Continua with countable number of arc-components

by

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Abstract. In this paper it is proved that continua with countable number of arc-components contain either free arcs or points having arbitrarily small neigborhoods with connected complements.

1. Introduction. All spaces under considerations are metric. By a neighborhood we mean an open set. The term continuum stands for a nonvoid compact connected space. By a free arc of a space X we mean an open arc which is an open subset of X.

Let $F \neq \emptyset$ be a subset of a space X and let A be another subset of X. The pair (X, F) is said to be colocally connected at A provided that for each neighborhood U of A in X there is a neighborhood $V \subset U$ of A in X such that $F \setminus V$ is contained in a single component of $X \setminus V$. In case $F = X$ we will simply say that X is *colocally connected at A* instead of saying that (X, X) is colocally connected at A. If $A = \{p\}$, then we say that (X, F) is *colocally connected at p*.

Recall that a continuum X is said to be *semi-locally connected at a point p* provided there are arbitrarily small neighborhoods V of p such that $X \setminus V$ has finitely many components (see [4, p. 19]). Clearly, colocal connectedness of X at a point p implies semi-local connectedness of X at p.

In this paper we prove that continua with countable number of arc-components contain either free arcs or points of colocal connectedness (see 3.3 and 3.6), hence such continua contain points of semi-local connectedness (see 3.8). If such a continuum X is a subset of a "nice" space, then we shall prove that there are points of semi-local connectedness (colocal connectedness in some cases) of X in the boundary of X (see 3.1, 3.5 and 3.7).

By a "nice" space we mean a topologically complete space M such that each point $p \in M$ has arbitrarily small neighborhoods U such that $U \setminus \{p\}$ is connected. Note that every open connected subset of M is arcwise connected.

2. Auxiliary lemmas.

2.1. Lemma. *Let G be an open connected and locally connected subset of a continuum X and let C be a component of* $\operatorname{Fr} G$. *Then there is a subset P of G homeomorphic to the closed half line such that $\overline{P} \setminus P \subset C$. Consequently, if C_0 and C_1 are different* *components of* $\mathrm{Fr}\,G$, *then there is a subset* R *of* G *homeomorphic to the real line such that* $\bar{R}\setminus R\subset C_0\cup C_1$ *and* $C_0\cap\bar{R}\neq\emptyset\neq C_1\cap\bar{R}$.

Proof. Since G is locally compact, connected and locally connected metric space, it is arcwise connected. Hence it suffices to prove the first assertion of the lemma.

One can easily construct a sequence $H_1, H_2, \ldots$ of open subsets of X such that $G\setminus H_1\neq\emptyset$, $\bar{H}_{n+1}\subset H_n$, $C\subset H_n\subset(1/n$-ball around $C)$ and $(\mathrm{Fr}\,H_n)\cap\mathrm{Fr}\,G=\emptyset$ for each $n\geqslant 1$. Since $G\cap\mathrm{Fr}\,H_n$ is a compact subset of G, there is a finite collection of locally connected continua $D_1^n, \ldots, D_{k_n}^n$ lying in G such that

$$G\cap\mathrm{Fr}\,H_n\subset D_1^n\cup\ldots\cup D_{k_n}^n$$

and

$$D_1^{n+1}\cup\ldots\cup D_{k_{n+1}}^{n+1}\subset H_n.$$

Let $p\in G\setminus H_1$ and $q\in C$ be arbitrary points.

