Some factorization theorems for closed subspaces

by

W. Kulpa (Katowice)

Abstract. There are given some theorems on factorization of a continuous map $f\colon X\to Y$ between completely regular spaces through continuous maps $X\xrightarrow{h}Z\to Y$ such that $h|M$ is a homeomorphism onto a closed subspace $hM\subset Z$, with $uZ = uM$, where $M\subset X$ is a fixed "well"-embedded closed subspace of the space X with infinite uniform weight $uM\geqslant uY$.

1. Preliminaries. The spaces considered in this note are assumed to be completely regular and the maps are assumed to be continuous. We use the notation of uniformity in the covering sense as in Isbell's book [4]. Some symbols and notations are taken from [6].

An infinite uniform weight of a space X is a cardinal number $uX = \inf\{\aleph_0 + \text{weight } \mathcal{U}\colon \mathcal{U}$ is a uniformity compatible with the topology on the space $X\}$. If $uX = \aleph_0$, then the space X is metrizable.

Denote by $\mathcal{U}_X^*$ the greatest uniformity compatible with the topology on the space X. For each subspace $M\subset X$, let $\mathcal{U}_X^*|M$ be a uniformity on M induced by $\mathcal{U}_X^*$, i.e. $\mathcal{U}_X^*|M = \{P|M\colon P\in\mathcal{U}_X^*\}$ where $P|M = \{u\cap M\colon u\in P\}$.

A subspace $M\subset X$ is said to be *u-subspace* iff $\mathcal{U}_M^* = \mathcal{U}_X^*|M$.

2. Factorization theorems. In this note we shall show that under some additional assumptions the following proposition is valid.

For each map $f\colon X\to Y$ and a closed u-subspace $M\subset X$, $uM\geqslant uY$, there exists a factorization of the map f through maps $X\underset{\text{onto}}{\overset{h}{\to}}Z\overset{g}{\to}Y$, i.e. $f = gh$, such that $h|M$ is a homeomorphism onto a closed subspace $hM\subset Z$ and $uM = uZ$.

THEOREM 1. *Let $M\subset X$ be a closed u-subspace of X with a complete compatible uniformity on M of weight equal to uM. Then for each map $f\colon X\to Y$, $uY\leqslant uM$, there exists a factorization of the map f through maps $X\underset{\text{onto}}{\overset{h}{\to}}Z\to Y$ such that $h|M$ is a homeomorphism onto a closed subspace $hM\subset Z$ with $uZ = uM$.*

Proof. Let $\mathcal{V}$ be a uniformity compatible with the topology on the space Y such that the weight $\mathcal{V}\leqslant uM$. The map $f\colon (X,\mathcal{U}_X^*)\to(Y,\mathcal{V})$ is uniform. Let $\mathcal{B}$ be a base for $\mathcal{V}$ with card $\mathcal{B}\leqslant uM$. Since M is a u-subspace of X, by a countable operation we can choose a pseudouniformity $\mathcal{U}\subset\mathcal{U}_X^*$ with weight $\mathcal{U} = uM$, $f^{-1}\mathcal{B}\subset\mathcal{U}$, and such that $\mathcal{U}|M$ is a complete uniformity compatible with the topology on M (for an indication of the proof see Proposition 3 in [6]).

Now, let $X_{\mathcal{U}} = \{[x]_{\mathcal{U}}\colon x\in X\}$, where $[x]_{\mathcal{U}} = \bigcap\{\mathrm{st}(x,P)\colon P\in\mathcal{U}\}$. The set $Z = X_{\mathcal{U}}$ is a partition of X into sets $[x]_{\mathcal{U}}$, $x\in X$. Put $h\colon X\to Z$, $x\mapsto[x]_{\mathcal{U}}$, and define a uniformity $\mathcal{U}^{\#}$ on the set Z, $\mathcal{U}^{\#} = \{P^{\#}\colon P\in\mathcal{U}\}$, $P^{\#} = \{Z-h(X-u)\colon u\in P\}$. The map $h\colon(X,\mathcal{U})\to(Y,\mathcal{V})$ is uniform. Since $h^{-1}\mathcal{U}^{\#} = \mathcal{U}$ and $f^{-1}\mathcal{V}\subset\mathcal{U}$, there exists a unique uniform map $g\colon(Z,\mathcal{U}^{\#})\to(Y,\mathcal{V})$, $g[x]_{\mathcal{U}} = fx$, such that $f = gh$ (for detailed proofs see [6]).

