In fact, there is an index n such that $T \cap E_n \neq \emptyset$. Let x be an arbitrary point of T. Since T is arcwise connected there is an arc $L \subseteq T$ joining x and E_n. Let $m > n$ be such that $\operatorname{diam} W_m < d(L, \mathrm{Fr}_M X)$ (see $(3)_m$). By $(1)_{m+1}$ the set W_m misses L. It follows that

$$x \in L \subset L \cup E_n \subset E_m \subset G .$$

Since x is an arbitrary point of T we have $T \subset G$, and consequently $\mathrm{Fr}_X G \subset \mathrm{Fr}_M X$.

Finally we shall prove (v). Suppose there exist a continuum $K \subset X$ meeting both G and $\mathrm{Fr}_X G$, and a point $c \in (\mathrm{Fr}_X G) \setminus K$. Let n be an index such that $E_n \cap K \neq \emptyset$. There is an index $m > n$ such that $c \in H_m \subset X \setminus (E_n \cup K)$. Since $c \notin E_m$, by $(4)_m$ we infer that X is aposyndetic at $E_{m-1} \cup \{a_m\}$ with respect to c. Since $c \in H_m \cap \mathrm{Fr}_X G \subset H_m \cap \mathrm{Fr}_M X$, by $(6)_m$ we have $W_m \subset H_m$. But this implies that $E_n \cup K \subset E_m$ because E_m is a component of $X \setminus W_m$ containing E and E is a subset of the continuum $E_n \cup K \subset X \setminus W_m$. Hence $K \subset E_m \subset G$ contrary to the assumption that $K \cap \mathrm{Fr}_X G \neq \emptyset$.

3. Main results. In this section we shall prove some results about the boundaries of continua lying in strongly locally connected spaces. Since each continuum can be treated as a nowhere dense subset of the Hilbert cube (which is strongly locally connected) each of the results gives some properties of the continuum itself. These are the "absolute" versions of the results. The "absolute" versions are given following the proofs of the "relative" versions.

3.1. THEOREM. *Let X be a nondegenerate continuum lying in a strongly locally connected space M and let E be either the empty set or a subcontinuum of X such that $(\mathrm{Fr}_M X) \setminus E \neq \emptyset$. Assume that there is no surjective map from X onto an indecomposable continuum sending E to a single point. Then there exists a continuum $K \subset X$ such that*

(i) $K \cap E = \emptyset$,

(ii) *K is a nowhere dense subset of $\mathrm{Fr}_M X$,*

(iii) *each continuum in X meeting both $(\mathrm{Fr}_M X) \setminus K$ and K contains K,*

(iv) *$(X, \mathrm{Fr}_M X)$ is colocally connected at K.*

Moreover, if E meets the closure of each arc-component of X, then K is a one-point set. Consequently, X is colocally connected at K.

3.2. Remark. Any two diferent subcontinua of X satisfying the conditions (ii), (iii) and (iv) are disjoint.

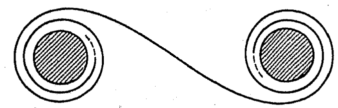
For let K_1 and K_2 be such continua. Suppose that $K_1 \cap K_2 \neq 0$ and $K_1 \setminus K_2 \neq \emptyset$. Let U be a neighborhood of K_2 in X such that $K_1 \setminus \bar{U} \neq \emptyset$ and $(\mathrm{Fr}_M X) \setminus U$ is contained in a component C of $X \setminus U$. By (ii) C meets $(\mathrm{Fr}_M X) \setminus K_1$ which contradicts (iii).

Proof of 3.1. Without loss of generality by 2.1 we can assume that $E \neq \emptyset$. Let A be a one-point set contained in E. By 2.3 the assumptions of 2.5 are fulfilled. Hence there is an open set $G \subset X$ satisfying the conclusion of 2.5. Setting $K = \mathrm{Fr}_X G$ we infer that all the conclusions of the theorem above are satisfied (see 2.6).

If E meets the closure of each arc-component of X, then G meets each arc-component of X. Let ab be an arc in X such that $b \in K$ and $ab \setminus \{b\} \subset G$. By condition (v) of 2.5 we infer that $K = \{b\}$. Finally, X is colocally connected at K by condition (iv) of the above theorem and 2.2.

This completes the proof.

3.3. Remark. Let M be the plane and let $X \subset M$ be the continuum pictured below. Observe that in this situation there is no continuum $K \subseteq \mathrm{Fr}_M X$ such that X is colocally connected at K. This shows that condition (iv) of 3.1 cannot be strengthened.



We say that a subset F of a continuum X is *spanned by a class* **K** *of subcontinua of* F provided that every subcontinuum of X meeting each member of **K** contains F.

3.4. COROLLARY. *Let X be a continuum lying in a strongly locally connected space M. Assume that there is no surjection from X onto an indecomposable continuum. Then $\mathrm{Fr}_M X$ is spanned by the class of subcontinua of $\mathrm{Fr}_M X$ satisfying conditions* (ii), (iii) *and* (iv) *of* 3.1.