For each $n\geqslant 1$ let $q_n\in H_n\cap G$ be a point such that $\lim_n q_n=q$. Let pq_n be an arc in G joining p and q_n. If $1\leqslant m\leqslant n$, then there exist an index $1\leqslant j\leqslant k_m$ and a point $x\in pq_n\cap D_j^m$ such that $pq_n=px\cup xq_n$ and $xq_n\subset\bar{H}_m$. In such a situation we will say that j is determined by q_n at the mth stage (in general there can be many such j's). There is a subsequence $q_1^{(1)}, q_2^{(1)}, \ldots$ of the sequence $q_1, q_2, \ldots$ such that each arc $pq_k^{(1)}$ determines the same index, say j_1, at the first stage (by the definition of a subsequence we have: if $q_r^{(1)}=q_k$ and $q_{r+1}^{(1)}=q_l$, then $k<l$). Similarily, there is a subsequence $q_1^{(2)}, q_2^{(2)}, \ldots$ of $q_1^{(1)}, q_2^{(1)}, \ldots$ such that each arc $pq_k^{(2)}$ determines the same index, say j_2, at the second stage. Repeating the procedure we can construct a sequence of sequences $\{q_k^{(1)}\}_{k=1}^{\infty}, \{q_k^{(2)}\}_{k=1}^{\infty}, \ldots$ and a sequence of indices $j_1, j_2, \ldots, 1\leqslant j_r\leqslant k_r$, such that $\{q_k^{(l+1)}\}_{k=1}^{\infty}$ is a subsequence of $\{q_k^{(l)}\}_{k=1}^{\infty}$ and each arc $pq_k^{(l)}$ determines j_l at the lth stage. Hence the sequence $q_1^{(1)}, q_2^{(2)}, q_3^{(3)}, \ldots$ converges to q and for each $m\leqslant n$ the arc $pq_n^{(n)}$ determines j_m at the mth stage. Without loss of generality we can assume that our original sequence $q_1, q_2, \ldots$ has the above properties; that is $\lim_n q_n=q$ and for each $m\leqslant n$ the arc pq_n determines j_m at the mth stage. Let A_1 be any arc in G joining q_1 and q_2. We shall show that for $n>1$ there exists an arc A_n in $H_{n-1}\cap G$ joining q_n and q_{n+1}. Observe that there are points $x\in pq_n\cap D_{j_n}^n$ and $y\in pq_{n+1}\cap D_{j_n}^n$ such that $xq_n\subset\bar{H}_n\cap G$ and $yq_{n+1}\subset\bar{H}_n\cap G$. To prove the existence of A_n it remains to note that $\bar{H}_n\subset H_{n-1}$ and $D_{j_n}^n$ is a locally connected continuum contained in $H_{n-1}\cap G$. Let $A=\bigcup_{n=1}^{\infty}A_n$. Clearly, $A\subset G$ and by a standard trick one can prove that A contains the required set P. This completes the proof.

2.2. Lemma. *Let* X *be a continuum with a countable number of arc-components. Assume that* A *is a subcontinuum of* X *irreducible between two points* a_0 *and* a_1, *and let* B *be an indecomposable subcontinuum of* A. *If* U *is a nonvoid open subset of* $\mathrm{Int}_A B$, *then there exists a continuum* $E\subset X$ *joining* a_0 *and* a_1 *such that* $U\setminus E\neq\emptyset$.

Proof. Let $U_1, U_2, \ldots$, be a sequence of open sets in X with diameters converging to zero such that the sets $U_n'=U_n\cap A$, $n=1,2,\ldots$, are nonvoid, contained in U, and form a base for open subsets of U. Let $p_1, p_2, \ldots$, be a sequence of points of X such that each point of X can be joined by an arc with some point of that sequence. For each pair of natural numbers m and n such that $p_m\notin U_n$ denote by P_{mn} the component of $X\setminus U_n$ containing p_m.

We claim that

$$X=\bigcup_{m,n}P_{mn}.$$

In fact, let $x\in X$. Then there is an arc L joining x with some point p_m. Since $U\setminus L\neq\emptyset$ (otherwise L would contain an uncountable collection of nondegenerate mutually disjoint continua), there is an index n such that $U_n\cap L=\emptyset$. It follows that $x\in L\subset P_{mn}$.