Let Z be a topological space with the topology induced by the uniformity $\mathcal{U}^{\#}$. Since $\mathcal{U}|M = h^{-1}\mathcal{U}^{\#}|M$ and $\mathcal{U}|M$ is a uniformity compatible with the topology on M, we have $h|M\colon M\to hM$ is a homeomorphism. Now, suppose that there exists a $z\in \mathrm{cl}_Z hM - hM$. Then

$$\{h^{-1}\mathrm{st}(z,Q)\cap M\colon Q\in\mathcal{U}^{\#}\} = \{\mathrm{st}(x,P)\cap M\colon P\in\mathcal{U}\}\quad\text{for } x\in h^{-1}z,$$

is a Cauchy filter in the complete uniform space $(M,\mathcal{U}|M)$. Hence there exists a point

$$y\in\bigcap\{\mathrm{cl}_M\mathrm{st}(x,P)\cap M\colon P\in\mathcal{U}\} = \bigcap\{\mathrm{st}(x,P)\cap M\colon P\in\mathcal{U}\}$$

(because $P'\succ_* P$ implies $\{\mathrm{cl}\,u\colon u\in P'\}\succ P$). Thus $hy = z$, $y\in M$, which contradicts $z\notin hM$.

A subspace $M\subset X$ is said to be a *G_m subspace* iff M is an intersection of m open sets in the space X. For the case $m = \aleph_0$ this means that M is G_δ in X.

THEOREM 2. *Let $M\subset X$ be a G_m and closed u-subspace of a normal space X. If $m\leqslant uM$, then for each map $f\colon X\to Y$, $uY\leqslant uM$, there exists a factorization of the map f through maps $X\overset{h}{\underset{\text{onto}}{\to}}Z\to Y$ such that $h|M$ is a homeomorphism onto a closed subspace $hM\subset Z$, $hM\cap h(X-M) = \emptyset$ and $uZ = uM$.*

Proof. Let $\mathcal{V}$ be a uniformity compatible with the topology on Y, with a base $\mathcal{B}\subset\mathcal{V}$, $\mathrm{card}\,\mathcal{B}\leqslant uM$, consisting of locally finite open coverings. Let $\mathcal{G}$ with $\mathrm{card}\,\mathcal{G}\leqslant uM$ be a family of open sets in X such that $M = \bigcap\{G\colon G\in\mathcal{G}\}$. The uniformity $\mathcal{U}_X^*$ has a base consisting of all the locally finite open coverings of X, because the space X is normal (see e.g. [4]). Hence it is possible to find by a countable operation (see Proposition 3 [6]) a pseudouniformity $\mathcal{U}\subset\mathcal{U}_X^*$ such that $f^{-1}\mathcal{B}\subset\mathcal{U}$, $\{G, X-M\}\in\mathcal{U}$ for each $G\in\mathcal{G}$, weight $\mathcal{U} = uM$ and $\mathcal{U}|M$ is a uniformity on M compatible with the topology.

Now, as in the previous theorem, define $Z = X_{\mathcal{U}}$ with the topology induced by $\mathcal{U}^{\#}$, $h\colon X\to Z$, $x\mapsto[x]_{\mathcal{U}}$, and $g\colon Z\to Y$, $[x]_{\mathcal{U}}\mapsto fx$.

To see that $hM\subset Z$ is a closed subspace of Z such that $h(X-M)\cap hM = \emptyset$, consider $x\notin M$. There exists a $G\in\mathcal{G}$ such that $x\notin G$. Put $P = \{G, X-M\}$, $z = hx$. Then $\mathrm{st}(z,P^{\#})\cap hM = \emptyset$. Since $P^{\#}\in\mathcal{U}^{\#}$, we have $hx\notin hM$ and $hx\notin\mathrm{cl}_Z hM$. This implies that $h(X-M)\cap hM = \emptyset$ and $\mathrm{cl}_Z hM = hM$.

