$\sum_n$-cofinalities of J_α

by

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Abstract. We study another aspect of the fine structure of Gödel's constructible universe L. We concentrate, in particular, on the behavior of definable (over Jensen's J_α) cofinalities and projecta for $\alpha>0$. It is shown that (1) for each $n<\omega$, the Σ_n-cofinality of J_α is Σ_n-regular over J_α; (2) for each $n<\omega$, if $\varkappa$ is the Δ_n-projectum of J_α, then $\omega\varkappa$ is either a limit of cardinals in J_α or else is Σ_n-regular over J_α; and (3) the number of possible values for $\Sigma_n(J_\alpha)$-cofinalities cannot exceed twice that for $\Delta_n(J_\alpha)$-projecta.

For every ordinal α, let L_α and J_α denote the αth-level of Gödel's and Jensen's hierarchies respectively. When attention is focused on those L_α's (hence J_α's) which are admissible sets, a satisfactory transfinite recursion theory results (cf. Ann. Math. Logic 4 (4) (1972)). The class of α-recursively enumerable sets has been the center of study, and various new techniques were invented to tackle problems related to this class. Apart from having to overcome the combinatorial problems (as in ordinary recursion theory) that come up in the use of priority argument, the non-Σ_n-admissibility, for $n>1$, of L_α makes it very difficult to lift a theorem in ordinary recursion theory to α-recursion theory for every admissible α.

Typical of the techniques being used in doing priority argument is the setting up of a short indexing set of requirements. This is done by exploiting to good use the relative positions of the $\Sigma_n(L_\alpha)$-cofinalities and $\Sigma_n(L_\alpha)$-projecta for $n\leqslant 3$. Experience has shown that a deeper understanding of the fine structure of L provides one with an invaluable tool to do generalized recursion theory. In particular, the concept of definable cofinalities and projecta play a major role in α-recursion theory. We were therefore led naturally to the study of their properties in a general setting. Thus we investigate the behavior of these ordinals from the viewpoint of Jensen's J_α for $\alpha>0$ (recall that $J_0 = \emptyset$). We prove in this paper that (1) the $\Sigma_n(J_\alpha)$-cofinalities and the $\Delta_n(J_\alpha)$-projecta are as regular as they should be (Theorems 1 and 2), and (2) a close relationship exists between the set of $\Sigma_n(J_\alpha)$-cofinalities and the set of $\Delta_n(J_\alpha)$-projecta (Theorem 3). This set of definable projecta enjoys the distinctive feature that each of its members is associated with a total, definable function of the same logical complexity. It therefore comes in sharp contrast with the set of $\Sigma_n(J_\alpha)$-projecta which are associated with partial functions. A further reason for considering this set is its naturalness with respect to $\Sigma_n(J_\alpha)$-cofinalities, in the sense of Lemma 1. We mention in passing a somewhat related work of Marek and Srebrny [3], in which for countable admissible sets L_α, the $\Sigma_n(L_\alpha)$-cofinalities are investigated in terms of the gaps of reals in the constructible universe.

Let $\alpha > 0$ be an ordinal.

DEFINITION. The $\Sigma_n(J_\alpha)$-*cofinality* (resp. $\Delta_n(J_\alpha)$-*projectum*), written $\sigma n\mathrm{cf}(\alpha)$ (resp. $\delta n\mathrm{p}(\alpha)$), is the least ordinal γ for which there exists a $\Sigma_n(J_\alpha)$-function mapping $\omega\gamma$ unboundedly into (resp. onto) $\omega\alpha$.

One can generalize the definition above by saying that if $\omega\varrho \leqslant \omega\alpha$, then the $\Sigma_n(J_\alpha)$-cofinality of ϱ is the least ordinal $\gamma \leqslant \varrho$ for which there is a $\Sigma_n(J_\alpha)$-function mapping $\omega\gamma$ unboundedly into $\omega\varrho$. If the least ordinal for which such functions exist is ϱ, then we say that ϱ is $\Sigma_n(J_\alpha)$-*regular*.