By the Baire theorem there exist two indices m_0 and n_0 such that $\mathrm{Int}_B(B\cap P_{m_0n_0})\neq\emptyset$. It follows that the continuum $P=P_{m_0n_0}$ meets all composants of B. Besides we can assume that U_{n_0} misses both a_0 and a_1. Let A_i, $i=0,1$, be the component of $A\setminus U_{n_0}$ containing a_i. Since $U_{n_0}\cap A=U_{n_0}'$, the set A_i meets $\mathrm{Fr}_A U_{n_0}'\subset\bar{U}\subset B$. It follows that A_i meets some composant C_i of B. Let $c_i\in A_i\cap C_i$. Since P meets C_i, there is a point $c_i'\in P\cap C_i$. Let D_i be a subcontinuum of C_i joining c_i and c_i'. It is easily seen that the continuum

$$E=A_0\cup D_0\cup P\cup D_1\cup A_1$$

has the required properties. This concludes the proof.

2.3. Lemma. *Let* X *be a continuum with a countable number of arc-components lying in a strongly locally connected space* M. *Let* U *be an open subset of* X *meeting* $\mathrm{Fr}_M X$ *and containing no free arc of* X. *Then each two points of* X *can be joined by a subcontinuum of* X *missing a point of* $U\cap\mathrm{Fr}_M X$.

Proof. Suppose the lemma fails. Then there are two points a_0 and a_1 in X such that

(1) each continuum in X joining a_0 and a_1 contains $U\cap\mathrm{Fr}_M X$.

First we shall prove that

(2) if A is a subcontinuum of X joining a_0 and a_1 and $(cd)\subset U$ is a free arc of A, then $(cd)\cap\mathrm{Fr}_M X=\emptyset$.

Suppose $r\in(cd)\cap\mathrm{Fr}_M X$. Since $U\cap\mathrm{Fr}_M X\subset A$ which follows from (1), there is a neighborhood W_1 of r in M such that $W_1\cap\mathrm{Fr}_M X\subset(cd)$. By the assumption about M we can assume in addition that W_1 is arcwise connected. Since U does not contain any free arc of X, there is a point $p_1\in W_1\cap X\setminus(cd)$. Let p_1r be an arc in W_1 joining p_1 and r. Since p_1 must belong to $\mathrm{Int}_M X$ and $r\in\mathrm{Fr}_M X$, there is a point, say x_1, which is the first point on the arc p_1r belonging to $\mathrm{Fr}_M X$. Observe that $x_1\in(cd)$ and $p_1x_1\setminus\{x_1\}\subset\mathrm{Int}_M X$. There is a neighborhood $W_2\subset W_1$ of x_1 in M such that $W_2\setminus\{x_1\}$ is arcwise connected. Let $p_2\in(p_1x_1\setminus\{x_1\})\cap W_2$. By [2, 2.1] there is a point $y\in(W_2\cap\mathrm{Fr}_M X)\setminus\{x_1\}$. Clearly, $y\in(cd)$. Let p_2y be an arc in $W_2\setminus\{x_1\}$ joining p_2 and y, and let x_2 be the first point of p_2y belonging to $\mathrm{Fr}_M X$. Clearly, $x_2\in(cd)\setminus\{x_1\}$ and the subarc p_2x_2 of p_2y is a subset of $\mathrm{Int}_M X\cup\{x_2\}$.

