Sentences with three quantifiers are decidable in set theory

by

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Abstract. We use a theorem of Ehrenfeucht, relating games to model theory, to prove that all closed prenex formulas with three quantifiers are decidable in set theory, with the predicates "=" and "ϵ" and the axioms of choice and regularity.

In this paper, we use a theorem of Ehrenfeucht (cf. [1]) to prove that in Zermelo–Fraenkel set theory, with the symbols "$\in$" and "$=$", and with the axiom of choice and the axiom of regularity (i.e. the axiom that all elements have ranks), all prenex sentences with three quantifiers are decidable. Our argument can be formalized in Zermelo–Fraenkel set theory with the axiom of choice.

We first present Ehrenfeucht's theorem. We speak of the following "N-game" played by two players with structures A and B for a language L. For each of the N moves, player 1 may pick an element from whichever of A and B he chooses, and player 2 must pick an element from the other structure. Note that neither player necessarily makes all his picks from the same structure. At the end of the game, $a_1, a_2, \ldots, a_N$ have been picked, in that order, from A, and $b_1, b_2, \ldots, b_N$ from B. Player 1 is the winner if and only if there is some quantifier-free formula $F(x_1, x_2, \ldots, x_N)$ in L such that $A \vdash F(a_1, a_2, \ldots, a_N)$ and $B \vdash \sim F(b_1, b_2, \ldots, b_N)$. Otherwise, player 2 is the winner.

THEOREM 1 (Ehrenfeucht). *If there is a closed prenex formula in L with N quantifiers which is satisfied by structure A and not structure B, then player* 1 *has a winning strategy in an N-game.*

Proof. If the hypothesis is satisfied, there is some N-quantifier prenex formula whose left-hand quantifier is $\exists$ which is satisfied in one structure but not the other. Call the formula $(\exists x_1)(Q_2 x_2) \ldots (Q_N x_N) F(x_1, x_2, \ldots, x_N)$ where each Q_i is either $\exists$ or V. For his first move, player 1 can pick an element e in whichever structure satisfies the formula, such that $(Q_2 x_2)(Q_3 x_3) \ldots (Q_N x_N) F(e, x_2, x_3, \ldots, x_N)$ is satisfied in that structure. But player 2 must pick e' in the other structure such that $\sim(Q_2 x_2)(Q_3 x_3) \ldots (Q_N x_N) F(e', x_2, x_3, \ldots, x_N)$ is satisfied in it. If a constant c is added to L and is assigned to e and e' to create structures A' and B', then

$(Q_2 x_2)(Q_3 x_3) \ldots (Q_N x_N) F(c, x_2, x_3, \ldots, x_N)$ is satisfied in one structure but not the other. Thus player 1 still has a prenex formula whose left-hand quantifier is existential which is satisfied in 1 structure and not the other, and has eliminated a quantifier. By using the same technique as in his first move for each succeeding move, player 1 can insure that at the end of N moves, $a_1, a_2, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ will have been picked from A and B respectively and exactly one of $A \vdash F(a_1, a_2, \ldots, a_N)$ and $B \vdash F(b_1, b_2, \ldots, b_N)$ will be true, so player 1 has a winning strategy. Q.E.D.

Before turning to our application of Ehrenfeucht's theorem, we prove a theorem of our own on the same subject.

THEOREM 2. *If A and B are structures for a finite language L which has no function symbols (but possibly constants), and A and B satisfy the same formulas with N variables, then player 2 has a winning strategy in an "N-game".*

Proof. First we define formulas V_N. If S is a finite set of variables, let $A(S)$ be the set of all atomic formulas that can be constructed with the constants and relation symbols in L and the variables in S. If X is a finite set of formulas, let $V(X)$ be the set of all conjunctions whose conjuncts include either F or $\sim F$, but not both, for each $F \in X$. Then $V(A(S))$ is a set of quantifier-free formulas. Let $\exists_n V(A(S))$ be the set of all formulas of the form $(\exists x_n)F$ where $F \in V(A(S))$. Let

$$V_1 = V(\exists_1 V(A\{x_1\})),$$
$$V_2 = V(\exists_2 V(\exists_1 V(A\{x_1, x_2\}))),$$
$$V_3 = V\big(\exists_3 V(\exists_2 V(\exists_1 V(A\{x_1, x_2, x_3\})))\big).$$

We hope this is enough to make the pattern for V_N clear. In each set V_N, A satisfies exactly one formula and B satisfies only the same one, since it satisfies the same formulas with N variables.