We fix the following notations: For each $n < \omega$, f_n is a $\Sigma_n(J_\alpha)$-function which maps $\omega\cdot\sigma n\mathrm{cf}(\alpha)$ unboundedly into $\omega\alpha$, and d_n is a $\Sigma_n(J_\alpha)$-function which maps $\omega\cdot\delta n\mathrm{p}(\alpha)$ onto $\omega\alpha$.

We assume that the reader is familiar with Jensen's work (cf. [2]), especially the following three important properties of J_α proved by him:

(1) For all n, every $\Sigma_n(J_\alpha)$-relation is $\Sigma_n(J_\alpha)$-uniformizable;

(2) For all n, $\delta n\mathrm{p}(\alpha)$ is the least ordinal γ for which there exists $A \subseteq \omega\gamma$ with $A \in \Delta_n(J_\alpha) - J_\alpha$; and

(3) For all n, there exists a $\Sigma_n(J_\alpha)$-master code $A_n \subseteq J_{\sigma n\mathrm{p}(\alpha)}$, where $\sigma n\mathrm{p}(\alpha)$ is the $\Sigma_n(J_\alpha)$-projectum (cf. [2]). There is also a $\Sigma_n(J_\alpha)$-injection g_n from $\omega\alpha$ into $\omega\cdot\sigma n\mathrm{p}(\alpha)$.

LEMMA 1. *For all* $n < \omega$, $\sigma n\mathrm{cf}(\alpha) \leqslant \delta n\mathrm{p}(\alpha)$.

The proof of Lemma 1 follows immediately from definition.

LEMMA 2. *For all* $n > m$, *if* $\sigma n\mathrm{cf}(\alpha) < \sigma m\mathrm{cf}(\alpha)$, *then* $\delta n\mathrm{p}(\alpha) \leqslant \sigma m\mathrm{cf}(\alpha)$ (1).

Proof. Define

$$\mathscr{R}(x, v) \text{ if and only if } x < \omega\cdot\sigma n\mathrm{cf}(\alpha) \text{ and } v < \omega\cdot\sigma m\mathrm{cf}(\alpha) \text{ and } f_n(x) < f_m(v).$$

$\mathscr{R}$ is clearly a $\Delta_n(J_\alpha)$-relation defined on $\omega\cdot\sigma m\mathrm{cf}(\alpha)$. Thus by (2), $\mathscr{R} \in J_\alpha$ except possibly when $\delta n\mathrm{p}(\alpha) \leqslant \sigma m\mathrm{cf}(\alpha)$. If the conclusion were false, one would then have a $\Delta_0(J_\alpha)$-uniformizing function t for $\mathscr{R}$, so that $f_m \circ t\colon \omega\cdot\sigma n\mathrm{cf}(\alpha) \to \omega\alpha$ will be a $\Sigma_m(J_\alpha)$-cofinality map, contradicting the fact that $\sigma n\mathrm{cf}(\alpha) < \sigma m\mathrm{cf}(\alpha)$.

LEMMA 3. *Let* $n = m+1$ *and assume that* $\sigma m\mathrm{p}(\alpha) \leqslant \sigma m\mathrm{cf}(\alpha)$. *Let* γ_0 *be the least ordinal* γ *for which there is a* $\Sigma_m(J_\alpha)$-*function mapping* $\omega\gamma$ *cofinally into* $\omega\text{-}\sigma m\mathrm{p}(\alpha)$. *Then* $\sigma n\mathrm{cf}(\alpha) \leqslant \gamma_0$.