Let $W_3 \subset W_2$ be a neighborhood of x_1 in M such that $W_3 \setminus \{x_1\}$ is arcwise connected and $W_3 \cap p_2 x_2 = \emptyset$. Again by [2, 2.1] there is a point $z \in (W_3 \setminus \{x_1\}) \cap \mathrm{Fr}_M X$. Let $p_3 \in (W_3 \setminus \{x_1\}) \cap p_1 x_1$ and let $p_3 z$ be an arc in $W_3 \setminus \{x_1\}$ between p_3 and z. Let x_3 be the first point on $p_3 z$ belonging to $\mathrm{Fr}_M X$. Clearly, $x_3 \in (cd)$. Let $p_3 x_3$ be the subarc of $p_3 z$ joining p_3 and x_3. Clearly, $p_3 x_3 \subset \mathrm{Int}_M X \cup \{x_3\}$. The points x_1, x_2 and x_3 are different and lie on (cd). Let $L \subset p_1 x_1 \setminus \{x_1\}$ be an arc containing p_1, p_2 and p_3. Let c_i be the first and let d_1 be the last point on cd belonging to $p_1 x_1 \cup p_2 x_2 \cup p_3 x_3$. Let cc_1 and $d_1 d$ denote the subarcs of cd and let $c_i \in p_i x_i$, $d_1 \in p_j x_j$. The continuum $K = [A \setminus (cd)] \cup cc_1 \cup p_i x_i \cup L \cup p_j x_j \cup d_1 d \subset X$ joins a_0 and a_1 and misses the point $x_k \neq x_i, x_j$ contrary to our supposition. This proves (2).

Consider the class $\boldsymbol{K}$ of all subcoutinua K of X satisfying the conditions

(3) K is a continuum irreducible between a_0 and a_1,

(4) the intersection of K with a component of $\mathrm{Int}_M X$ is either void or homeomorphic to the closed half real line or homeomorphic to the line.

We shall show that

(5) $\boldsymbol{K} \neq \emptyset$.

Let $G_1, G_2, \ldots$ be the sequence of all components of $\mathrm{Int}_M X$. We shall construct a sequence of continua $B_0, B_1, \ldots$ in X such that for each $n \geqslant 1$ we have

(6) the points a_0 and a_1 belong to B_{n-1},

(7) $B_n \subset B_{n-1}$,

(8) $B_n \cap G_n$ is homeomorphic to a closed connected subset of the line,

(9) each continuum in B_n joining a_0 and a_1 contains $B_n \cap G_n$,

(10) $B_{n-1} \cap G_j$ is either void or equal to G_j for $j \geqslant n$.

Let $B_0 = X$ and assume the sets $B_0, \ldots, B_{n-1}$ have been constructed. In case where $B_{n-1} \cap G_n = \emptyset$ let $B_n = B_{n-1}$. Now assume $B_{n-1} \cap G_n \neq \emptyset$. Then by (10) we have $\bar{G}_n \subset B_{n-1}$. Let P_0 and P_1 denote the following sets. If $a_0 \in G_n$, then $P_0 = \emptyset$, if $a_0 \notin G_n$, then P_0 is the component of $B_{n-1} \setminus G_n$ containing a_0. The set P_1 is defined analogously. Observe that for $j > n$ the set $P_0 \cap G_j$ is either void or equal to G_j. The same holds for P_1. Consider three cases:

(i) $a_0, a_1 \in G_n$. Then let B_n be an arc in G_n joining a_0 and a_1.

(ii) $a_0 \in G_n$ and $a_1 \notin G_n$ (or $a_0 \notin G_n$ and $a_1 \in G_n$).

Let C be the component of $\mathrm{Fr}_M G_n = \mathrm{Fr}_X G_n$ meeting P_1 (or P_0). By 2.1 there is a set $P \subset G_n$ containing a_0 (or a_1) and homeomorphic to the closed half line such that $\bar{P} \setminus P \subset C$. Let $B_n = P \cup C \cup P_1$ (or $B_n = P \cup C \cup P_0$).

(iii) $a_0, a_1 \notin G_n$. If $P_0 = P_1$, then let $B_n = P_0$. Otherwise, let C_0 and C_1 be the components of $\mathrm{Fr}_X G_n$ meeting P_0 and P_1, respectively. Since $C_0 \neq C_1$, then by Lemma 2.1 there is a set $R \subset G_n$ homeomorphic to the real line such that $\bar{R} \setminus R \subset C_0 \cup C_1$ and $\bar{R} \cap C_0 \neq \emptyset \neq \bar{R} \cap C_1$. Then let $B_n = P_0 \cup R \cup P_1$. The properties (6), (7), (8), (9) and (10) are easily provable, which completes the construction of B_n's.