LEMMA. *If A and B are arbitrary structures which satisfy the same formula in each V_K for $K \leqslant N$, then player 2 has a winning strategy in an N-game.*

Proof. We assume that in a 1-game player 1 picks his first element to be a_1 from A, and there is no loss of generality in assuming so. Then A satisfies one formula $P(a_1)$ in $V(A\{a_1\})$ and it satisfies the corresponding formula $(\exists x_1)P(x_1)$ in $\exists_1 V(A\{x_1\})$, and so does B, so player 2 can choose b_1 in B so that B satisfies the formula in $V(A\{b_1\})$ similar to the one that A satisfies in $V(A\{a_1\})$. So player 2 has a winning strategy in a 1-game.

In order to use induction, we assume that for some K less than N, player 2 has a winning strategy in a K-game. Then since A and B satisfy the same formula in V_{K+1}, if a $(K+1)$-game is played, suppose player 1 picks a from A on his first move. Exactly one of the conjuncts in the formula in V_{K+1} that A satisfies is of the form $(\exists x_{K+1})P(x_1, \ldots, x_{K+1})$ and has the property that $P(x_1, x_2, \ldots, x_K, a)$ is satisfied in A. (To see this, note that in the expansion A' of A, in which the constant a is assigned to the element a, $P(x_1, x_2, \ldots, x_K, a)$ is the one true formula in V_K). Then player 2 can pick b in B such that $P(x_1, x_2, \ldots, x_K)$ is satisfied in B. Now if the constant c is added to L to form L' and it refers to a in A and b in B, then A and B become structures A' and B' for L' which satisfy the same formula $P(x_1, x_2, \ldots, x_K, c)$ in V_K. So the last K moves of the $(K+1)$-game amount to a K-game played with the two new structures A' and B' for L', so the induction is complete. This proves the lemma and thereby Theorem 2.

The following strengthening of Theorem 2 is false: If A and B are structures for a finite language L with no function symbols or constants, which satisfy the same N-quantifier formulas, then player 2 has a winning strategy in an N-game.

To see this, consider the language L containing only the relation symbol R. Let A have $\{1, 2\}$ as its universe and let the universe of B be $\{1, 2, 3\}$. Define R by $R(x, y) \leftrightarrow x < y$. Then it can be easily verified that A and B satisfy the same 2-quantifier formulas, but player 1 has a winning strategy in a 2-game, in which his first move is to pick the element 2 in B.

Before turning to our main theorem we stipulate that by "set theory" we mean the usual axioms but not the axiom of choice and the continuum hypothesis. The axioms of substitution and well-foundedness are included, however.

Our example of an application of Ehrenfeucht's theorem to prove a decidability result is:

THEOREM 3. *All 3-quantifier closed prenex formulas in set theory, with both $\in$ and $=$, are decidable in set theory with the axioms of choice and regularity.*

We will prove Theorem 3 by showing that in a 3-game played with any 2 models for set theory, player 2 has a winning strategy.

We first make some preliminary remarks. Since the axiom of choice is independent from set theory and can be expressed by an 8-quantifier formula ([1]), there is an obvious limit on possible extensions of Theorem 3. We conjecture that all 7-quantifier closed formulas in set theory are decidable. But it would be hopeless to try to use the game-theory technique even for the case of 5-quantifier formulas as the following example shows.

EXAMPLE 1. If A is a model for set theory which satisfies the axiom of choice and B is a model which does not, then player 1 has a winning strategy in a 5-game played with A and B.