Proof. We know that $\sigma n\mathrm{cf}(\alpha) \leqslant \delta n\mathrm{p}(\alpha) \leqslant \sigma m\mathrm{p}(\alpha)$. Let f be a $\Sigma_m(J_\alpha)$-function

(1) A special case appeared as Lemma 1.3 of [1].

mapping $\omega\gamma_0$ cofinally into $\omega\cdot\sigma m\mathrm{p}(\alpha)$ and let g_m be as given in (3). Define $\mathscr{R}(x, v)$ if and only if $x < \omega\gamma_0$ and $v > f(x)$ and $v \in \mathrm{Range}(g_m)$ and

$$(\forall \zeta < v)(\forall w, z)(g_m(w) = \zeta \ \& \ g_m(z) = v \to w < z)\,.$$

We note that for any $\varrho < \omega\cdot\sigma m\mathrm{p}(\alpha)$, $(g_m''\omega\alpha) \cap \varrho$ is, by the definition of $\sigma m\mathrm{p}(\alpha)$, an element of J_α. Let $K_\varrho = (g_m''\omega\alpha) \cap \varrho$. Since $\sigma m\mathrm{cf}(\alpha) \geqslant \sigma m\mathrm{p}(\alpha)$, the set $(g_m^{-1})''K_\varrho$ is bounded below $\omega\alpha$ so that there must be a v satisfying $\mathscr{R}(x, v)$. As g_m is $\Delta_m(J_\alpha)$, we see that $\mathscr{R}(x, v)$ is actually $\Sigma_n(J_\alpha)$. Let t be $\Sigma_n(J_\alpha)$ and uniformize $\mathscr{R}(x, v)$. Then $g_m^{-1} \circ t$ is $\Sigma_n(J_\alpha)$ and maps $\omega\gamma_0$ cofinally into $\omega\alpha$.

LEMMA 4. *Let* $m \geqslant 1$ *and* $n = m+1$. *Suppose that* $\omega\gamma < \omega\beta \leqslant \omega\cdot\sigma m\mathrm{cf}(\alpha)$. *If* $t\colon \omega\gamma \to \omega\beta$ *is a* $\Sigma_n(J_\alpha) - J_\alpha$ *function, then there is a* $\Sigma_n(J_\alpha)$-*function mapping* $\omega\gamma$ *cofinally into* $\omega\alpha$.

Proof. By induction, we may assume that for $m > 1$, $t \in \Sigma_n(J_\alpha) - \Sigma_m(J_\alpha)$ (for $m = 1$, it is enough to assume $t \notin J_\alpha$). Without loss of generality, let $t(x) = z$ be defined by

$$(\exists w_1)(\forall w_2 \geqslant w_1)\,\varphi(w_2, x, z)\,,$$

where $\varphi(w_2, x, z)$ is $\Sigma_{m-1}(J_\alpha)$. If the set $A = \{w_1 |\ (\forall w_2 \geqslant w_1)\,\varphi(w_2, x, z)\}$ has no bounded (in $\omega\alpha$) subset B such that as x ranges over $\omega\gamma$, an element w_1 in B can be found to satisfy $(\forall w_2 \geqslant w_1)\,\varphi(w_2, x, z)$, then already a $\Sigma_n(J_\alpha)$-cofinality map from $\omega\gamma$ into $\omega\alpha$ can be found by first defining $\mathscr{R}(x, \langle z, w_1\rangle)$ if and only if

$$(\forall w_2 \geqslant w_1)\,\varphi(w_2, x, z)\,,$$

and then (as $\mathscr{R}$ is clearly $\Sigma_n(J_\alpha)$) taking a $\Sigma_n(J_\alpha)$-function u which uniformizes $\mathscr{R}$. This function u satisfies our requirement.

Suppose on the other hand that a bounded subset B of A as described exists, and let b be its bound. Then for all $x < \omega\gamma$, $z < \omega\beta$, $t(x) = z$ if and only if $(\forall w_2 \geqslant b)\,\varphi(w_2, x, z)$. So if $m > 1$, $t(x) \neq z$ if and only if

$$(\exists w_2 \geqslant b)(\forall w_3 \geqslant w_2)\,\psi(w_3, x, z)\,,$$

where $\psi(w_3, x, z)$ is $\Sigma_{m-2}(J_\alpha)$ (if $m = 1$, $\varphi(w_2, x, z)$ would be $\Delta_0(J_\alpha)$, so that $t(x) \neq z$ is $\Sigma_1(J_\alpha)$).