Consider the set $B = \bigcap_n B_n$. By (6) and (7), B is a continuum in X containing a_0 and a_1. Let K be a continuum in B irreducible between a_0 and a_1. By (1), (8) and (9) continuum K satisfies (4), which proves (5).

Now we prove that

(11) each nondegenerate layer T of a continuum $K \in \boldsymbol{K}$ is contained in $\mathrm{Fr}_M X$.

Suppose $T \cap \mathrm{Int}_M X \neq \emptyset$. Then there is a component G of $\mathrm{Int}_M X$ such that $T \cap G \neq \emptyset$. Since T is nondegenerate by (4) there is a free arc L of K wholly contained in T. By [3, Th. 4, p. 216] the layer T is a union of a countable number of nowhere dense subcontinua of K and indecomposable continua. Since L is an open subset of K, there is an indecomposable subcontinuum of T meeting L. Then $L \cap T$ is an open nonvoid subset of an indecomposable continuum contained in an arc, which is impossible. Hence (11) follows.

Let us prove that

(12) if T is a layer of a continuum $K \in \boldsymbol{K}$, then $\mathrm{Int}_K(T \cap U) = \emptyset$.

By [3, Th. 4, p. 216] the layer T is a union of a countable number of nowhere dense subcontinua of K and indecomposable continua. Suppose $\mathrm{Int}_K(T \cap U) \neq \emptyset$. Since it is an open nonvoid subset of K, by the Baire theorem there is an indecomposable subcontinuum D of T such that $\mathrm{Int}_K(D \cap U) \neq \emptyset$. By (3) and 2.2 it follows that there is a continuum $E \subset X$ joining a_0 and a_1 such that $\mathrm{Int}_K(D \cap U) \setminus E \neq \emptyset$. By (11) we have $D \subset T \subset \mathrm{Fr}_M X$. It follows that $U \cap \mathrm{Fr}_M X \setminus E \neq \emptyset$, contrary to (1). This proves (12).

For $K \in \boldsymbol{K}$ let $g_K\colon K \to [0, 1]$ be the (continuous) map considered in [3, § 48, IV] such that for each $t \in [0, 1]$ the set $g_K^{-1}(t)$ is a layer of K.

Let us prove that

(13) only a countable number of layers of a continuum $K \in \boldsymbol{K}$ meets $U \cap \mathrm{Fr}_M X$.

Suppose it is not. Then there are three numbers $t_0 < t_1 < t_2$ such that $g_K^{-1}(t_i)$ meets $U \cap \mathrm{Fr}_M X$, $i = 0, 1, 2$, and there is an arc-component of X meeting $g_K^{-1}(t_0)$ and $g_K^{-1}(t_2)$.

Let $L \subset X$ be an arc joining $g_K^{-1}(t_0)$ and $g_K^{-1}(t_2)$. By (1) the continuum

$$A = g_K^{-1}([0, t_0]) \cup L \cup g_K^{-1}([t_2, 1])$$

must contain $U \cap \mathrm{Fr}_M X$. This implies that

$$\emptyset \neq g_K^{-1}(t_1) \cap U \cap \mathrm{Fr}_M X \subset L \setminus \left(g_K^{-1}([0, t_0]) \cup g_K^{-1}([t_2, 1])\right).$$

Clearly, there is a free arc L_1 of A contained in U such that $L_1 \cap \mathrm{Fr}_M X \neq \emptyset$, contrary to (2). This completes the proof of (13).