A space X is *m-feathered* iff there exists a family $\mathcal{P}$, with $\mathrm{card}\,\mathcal{P}\leqslant m$, of coverings of X with open sets in the Čech–Stone compactification βX such that for each $x\in X$, $\bigcap\{\mathrm{st}(x,P)\colon P\in\mathcal{P}\}\subset X$. For the case $m = \aleph_0$, the space X is said to be *feathered* or a *p-space* (see Archangel'skiĭ [1]). Each G_m subspace of βX subspace is m-feathered.

A map $h\colon X\overset{\text{onto}}{\to}Z$ is said to be *perfect* iff $h^{-1}z$ is compact for each $z\in Z$ and $hM\subset Z$ is closed for each closed set $M\subset X$.

THEOREM 3. *Let $M\subset X$ with $uM\leqslant m$ be a closed subspace of an m-feathered paracompact space X. Then for each map $f\colon X\to Y$, $uY\leqslant uM$ there exists a factorization through maps $X\overset{h}{\to}Z\to Y$ such that $h\colon X\to Z$ is a perfect map onto an m-feathered paracompact space Z where $uZ = uM$ and $h|M$ is a homeomorphism.*

Proof. Let $\mathcal{B}$ with $\mathrm{card}\,\mathcal{B}\leqslant uM$ be a base for a uniformity compatible with the topology on Y. By a countable operation we can choose a pseudouniformity $\mathcal{U}\subset\mathcal{U}_X^*$ such that $\mathcal{U}|M$ is compatible with the topology on M, $f^{-1}\mathcal{V}\subset\mathcal{U}$, $\mathcal{P}|X\subset\mathcal{U}$, and for each $P\in\mathcal{P}$ there exists a locally open covering $Q\in\mathcal{U}$ such that $Q^*\succ P$ where $Q^* = \{v^*\colon v\in Q\}$, $v^* = \mathrm{cl}_{\beta X}\bigcup\{u\subset\beta X\colon u\cap X = v,\ u \text{ is open in } \beta X\}$. From the compactness of βX it follows that $\{\mathrm{st}(x,P)\colon P\in\mathcal{U}\}$ is a base of $[x]_{\mathcal{U}}$ and $[x]_{\mathcal{U}}$ is compact. In virtue of $h^{-1}\mathcal{U}^{\#} = \mathcal{U}$; we infer that $h\colon X\to Z$, $x\mapsto[x]_{\mathcal{U}}$ is a perfect map. By a theorem of Michael [8] the space Z is paracompact and the space Z is m-feathered because $h(\beta X-X)\subset\beta Z-Z$ for each perfect map.

3. Some remarks and applications to the factorization theorems. Dowker [2] (see also Katětov [5]) has proved that for each closed subspace $M\subset X$ of a collectionwise normal space X and for each locally finite closed covering $\{F_s\colon s\in S\}$ of the space M and for each open in X covering $\{U_s\colon s\in S\}$ of the space M such that $F_s\subset U_s$, $s\in S$, there exists a locally open in X family $\{V_s\colon s\in S\}$ such that $F_s\subset V_s\cap M\subset U_s$. Since for each space M the uniformity $\mathcal{U}_M^*$ has a base consisting of locally finite closed coverings, the result of Dowker implies the following

LEMMA. *Each closed subset M of a collectionwise normal space X is a u-subspace.*

By putting in the factorization theorems $Y = \{\text{point}\}$, $uM = \aleph_0$, we find that for each metrizable space $M\subset X$ under some additional assumptions (as in the theorems) there exists a map $f\colon X\to Z$ onto a metrizable space Z such that $h|M$ is a homeomorphism onto a closed subspace $hM\subset Z$. Now, notice that for each pair (X, M), $M\subset X$, for which there exists a map $h\colon X\to Z$ onto a metrizable space such that $h|M$ is a homeomorphism onto a closed subspace $hM\subset Z$, the Dugundji Extension Theorem is valid, i.e. there exists a simultaneous extender $l\colon C_R(M)\to C_R(X)$, where R means a locally convex vector space (see [7]). Lutzer and Martin have proved in [7] that in M is a closed subset of collectionvise normal space X, then the Dugundji Extension Theorem is valid for the pair (X, M). Theorem 2 also implies the result of Lutzer and Martin.

Theorems 1 and 3 give some new results for the question when a pair (X, M), where M is a metrizable subspace, satisfies the Dugundji Extension Theorem. From

Theorems 1 and 3 and from the lemma it follows that if M is a complete metrizable closed subspace of a collectionwise normal space X or M is a closed subset of a paracompact p-space, then the Dugundji Extensions Theorem is valid for the pair (X, M).