Proof of Example 1. Player 1's first move is to choose a set b_1 from B whose members are disjoint and for which there is no choice set in B. Then player 2 must choose a set a_1 which is non-empty and whose elements are disjoint, or else he will lose before his 5th move. Because if he chooses the empty set he will lose on the next move and if 2 different elements a_2 and a_3 of a_1 are not disjoint, then player 1 can choose these elements on his next two moves, while player 2 will have to counter by choosing two different elements of b_1. And then when player 1 chooses an element from $a_2 \cap a_3$, player 2 will lose on his next move.

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(1) $(\forall x_1)\big((\forall x_2)(\forall x_3)(x_2 \in x_1 \wedge x_3 \in x_1 \to (\forall x_4)(x_4 \notin x_2 \vee x_4 \notin x_3))$
$\to (\exists x_5)(\forall x_6)\big(x_6 \in x_1 \to (\exists x_7)\big(x_7 \in x_6 \wedge x_7 \in x_5 \wedge (\forall x_8)(x_8 \in x_6 \wedge x_8 \in x_5 \to x_8 = x_7)\big)\big)\big).$

For his second move, player 1 chooses a_2 in A, the image of the choice function of a_1. Player 2 must choose a set b_2 from B such that either (1) b_2 contains more than 1 element from some element b of b_1 or (2) b_2 fails to contain any elements from some element b' of b_1. This is because there is no choice set for b_1.

If (1) is true, player 1 can choose b on his 3rd move and then choose 2 different elements from $b_2 \cap b$ on his 4th and 5th moves, which will make him win. If (2) is true, player 1 can choose b' on this 3rd move, forcing player 2 to choose an element a_3 of a_1. Then player 1 can win on his 4th move by choosing an element of $a_3 \cap a_2$. Q.E.D.

We conjecture that player 2 does have a winning strategy in a 4-game but will not attempt to prove this as it seems too difficult. But to get an idea of the variety of 3-quantifier formulas in set theory, consider that with variables x, y, z and relation symbols "$\in$" and "$=$" we could construct 18 atomic formulas, 2^{18} different valuations for the 18 formulas and $(2)^{2^{18}}$ different truth tables. So there are $(2)^{2^{18}}$ nonequivalent quantifier-free compound statements containing 3 variables.

We note that the axiom of extensionality has 3 quantifiers, the axiom of the empty set has 2, and the axioms of infinity, regularity and the power set have 4.

Proof of Theorem 3. We note that if player 1 has a winning strategy in a 3-game, then he has a winning strategy in which he does not pick elements on the 2nd or 3rd move which have already been picked by either player. Because if player 1 picks an element that was already picked, then player 2 can pick the other element that was picked on the same move, so that player 1's move is wasted. If, for example, there is some formula $F(x_1, x_2, x_3)$ such that $A \vdash F(a_1, a_2, a_3)$ and and $B \vdash \sim F(b_1, b_2, b_3)$, then there is a formula $F'(x_1, x_2)$ such that $A \vdash F'(a_1, a_3)$ and $B \vdash \sim F'(b_1, b_3)$, and of course $F'(x_1, x_2)$ is usable in a 3-game.

So we need only to show that if player 1 is required to pick new elements each time, player 2 has a winning strategy. As a preliminary, we chart the 9 ways that, if a and b are unequal elements of a model for set theory, a third unequal element c may be related to a and b (see Table 1).

Table 1

	$c \in b$?	$b \in c$?	$c \in a$?	$a \in c$?
1	No	No	No	No
2	No	No	No	Yes
3	No	No	Yes	No
4	No	Yes	No	No
5	No	Yes	No	Yes
6	No	Yes	Yes	No
7	Yes	No	No	No
8	Yes	No	No	Yes
9	Yes	No	Yes	No

DEFINITION. $P_M(a, b)$ is the subset of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ which contains the numbers of the lines in Table 1 which can be satisfied by a and b and some element c in the model M.

DEFINITION. $P_M(b)$ is the ordered triple $\langle P_{M,1}(b), P_{M,2}(b), P_{M,3}(b)\rangle$ such that $P_{M,1}(b)$ is the set containing each set $P_M(a, b)$ such that $a \in b$, $P_{M,2}(b)$ contains each set $P_M(a, b)$ such that $b \in a$, and $P_{M,3}(b)$ contains each element $P_M(a, b)$ such that $a \notin b$, $b \notin a$ and $a \neq b$.