Fix a continuum $K_0 \in \boldsymbol{K}$ (see (5)). Let $T_1, T_2, \ldots$ be all the layers of K_0 each of which meets $U \cap \mathrm{Fr}_M X$ (see (13)). Let $F_n = T_n \cap U$ for $n \geqslant 1$. By (1) and (11) we have $\bigcup_n F_n = U \cap \mathrm{Fr}_M X$ and since $U \cap \mathrm{Fr}_M X$ satisfies the Baire theorem, there is an index m such that the interior of F_m in $U \cap \mathrm{Fr}_M X$ is nonvoid, hence

(14) there is an open set V in M such that $\emptyset \neq V \cap \mathrm{Fr}_M X \subset F_m = T_m \cap U$.

By (14), (11) and [2, 2.1] it follows that

(15) T_m is a nondegenerate subcontinuum of $\mathrm{Fr}_M X$.

Let Y_i be a component of $K_0 - T_m$ containing the point a_i provided $a_i \notin T_m$; otherwise let $Y_i = \emptyset$. By (12) we infer that

(16) $T_m \cap U \subset \overline{Y}_0 \cup \overline{Y}_1$.

Let us prove the following proposition

(17) If $\overline{Y}_i \cap V \cap T_m \neq \emptyset$ then there exist an arc $r_i s_i \subset \mathrm{Int}_M X \cup \{r_i\}$ and a continuum $D_i \subset Y_i$ such that $r_i \in T_m \cap V$ and $E_i = D_i \cup r_i s_i$ is a continuum irreducible between r_i and a_i satysfying (4) (where K is replaced by E_i).

Let $q \in \overline{Y}_i \cap V \cap T_m$. Let $V_1 \subset V$ be an arcwise connected neighborhood of q. There is a point $p \in Y_i \cap V_1$. Let pq be an arc in V_1 joining p and q. Observe that by (14) we have $p \in \mathrm{Int}_M X$ and by (11) and (15) we have $q \in \mathrm{Fr}_M X$. Let r_i be the first point on pq (going from p to q) belonging to $\mathrm{Fr}_M X$ and let pr_i be the subarc of pq. By (14) we have $r_i \in T_m \cap V$ and $pr_i \subset (\mathrm{Int}_M X) \cup \{r_i\}$. By [3, § 48, IV] there is a continuum $D \subset Y_i$ joining p and a_i. Let s_i be the first point on $r_i p$ (going from r_i to p) belonging to D. Let D_i be a subcontinuum of D irreducible between s_i and a_i. One easily verifies that a continuum $E_i = D_i \cup r_i s_i$ satisfies (4), which proves (17).

In case where $\overline{Y}_i \cap V \cap T_m = \emptyset$ set $E_i = \overline{Y}_i$. By (16) it follows that $E_0 \cup T_m \cup E_1 \subset X$ is a continuum containing a_0 and a_1. Let $K_1 \subset E_0 \cup T_m \cup E_1$ be a continuum irreducible between a_0 and a_1. Observe that no component of $\mathrm{Int}_M X$ intersects both Y_0 and Y_1, which implies that there is no such component intersecting both E_0 and E_1. It follows from (17) that $K_1 \in \mathbf{K}$.

Since $T_m \subset \mathrm{Fr}_M X$ (see (15)), by (1) we infer that $T_m \cap U \subset K_1$. Since $K_1 \subset E_0 \cup T_m \cup E_1$, we get

(18) $U \cap T_m \setminus (E_0 \cup E_1) = U \cap K_1 \setminus (E_0 \cup E_1)$.

By (14) we have $\emptyset \neq V \cap T_m \subset U \cap T_m$. By (17) the set $(V \cap T_m) \cap (E_0 \cup E_1)$ contains at most two points. Since by (15) the set T_m is a nondegenerate continuum (and V is open) we infer that $\emptyset \neq V \cap T_m \setminus (E_0 \cup E_1) \subset U \cap T_m \setminus (E_0 \cup E_1)$. Hence by (18) the set $H = U \cap K_1 \setminus (E_0 \cup E_1)$ is a nonvoid open subset of K_1 contained in $\mathrm{Fr}_M X$. By (13) it follows that H is contained in a union of countably many layers of K_1. From the Baire theorem it follows that there is a layer T of K_1 such that $\mathrm{Int}_{K_1}(T \cap H) \neq \emptyset$. This implies that $\mathrm{Int}_{K_1}(T \cap U) \neq \emptyset$, contrary to (12). This completes the proof of the lemma.