In [3] we give an example of a hereditarily paracompact space and a closed separable metric space $M \subset X$ such that the Dugundji Extension Theorem is not valid for the pair (X, M). This implies that the theorems on the factorization of a map $f\colon X \to Y$ through $X \overset{h}{\to} Z \to Y$, $uZ = uM \geqslant uY$, such that $h|M$ is a homeomorphism onto a closed subspace $hM \subset Z$, cannot be obtained without additional assumptions on embedding the set $M \subset X$, even if we assume that the space X is nice.

Another corollary which can be obtained from Theorem 3 by putting $M = \{\text{point}\} = Y$, is a result of Archangel'skiĭ [1] stating that each paracompact p-space has a perfect map onto a metrizable space.

References

[1] А. Б. Архангельский, *Об одном классе пространств содержащием все метрические и все локально бикомпактное пространства*, Матем. Сборник 67 (109) (1965), pp. 55–85.

[2] C. H. Dowker, *On a theorem of Hanner*, Arkiv för Matematik 2 (1952), pp. 307–313.

[3] R. Heath and D. Lutzer, *Dugundji extension theorems for linearly ordered spaces*, Pacific J. Math. 55 (1974), pp. 419–426.

[4] J. R. Isbell, *Uniform Spaces*, Providence 1964.

[5] M. Катѐтов, *Extensions of locally finite covers*, Colloq. Math. 6 (1958), pp. 145–151.

[6] W. Kulpa, *Factorization and inverse expansion theorems*, Colloq. Math. 21 (1970), pp. 217–227.

[7] D. Lutzer and H. Martin, *A note on the Dugundji extension theorem*, Proc. Amer. Math. Soc. 45 (1975), pp. 137–139.

[8] E. Michael, *Another note on paracompact spaces*, Proc. Amer. Math. Soc. 8 (1957), pp. 822–828.

SILESIAN UNIVERSITY, Katowice

Accepté par la Rédaction le 19. 7. 1976

Uniform homotopy

by

Allan Calder (St. Louis, Miss.)

Abstract. For X a finite dimensional normal space and Y a compact space, let $\beta^*\colon [\beta X, Y] \to [X, Y]$ be the natural quotient map from the set of uniform homotopy classes of maps from X to Y to the set of homotopy classes of maps. It is shown that if there exists a topological group G with the structure of a CW-complex of finite type such that its classifying space dominates Y then $\ker\beta^* = 0$. Hence if β^* is a homomorphism, for suitable group structures on the sets, maps from X to Y are homotopic if and only if they are uniformly homotopic. The condition that Y be compact can be removed when considering bounded maps.

This paper is concerned with the problem of when are uniform homotopy and homotopy equivalent. More specifically, under what conditions on the spaces X and Y are a pair of maps $X \to Y$ homotopic if and only if they are uniformly homotopic? This is known to be the case when X is countably compact and Y is compact [5], [16]. On the otherhand, if Y is the circle S^1 and X is not pseudocompact then there is a map from X to Y which is homotopic to a constant map but not uniformly so [5], [8] [25, p. 225].

In fact the techniques used allow consideration of a slightly different problem, namely, "when does the embedding $\beta\colon X \to \beta X$ of a space into its Stone–Čech compactification induce a bijection β^* between the set $[\beta X, Y]$, of homotopy classes of maps from βX to Y, and the set $[X, Y]$ of homotopy classes of maps from X to Y". This is equivalent to the original problem for Y compact.

For X a finite dimensional (covering dimension) normal space and Y a space dominated by a CW-complex B, where B is the classifying space of a group G and G is a CW-complex of finite type it is shown that β^* is onto and has the null class as "kernel". Hence if β^* is a homomorphism then β^* is a bijection. If in addition to all this, Y is also compact then every pair of maps from X to Y are homotopic if and only if they are uniformly homotopic.

As corollaries we have that if X is a finite dimensional normal space then the Čech cohomology of X (based on locally finite covers) is isomorphic to that of βX in all dimensions over a finite coefficient group and in dimensions higher than 1 over a finitely generated abelian coefficient group, and that a map from X into an n-sphere $(n > 1)$ is uniformly homotopic to a constant map if and only if it is null homotopic.