3.5. COROLLARY. *Let E be either the empty set or a proper subcontinuum of a continuum X. Assume that there is no surjection from X onto an indecomposable continuum sending E to a single point. Then there is a continuum $K \subset X$ such that*

(i) $K \cap E = \emptyset$,

(ii) $\mathrm{Int} K = \emptyset$,

(iii) *each subcontinuum of X meeting both K and $X \setminus K$ contains K,*

(iv) *X is colocally connected at K.*

Moreover, if E meets the closure of each arc component of X, then K is a one-point set.

3.6. COROLLARY. *Let X be a continuum which can not be mapped onto an indecomposable continuum. Then X is spanned by the class of subcontinua of X satisfying conditions* (ii), (iii) *and* (iv) *of* 3.5.

3.7. THEOREM. *Let X be an arcwise connected continuum lying in a strongly locally connected space M. Let C be the subset of $\mathrm{Fr}_M X$ consisting of all points at which X is colocally connected. Then C spans $\mathrm{Fr}_M X$.*

Proof. Let E be a subcontinuum of X containing C. Suppose $(\mathrm{Fr}_M X) \setminus E \neq \emptyset$. Since there is no surjection from X onto an indecomposable continuum, by 3.1 there is a point $p \in (\mathrm{Fr}_M X) \setminus E$ at which X is colocally connected, contrary to the choice of E.

3.8. Corollary. *If X is an arcwise connected continuum, then the set of all points at which X is colocally connected spans X.*

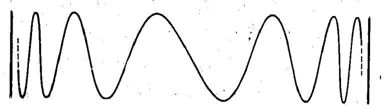
3.9. Remark. The above corollary fails for continua with two arc-components. The well-known ($\sin 1/x$)-curve is such an example.

3.10. Corollary. *Let X be a continuum with two arc-components lying in a strongly locally connected space M. Then there is a point $p \in \mathrm{Fr}_M X$ at which X is colocally connected.*

Proof. Let A and B be the arc-components of X. There is a point $x \in \bar{A} \cap \bar{B}$. Let $E = \{x\}$. Clearly, $\mathrm{Fr}_M X \neq \emptyset$ (otherwise X would be a locally connected continuum). By 2.1 we have $(\mathrm{Fr}_M X)\setminus E \neq \emptyset$. Since there is no surjection from X onto an indecomposable continuum, by 3.1 we get a point in $\mathrm{Fr}_M X$ with the desired properties.

3.11. Corollary. *Every continuum with two arc-components contains a point at which it is colocally connected.*

3.12. Remark. The above corollary fails for continua with three arc-components. The continuum pictured below is such an example.



In [7] some results similar to the above ones are established for continua with countable number of arc-components.

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A hereditarily normal strongly zero-dimensional space containing subspaces of arbitrarily large dimension

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Abstract. We construct a hereditarily normal space X with $\dim X = 0$ containing for every $n = 1, 2, \ldots$ a perfectly normal subspace X_n such that $\dim X_n = \mathrm{Ind}\, X_n = n$ and $\mathrm{loc}\dim X_n = 0$.

There was an old problem raised by E. Čech [3] whether the covering dimension dim is monotone in the class of hereditarily normal spaces; the analogous problem for the large inductive dimension Ind was raised by C. H. Dowker [4] (see also [1; Ch. VII] and [14; Problem 11-14]).

Under the assumption of an existence of Souslin's continuum V. V. Filippov [10] solved these problems in the negative exhibiting a hereditarily normal space X with $\dim X = 0$ containing for $n = 1, 2, \ldots$ a subspace X_n with $\dim X_n = \mathrm{Ind}\, X_n = n$, and later on the authors [18] constructed (using only the usual set theory) a hereditarily normal space X with $\dim X = 0$ containing a subspace Y with $\dim Y = \mathrm{Ind}\, Y = 1$ (1).

In this paper we improve our previous result [18] by a construction of a hereditarily normal space X with $\dim X = 0$ containing for $n = 1, 2, \ldots$ a subspace X_n with $\dim X_n = \mathrm{Ind}\, X_n = n$. This construction is in fact very similar to our former construction [18; Sec. 3]. However, to obtain the stronger result we needed another approach to the dimensional properties of this construction (exhibited in [17]) and some special results on the structure of complete metrizable spaces (proved in [20]) to apply this idea.

1. Terminology, notation and auxiliary results.

1.1. Our terminology follows [5]. We shall denote by N the set of natural numbers, by I the real unit interval $[0, 1]$ and by I^n the unit n-dimensional cube; ∂I^n stands for the boundary of the cube I^n, i.e., the points in I^n at least one of whose

(1) It is worth while to notice that compact hereditarily normal spaces missing the monotonicity of the dimensions dim and Ind were constructed, under some set-theoretical hypothesis stronger than the continuum hypothesis, by V. V. Fedorčuk [6], [7] and A. Ostaszewski [15], and, more recently, under the continuum hypothesis, by V. V. Fedorčuk [8] and E. Pol [16]; these examples, especially those in [6] and [15] have many further very interesting properties.