3. Main results. This section contains our main results.

3.1. THEOREM. *Let X be a continuum with a finite number of arc-components lying in a strongly locally connected space M. Denote by T the union of all free arcs of X. Let $C_1, C_2, \ldots, C_n$ be a finite collection of subcontinua of X containing all points of $\mathrm{Fr}_M X$ at which X is colocally connected. Then*

$$\mathrm{Fr}_M X \subset \overline{T} \cup C_1 \cup \ldots \cup C_n .$$

Proof. Suppose the theorem fails. Let A be a finite set meeting each arc-component of X. Let $U = X \setminus (A \cup \overline{T} \cup C_1 \cup \ldots \cup C_n)$. Since X is nondegenerate, there is no point of $\mathrm{Fr}_M X$ isolated in $\mathrm{Fr}_M X$ (see [2, 2.1]). Hence by our supposition we have $U \cap \mathrm{Fr}_M X \neq \emptyset$. Using finitely many times Lemma 2.3 we can find a continuum $E \subset X$ such that $A \cup C_1 \cup \ldots \cup C_n \subset E$ and $(\mathrm{Fr}_M X) \setminus E \neq \emptyset$. Note that E meets each arc-component of X. Since X can not be mapped onto any indecomposable continuum, by [2, Th. 3.1] there is a point $p \in (\mathrm{Fr}_M X) \setminus E$ at which X is colocally connected. This implies that $p \in (\mathrm{Fr}_M X) \setminus (C_1 \cup \ldots \cup C_n)$, which is a contradiction. This completes the proof.

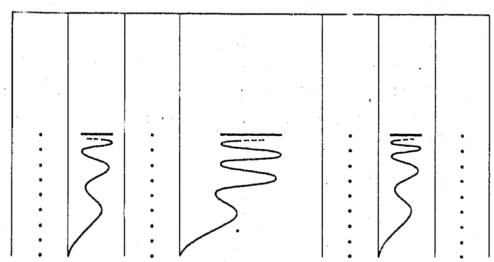
3.2. Remark. One easily sees that in the statement of 3.1 one could replace T by the union of free arcs of X which separate X.

Considering X as a nowhere dense subset of the Hilbert cube (which is strongly locally connected) we obtain from 3.1 the following.

3.3. COROLLARY. *Let X be a continuum with a finite number of arc-components. Denote by T the union of all free arcs of X separating X. Let $C_1, \ldots, C_n$ be a finite collection of subcontinua of X containing all points of X at which X is colocally connected. Then*

$$X = \overline{T} \cup C_1 \cup \ldots \cup C_n .$$

3.4. Remark. The above corollary fails for continua with a countable number of arc-components. Such an example can be obtained taking the union of the Cantor brush and an infinite sequence of $(\sin 1/x)$-curves as indicated below (comp. [1, Ex. 4.7]).



3.5. THEOREM. *Let X be a continuum with a countable number of arc-components lying in a strongly locally connected space M. Let $C_1, C_2, \ldots, C_n$ be a finite collection of subcontinua of X such that $C_1 \cup \ldots \cup C_n$ contains the union of all free arcs of X and all points of X at which X is colocally connected. Then*

$$\mathrm{Fr}_M X \subset C_1 \cup \ldots \cup C_n .$$

Proof. Suppose the theorem fails. Using 2.3 finitely many times one can find a continuum E such that $C_1 \cup \ldots \cup C_n \subset E$ and $(\mathrm{Fr}_M X) \setminus E \neq \emptyset$. Let A be a countable subset of X meeting each arc-component of X.