Now for each $z < t(x)$, a $w_z \geqslant b$ exists giving $(\forall w_3 \geqslant w_z)\psi(w_3, x, z)$ (if $m = 1$, a w_z exists giving $\sim\varphi(w_z, x, z)$). For $z < t(x)$, define $\mathscr{R}_x(z, w_z)$ if and only if for all $w_3 \geqslant w_z$, $\psi(w_3, x, z)$. Then $\mathscr{R}_x(z, w_z)$ is $\Sigma_m(J_\alpha)$ for each $x < \omega\gamma$ (if $m = 1$, define $\mathscr{R}_x(z, w_z)$ if and only if $\sim\varphi(w_z, x, z)$. Then $\mathscr{R}_x(z, w_z)$ is $\Delta_0(J_\alpha)$, hence $\Delta_1(J_\alpha)$). Let $h_x\colon t(x) \to \omega\alpha$ be $\Sigma_m(J_\alpha)$ and uniformize $\mathscr{R}_x$. As $t(x) < \omega\beta \leqslant \omega\cdot\sigma m\mathrm{cf}(\alpha)$, $h_x''t(x)$ is bounded in $\omega\alpha$. Let σ_x denote the least ordinal greater than $h_x''t(x)$. If, as x ranges over $\omega\gamma$, the σ_x's remain bounded in $\omega\alpha$ by a σ^*, then for all $x < \omega\gamma$, and $z < t(x)$, we have $(\forall w_3 \geqslant \sigma^*)\psi(w_3, x, z)$, so that $t(x) = z$ if and only if

$$(\exists w_3 \geqslant \sigma^*) \sim\psi(w_3, x, z) \quad \text{and} \quad (\forall v < z)(\forall w_3 \geqslant \sigma^*)\psi(w_3, x, v)\,,$$

implying that t is $\Sigma_m(J_\alpha)$, contradicting our assumption (if $m = 1$, a simple calculation shows that if σ^* exists, then t is in J_α, which is also a contradiction). Now define $\mathscr{R}(x, u)$ if and only if $x<\omega\gamma$ and $(\forall z<t(x))(h_x(z)<u)$. Notice that this is equivalent to

$$x<\omega\gamma \quad \text{and} \quad (\exists z_0)(\forall z<z_0)(\forall w_2\geqslant b)(\varphi(w_2, x, z_0)\,\&\, h_x(z)<u)\,.$$

As h_x is total on $t(x)$, it is $\Delta_m(J_\alpha)$, so that $\mathscr{R}(x, u)$ is easily checked to be $\Sigma_n(J_\alpha)$. Finally, any $\Sigma_n(J_\alpha)$-function uniformizing $\mathscr{R}(x, u)$ maps $\omega\gamma$ cofinally into $\omega\alpha$.

THEOREM 1. *For all $n<\omega$, if $\sigma n\,\mathrm{cf}(\alpha)<\omega\alpha$, then it is $\Sigma_n(J_\alpha)$-regular.*

Proof. The proof for $n = 0$ is standard. One notes that if there is a $\Sigma_0(J_\alpha)$-function mapping an initial segment γ of $\omega\alpha$ unboundedly into $\omega\alpha$, then there is a strictly increasing $\Sigma_0(J_\alpha)$-function mapping $\zeta\leqslant\gamma$ unboundedly into $\omega\alpha$. Next $\sigma 1\,\mathrm{cf}(\alpha) = \sigma 0\,\mathrm{cf}(\alpha)$. The proof is just a relativization (to $\omega\cdot\sigma 0\,\mathrm{cf}(\alpha)$) of the basic fact in Kripke–Platek set theory: Σ_0-replacement implies Σ_1-replacement. Now if there is a $\Sigma_1(J_\alpha)$-function mapping a proper initial segment of $\omega\cdot\sigma 1\,\mathrm{cf}(\alpha)$ unboundedly into $\omega\cdot\sigma 1\,\mathrm{cf}(\alpha)$, then as for the $n = 0$ case one can define a strictly increasing $\Sigma_1(J_\alpha)$-function from a proper initial segment of $\omega\cdot\sigma 1\,\mathrm{cf}(\alpha)$ unboundedly into $\omega\alpha$, yielding a contradiction. Hence $\sigma 1\,\mathrm{cf}(\alpha)$ is $\Sigma_1(J_\alpha)$-regular.