Now suppose that a 3-game is to be played with a model r for Zermelo–Fraenkel set theory minus the axiom of choice, but with the axiom of regularity and a model s for Zermelo–Fraenkel set theory minus the axiom of choice and the axiom of regularity. We will show that no matter what element r_1 is chosen in r by player 1, player 2 can choose s_1 in s such that $P_r(r_1) = P_s(s_1)$. This will prove Theorem 3, since player 2 can then win a 3-game no matter what player 1 does on his 2nd and 3rd moves. It is tedious but involves no difficulty to verify that if r_1 is $\{\ \}$ or $\{\{\ \}\}$ or $\{a\}$ where $a \neq \{\ \}$, then if player 2 chooses s_1 to be $\{\ \}$ or $\{\{\ \}\}$ or $\{a'\}$, where $a' \neq \{\ \}$, respectively, then $P_r(r_1) = P_s(s_1)$. So we need only consider the case in which r_1 contains at least 2 elements. For such a case we present a method of constructing s_1 such that s_1 will have at least 2 elements, which insures that $P_{r,2}(r_1) = P_{s,2}(s_1)$. We leave this for the reader to verify. We note that $P_{r,1}(r_1)$ contains only members of the form $\{1, 2, 4, 5\} \cup S$, where S is some subset of $\{3, 7, 8, 9\}$, so for each of the 16 possible members of $P_{r,1}(r_1)$, the reader must verify that our method of construction insures that it is a member of $P_{r,1}(r_1)$ if and only if it is a member of $P_{s,1}(s_1)$. We also note that if $r_2 \in e \in r_1$ and $r_1 \neq r_2$, then $P_r(r_2, r_1)$ is of the form $\{1, 2, 4, 5, 8\} \cup S$ where S is some subset of $\{3, 7, 9\}$. But if s_1 has 2 or more members and player 1 picks r_2 or s_2 on his 2nd move so that $r_2 \notin r_1$ and $r_1 \notin r_2$ and $r_1 \neq r_2$ and there is no e such that $r_2 \in e \in r_1$, or $s_1 \notin s_2$ and $s_2 \notin s_1$ and $s_1 \neq s_2$ and there is no e such that $s_2 \in e \in s_1$, then player 2 can pick s_2 or r_2 from the other model such that $P_r(r_1, r_2) = P_s(s_1, s_2)$. So we are only concerned with those elements of $P_{r,3}(r_1)$ of the form $\{1, 2, 4, 5, 8\} \cup S$ where S is some subset of $\{3, 7, 9\}$, and the reader can verify that each of 8 possible members of $P_{r,3}(r_1)$ of this type is in $P_{r,3}(r_1)$ if and only if it is in $P_{s,3}(s_1)$.

To aid the reader, we mention that each different element of $P_{r,1}(r_1)$ corresponds to a different set of answers to the four questions about some member a of r_1:

1. Does $a \cap r_1 = \{\ \}$?
2. Does $a - r_1 = \{\ \}$?
3. Is a a member of a member of r_1?
4. Is there some b in r_1 such that $b \neq a$, $b \notin a$, and $a \notin b$?

By thoroughly mastering the method of construction of s_1 the reader can see that any set of answers to questions 1, 2, 3 and 4 corresponds to some member a of r_1 if and only if there is some member a' in s_1 with the corresponding properties (i.e. 1-4 with a replaced by a' and r_1 by s_1).

Similarly, each element of $P_{r,3}(r_1)$ with which we are concerned corresponds to a different set of answers to the three questions about some member a of a member of r_1 such that a is not a member of r_1:

1. Does $a \cap r_1 = \{\ \}$?
2. Does $a - r_1 = \{\ \}$?
3. Is there some b in r_1 such that $b \neq a$, $b \notin a$ and $a \notin b$?

The remark made about 1-4 with respect to members a' of s_1 is true for 1-3 as well, with respect to members a' of members of s_1 such that a' is not a member of s_1.

We will speak of an "s_1-list" that player 2 begins making after player 1 chooses r.