Let a be an arbitrary point of A and let C be a subcontinuum of X containing E such that $(\mathrm{Fr}_M X)\setminus C \neq \emptyset$. Since X is nondegenerate, there is no point of $\mathrm{Fr}_M X$ isolated in $\mathrm{Fr}_M X$ (see [2, 2.1]); hence $(\mathrm{Fr}_M X)\setminus(C \cup \{a\}) \neq \emptyset$. By 2.3 there is a continuum $D \subset X$ containing $C \cup \{a\}$ such that $(\mathrm{Fr}_M X)\setminus D \neq \emptyset$. Since X can not be mapped onto any indecomposable continuum, by [2, 2.3] there is a point $p \in [\mathrm{Fr}_M X)\setminus D$ such that X is aposyndetic at D with respect to p. Hence X is aposyndetic at $C \cup \{a\}$ with respect to p.

Thus the assumptions of Lemma 2.5 in [2] are fulfilled. It follows from that lemma that there is a proper connected open subset G of X satisfying the following conditions:

(1) $A \cup E \subset G$,

(2) $\mathrm{Fr}_X G \subset \mathrm{Fr}_M X \subset \bar{G}$,

(3) $\mathrm{Fr}_X G$ is a continuum at which $\bar{G}$ is colocally connected,

(4) each subcontinuum of X meeting both G and $\mathrm{Fr}_X G$ contains $\mathrm{Fr}_X G$.

Let x be a point of $\mathrm{Fr}_X G$ and let $a \in A$ be a point belonging to the arc-component of X containing x. Let ax be an arc in X joining a and x. By (1) we have $a \in G$. Let y be the first point on ax belonging to $\mathrm{Fr}_X G$. Denote by ay the subarc of ax between a and y. Since $ay \subset X$, by (4) it follows that $\mathrm{Fr}_X G \subset ay \cap \mathrm{Fr}_X G = \{y\}$. Hence $\mathrm{Fr}_X G = \{y\}$. By (2) and (3) we infer that $(X, \mathrm{Fr}_M X)$ is colocally connected at y. By [2, 2.2] it follows that X is colocally connected at y. But by (1) and (2) we have $y \in (\mathrm{Fr}_M X)\setminus E \subset (\mathrm{Fr}_M X)\setminus(C_1 \cup \ldots \cup C_n)$, which is a contradiction completing the proof.

Considering X as a nowhere dense subset of the Hilbert cube we obtain by 3.5 the following.

3.6. Corollary. *Let X be a continuum with a countable number of arc-components. Let $C_1, \ldots, C_n$ be a finite number of subcontinua of X such that $C_1 \cup \ldots \cup C_n$ contains both the union of all free arcs of X and all the points of X at which X is colocally connected. Then*

$$X = C_1 \cup \ldots \cup C_n .$$

Since every point on a free arc of X is a point of semi-local connectedness of X we obtain from 3.5 and 3.6 two following corollaries.

3.7. Corollary. *Let X be a continuum with a countable number of arc-components lying in a strongly locally connected space M. Let $C_1, \ldots, C_n$ be a finite number of subcontinua of X such that all points of semi-local connectedness of X lying in $\mathrm{Fr}_M X$ belong to $C_1 \cup \ldots \cup C_n$. Then*

$$\mathrm{Fr}_M X \subset C_1 \cup \ldots \cup C_n .$$

3.8. Corollary. *Let X be a continuum with a countable number of arc-components and let $C_1, \ldots, C_n$ be a finite collection of subcontinua of X such that all points of semi-local connectedness of X belong to $C_1 \cup \ldots \cup C_n$. Then*

$$X = C_1 \cup \ldots \cup C_n .$$

3.9. Remark. Clearly, the above corollary fails for all continua. There exists even an example of a snake-like hereditarily decomposable continuum having no points of semi-local connectedness (see [3, p. 191, Ex. 4]).

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