Let $m\geqslant 1$ and $n = m+1$. For the sake of contradiction, suppose that $\omega\varkappa<\omega\cdot\sigma n\,\mathrm{cf}(\alpha)$ and c is a $\Sigma_n(J_\alpha)$-function mapping $\omega\varkappa$ cofinally into $\omega\cdot\sigma n\,\mathrm{cf}(\alpha)$.

Case (i). $\sigma m\mathrm{p}(\alpha)\leqslant\sigma m\,\mathrm{cf}(\alpha)$. Let γ_0 be as in Lemma 3. Then $\sigma n\,\mathrm{cf}(\alpha)\leqslant\gamma_0$. Let $A_m\subseteq J_{\sigma m\mathrm{p}(\alpha)}$ be a $\Sigma_m(J_\alpha)$-master code given by (3). Define $\mathscr{R}(x, v)$ if and only if

$$x<\omega\varkappa \quad \text{and} \quad c(x)<v \quad \text{and} \quad (\forall z<c(x))(f_n(z)<f_n(v))\,,$$

where f_n is as defined in the beginning of this paper. As $\{\langle z, v\rangle|\ f_n(z)<f_n(v)\}$ is a $\Delta_n(J_\alpha)$-subset of $\omega\cdot\sigma m\mathrm{p}(\alpha)$, it is actually $\Delta_1(\langle J_{\sigma m\mathrm{p}(\alpha)}, A_m\rangle)$. Thus $\mathscr{R}(x, v)$ if and only if

$$x<\omega\varkappa \quad \text{and} \quad c(x)<v \quad \text{and} \quad (\forall z<c(x))(\exists w)\,\varphi(z, v, w)\,,$$

where $\varphi(z, v, w)$ is $\Delta_0(\langle J_{\sigma m\mathrm{p}(\alpha)}, A_m\rangle)$ and therefore $\Sigma_m(J_\alpha)$.

For each $x<\omega\varkappa$, $c(x)<\omega\cdot\sigma n\,\mathrm{cf}(\alpha)\leqslant\omega\gamma_0$, so that by $\Delta_0(\langle J_{\sigma m\mathrm{p}(\alpha)}, A_m\rangle)$-replacement on $c(x)$, one has $\mathscr{R}(x, v)$ if and only if

$$(\exists u)(\forall z<c(x))(\exists w<u)\,\varphi(z, v, w)\,.$$

So $\mathscr{R}(x, v)$ is $\Sigma_1(\langle J_{\sigma m\mathrm{p}(\alpha)}, A_m\rangle)$ (notice that $c(x)<v$ is $\Sigma_n(J_\alpha)$ and therefore $\Sigma_1(\langle J_{\sigma m\mathrm{p}(\alpha)}, A_m\rangle)$), and we conclude that it is $\Sigma_n(J_\alpha)$. Let $t\colon \omega\varkappa\to\omega\cdot\sigma n\,\mathrm{cf}(\alpha)$ be $\Sigma_n(J_\alpha)$ and uniformize $\mathscr{R}(x, v)$. Then $f_n\circ t$ is $\Sigma_n(J_\alpha)$ and maps $\omega\varkappa$ cofinally into $\omega\alpha$. Since $\omega\varkappa<\omega\cdot\sigma n\,\mathrm{cf}(\alpha)$, we have a contradiction.

Case (ii). $\sigma m\,\mathrm{cf}(\alpha)<\sigma m\mathrm{p}(\alpha)$. By the method of Lemma 2, we can get a $\Sigma_n(J_\alpha)$-map t from $\omega\cdot\sigma n\,\mathrm{cf}(\alpha)$ into $\omega\cdot\sigma m\,\mathrm{cf}(\alpha)$ cofinally. Thus $t(x) = z$ if and only if $(\exists w)\,\varphi(w, x, z)$, where $\varphi(w, x, z)$ is $\Delta_0(\langle J_{\sigma m\mathrm{p}(\alpha)}, A_m\rangle)$. If $\Delta_0(\langle J_{\sigma m\mathrm{p}(\alpha)}, A_m\rangle)$-replacement holds for the formula φ on the initial segment $\omega\cdot\sigma n\,\mathrm{cf}(\alpha)$, we can repeat the argument of case (i) to obtain a contradiction. On the other hand, suppose that $\omega\tau<\omega\cdot\sigma n\,\mathrm{cf}(\alpha)$ is the least ordinal where such a replacement operation fails. If $t|\omega\tau$ is an element of J_α, it is actually in $J_{\sigma m\mathrm{p}(\alpha)}$ because $\omega\cdot\sigma m\mathrm{p}(\alpha)$ is (obviously) a cardinal of J_α and $t|\omega\tau$ is a bounded subset of $\omega\cdot\sigma m\mathrm{p}(\alpha)$. If this happens, then $(\exists w)\,\varphi(w, x, z)$ defined on $\omega\tau\times\omega\cdot\sigma m\,\mathrm{cf}(\alpha)$ would actually be $\Delta_0(J_{\sigma m\mathrm{p}(\alpha)})$, which immediately implies that $\Delta_0(J_{\sigma m\mathrm{p}(\alpha)})$-replacement holds for $\varphi(w, x, z)$ on the initial segment $\omega\tau$ (for $x<\omega\tau$), since $\omega\cdot\sigma m\mathrm{p}(\alpha)$ is a cardinal in J_α and hence admissible. Thus $t|\omega\tau$ is not in J_α, and so by Lemma 5 there is a $\Sigma_n(J_\alpha)$-function mapping $\omega\tau<\omega\cdot\sigma n\,\mathrm{cf}(\alpha)$ cofinally into $\omega\alpha$. Again this is a contradiction.

COROLLARY 1. *For all $n<\omega$, $\sigma n\,\mathrm{cf}(\alpha)<\delta n\mathrm{p}(\alpha)$ if $\delta n\mathrm{p}(\alpha)$ is not $\Sigma_n(J_\alpha)$-regular.*

Proof. By Lemma 1, $\sigma n\,\mathrm{cf}(\alpha)\leqslant\delta n\mathrm{p}(\alpha)$. By Theorem 1, the conclusion follows.

Remark 1. The converse of Corollary 1 is false. For example, take $\omega = \omega_1^L+\omega$, then $1 = \sigma 1\,\mathrm{cf}(\alpha)<\omega_1^L = \delta 1\mathrm{p}(\alpha)$.

Remark 2. The method of proof of Theorem 1 can also be used to show that for all $n<\omega$, there is a strictly increasing $\Sigma_n(J_\alpha)$-function mapping $\omega\cdot\sigma n\,\mathrm{cf}(\alpha)$ cofinally into $\omega\alpha$.

THEOREM 2. *For all $n<\omega$, $\omega\cdot\delta n\mathrm{p}(\alpha)$ is either a limit of cardinals in J_α or is $\Sigma_n(J_\alpha)$-regular.*

Proof. Suppose for the sake of contradiction that β is the largest cardinal less than $\omega\cdot\delta n\mathrm{p}(\alpha)$ and $\gamma<\delta n\mathrm{p}(\alpha)$ is the least ordinal for which there exists a $\Sigma_n(J_\alpha)$-function f mapping $\omega\gamma$ cofinally into $\omega\cdot\delta n\mathrm{p}(\alpha)$. Let $g\colon \beta\to\omega\gamma\times\beta$ be a $\Delta_0(J_\alpha)$-bijection. Such a function exists since $\omega\gamma\leqslant\beta$. Define $\mathscr{R}(x, u)$ if and only if

$$x<\omega\gamma \text{ and for some } \varrho<\omega\cdot\delta n\mathrm{p}(\alpha),\ f(x)<\varrho \text{ and } u \text{ is a surjection of } \beta \text{ onto } \varrho.$$

Since $\varrho<\omega\cdot\delta n\mathrm{p}(\alpha)$, the maps u are $\Delta_0(J_{\delta n\mathrm{p}(\alpha)})$. Hence $\mathscr{R}(x, u)$ is a $\Sigma_n(J_\alpha)$-relation such that for all $x<\omega\gamma$, $\mathscr{R}(x, u)$ holds for some u in $J_{\delta n\mathrm{p}(\alpha)}$ (notice that if $\omega\cdot\delta n\mathrm{p}(\alpha)<\omega\alpha$, then it is a cardinal in J_α, so that by the absoluteness of Δ_0-formulas, a u in $J_{\delta n\mathrm{p}(\alpha)}$ satisfying the relation can be found). We also note that as x ranges through $\omega\gamma$, the corresponding u's will range through projections of β onto ϱ, for unboundedly many ϱ's in $\omega\cdot\delta n\mathrm{p}(\alpha)$. Let t be a $\Delta_n(J_\alpha)$-uniformization for $\mathscr{R}$. Let $g(v) = \langle x, \zeta\rangle$ and let $\Gamma(v) = [t(x)](\zeta)$. Then $\Gamma\colon \beta\to\omega\cdot\delta n\mathrm{p}(\alpha)$ is a $\Delta_n(J_\alpha)$-projection. The composition map $d_n\circ\Gamma$ then gives a $\Delta_n(J_\alpha)$-projection of β onto $\omega\alpha$, which is not possible since $\beta<\omega\cdot\delta n\mathrm{p}(\alpha)$. Thus $\omega\cdot\delta n\mathrm{p}(\alpha)$ is either a limit of cardinals in J_α or is $\Sigma_n(J_\alpha)$-regular.

Let k be the smallest positive integer for which there exist ordinals $\varrho_k<\ldots<\varrho_2<\varrho_1\leqslant\omega\alpha$ such that (i) for all $0<i\leqslant k$, $\varrho_i = \sigma m\,\mathrm{cf}(\alpha)$ for some m, and (ii) for all $m<\omega$, $\sigma m\,\mathrm{cf}(\alpha) = \varrho_i$ for some i, $0<i\leqslant k$. Similarly, let q be the least positive integer obtained by changing $\sigma m\,\mathrm{cf}(\alpha)$ in (i) and (ii) above to $\delta m\mathrm{p}(\alpha)$. The next theorem shows that these two numbers are closely related.

THEOREM 3. *$k\leqslant 2q$, and, in particular, if $q>1$ and if whenever $\omega<\omega\cdot\delta n\mathrm{p}(\alpha)<\omega\alpha$ implies that $\omega\cdot\delta n\mathrm{p}(\alpha)$ is a successor cardinal in J_α, then $k\leqslant q$.*

Proof. Divide $\varrho_k < \varrho_{k-1} < \ldots < \varrho_2$ into pairs $\{\varrho_2, \varrho_3\}, \{\varrho_4, \varrho_5\}, \ldots, \{\varrho_{k-1}, \varrho_k\}$ if k is odd, and into $\{\varrho_2, \varrho_3\}, \{\varrho_4, \varrho_5\}, \ldots, \{\varrho_{k-2}, \varrho_{k-1}\}$ if k is even. Let t be the number of pairs. By Lemma 2, there would be at least t values taken on by the set of definable projecta. But $k = 2t+1$ if k is odd and $k = 2t+2$ if k is even. Since whenever $\sigma m\mathrm{cf}(\alpha) = \varrho_1$ then $\delta m\mathrm{p}(\alpha) \geqslant \varrho_1$, we see that $q \geqslant 1+t$. Hence $k \leqslant 2q$.

Now suppose that for all n, if $\omega < \omega \cdot \delta n\mathrm{p}(\alpha) < \omega\alpha$ then $\omega \cdot \delta n\mathrm{p}(\alpha)$ is a successor cardinal of J_α. By Theorem 2, it must also be $\Sigma_n(J_\alpha)$-regular. Let $m < n$. If $\sigma m\mathrm{cf}(\alpha) \leqslant \delta n\mathrm{p}(\alpha)$, then applying the method used in the proof of Lemma 2, there is a $\Sigma_n(J_\alpha)$-function mapping $\omega \cdot \sigma n\mathrm{cf}(\alpha)$ cofinally into $\omega \cdot \sigma m\mathrm{cf}(\alpha)$. Since $\omega \cdot \delta n\mathrm{p}(\alpha)$ is $\Sigma_n(J_\alpha)$-regular (we assume here that it is less than $\omega\alpha$), $\sigma m\mathrm{cf}(\alpha)$ must be less than $\delta n\mathrm{p}(\alpha)$. On the other hand, the graph of the $\Sigma_n(J_\alpha)$-cofinality function is $\Delta_n(J_\alpha)$, so that necessarily $\sigma m\mathrm{cf}(\alpha) = \sigma n\mathrm{cf}(\alpha)$. Thus for all $m < n$: $(*)$ if $\omega < \omega \cdot \delta n\mathrm{p}(\alpha) < \omega\alpha$, then either $\sigma m\mathrm{cf}(\alpha) = \sigma n\mathrm{cf}(\alpha)$ or $\delta n\mathrm{p}(\alpha) < \sigma m\mathrm{cf}(\alpha)$.

Let $i > 1$. If $\sigma m\mathrm{cf}(\alpha) = \varrho_i$ and $\sigma(m+1)\mathrm{cf}(\alpha) = \varrho_{i+1}$, then

$$\sigma(m+1)\mathrm{cf}(\alpha) \leqslant \delta(m+1)\mathrm{p}(\alpha) < \sigma m\mathrm{cf}(\alpha) \leqslant \delta m\mathrm{p}(\alpha).$$

From this one concludes immediately that $k \leqslant q$.

COROLLARY 2. *Suppose that $q > 1$ and that for all $n < \omega$, if $\omega < \omega \cdot \delta n\mathrm{p}(\alpha) < \omega\alpha$ then $\omega \cdot \delta n\mathrm{p}(\alpha)$ is a successor cardinal in J_α. Then $k = q$ if and only if for all $n = m+1$, whenever $\delta n\mathrm{p}(\alpha) < \delta m\mathrm{p}(\alpha)$, one also has $\delta n\mathrm{p}(\alpha) < \sigma m\mathrm{cf}(\alpha)$.*

Proof. By Theorem 3, we know that $k \leqslant q$. If $\delta n\mathrm{p}(\alpha)$ less than $\delta m\mathrm{p}(\alpha)$ implies that $\delta n\mathrm{p}(\alpha) < \sigma m\mathrm{cf}(\alpha)$, then one has in turn $\sigma n\mathrm{cf}(\alpha) \leqslant \delta n\mathrm{p}(\alpha) < \sigma m\mathrm{cf}(\alpha)$, so that $q \leqslant k$.

Conversely, suppose that $q \leqslant k$. Let $n = m+1$ be the least number with $\sigma m\mathrm{cf}(\alpha) \leqslant \delta n\mathrm{p}(\alpha) < \delta m\mathrm{p}(\alpha)$. As $\delta n\mathrm{p}(\alpha)$ is $\Sigma_n(J_\alpha)$-regular, one obtains, as in the proof of Theorem 3, either $\sigma n\mathrm{cf}(\alpha) = \sigma m\mathrm{cf}(\alpha) = \delta n\mathrm{p}(\alpha)$ or $\sigma n\mathrm{cf}(\alpha) = \sigma m\mathrm{cf}(\alpha) < \delta n\mathrm{p}(\alpha)$. Thus up to n, the set of definable projecta takes on one more value than the set of definable cofinalities. This difference of at least one value will persist, by Lemma 2 and $(*)$ of Theorem 3, as we go through the set of natural numbers. Thus $q \geqslant k+1$, which contradicts our assumption.

Remark 3. The inequality in Theorem 3 is the best possible. To see this, note that for $\alpha = \aleph_\omega^L$, $k = 2$ and $q = 1$, yielding inequality in the theorem. On the other hand, for $\alpha = \omega_1^{\mathrm{CK}}$ (Church–Kleene first non-recursive ordinal), $k = q = 2$, yielding strict equality in the theorem.

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