The idea behind the construction of the s_1-list is to consider each possible combination of answers that could be given to 1-4 by a member a of r_1 and each combination of answers that could be given to 1′-3′ by a member a of a member of r_1 such that a is not a member of r_1, and for each combination of answers to add an element a' of s or an element with a member a', respectively, to the s_1-list and make such notations as insure that s_1 will have a member a', or a member a' of a member, such that a' is not a member, such that a' and s_1 give the corresponding combination of answers to 1-4 or 1′-3′. Also, this must be done in a such a way that s_1 does not have any members which give combinations of answers to 1-4 that do not occur in r_1, and that s_1 does not have any non-members which are members of members which give combinations of answers to 1′-3′ that do not occur in r_1.

The s_1-list may be infinite but is constructed in such a way that a finite formula could be constructed to correspond to the open sentence "x is a member of the s_1-list" and such that there is an element s_1 in s which contains precisely those elements in s which satisfy the formula, and such that $P_s(s_1) = P_r(r_1)$.

The method by which the s_1-list is constructed is described inductively by using the fact that all elements in r have ranks, and the fact that $P_r(r_1) = P_s(s_1)$ is proved inductively.

The first stage in constructing the s_1-list is as follows. If the element of smallest rank in r_1 has no members, then we put the element in s which has no members on the s_1-list. If the element or elements of smallest rank in r_1 do have members, then the set of all such elements can be divided into "types" according to the list of answers that each element gives to questions 1-4 and the set of lists of answers that its members give to questions 1′-3′.

DEFINITION. An element of r or s is said to have *finite rank* if there are only finitely many elements in r or s, respectively, with smaller rank, and infinite rank otherwise. Note that an element of r or s could be an integer with respect to the model r or s, respectively, but be of infinite rank.

LEMMA 1. *The first stage of the s_1-list, consisting of elements of s together with the following designations for certain members and members of members of the list*:

(a) *sometimes contained in members of the s_1-list*,

(b) *never contained in members of the s_1-list*,

(c) *sometimes not contained in members of the s_1-list to be added in later stages*,

(d) *always contained in members of the s_1-list to be added in later stages*,

can be constructed such that if all members added to the s_1-list in later stages have greater rank than those added in the first stage and all designations for elements and members of elements added in the first stage are fulfilled, then each type of element which appears α times with the smallest rank β of elements in r, appears the minimum of α and 10 times in the first stage of the s_1-list and has the finite rank β if β is finite and has some infinite rank if β is infinite.

We omit the proof because it is just a straight-forward verification for all the possible types that might appear with smallest rank in r_1.

In order to complete an induction started by Lemma 1, we need the following:

LEMMA 2. *If after N stages, the set S_N of elements on the s_1-list together with designations of the type in Lemma 1 for certain members and members of members of the list, has the properties that*:

(a) *there is an element in s such that precisely the elements which are in S_N are members of it*,

(b) *for each of the 4 designations mentioned in Lemma 1, there is an element in s such that precisely the elements and members of elements which are on the s_1-list and have that designation are members of it*,

(c) *there is an element $\mathscr{D}_N$ which is an ordinal in r such that if all the designations of elements and members of elements thus far on the s_1-list are fulfilled and only elements of greater rank are later added to the s_1-list, then if α is the number or cardinality of the set of elements of any type which have rank less than $\mathscr{D}_N$ in r_1, then at least* $\min\{\alpha, 10\}$ *elements of that type will be in S_N*,

(d) *there is a set A_N of members and members of members of r_1 and a 1 to 1 mapping from A_N onto the set of those elements and members of elements thus far on the s_1-list which have designations of the type mentioned in* (a) *or* (c) *of Lemma 1 such that each member m of A_N is contained by a member of r_1 with rank equal or greater than $\mathscr{D}_N$ if its image has designation* (a), *and there is a member of r_1 with rank equal or greater than $\mathscr{D}_N$ which does not contain m if its image has designation* (c),

(e) *if the set of ranks of elements of r_1 which are elements of elements of r_1 is co-final with $\mathscr{D}_N$, then there is a set of elements on the s_1-list which are designated "sometimes contained in members of the s_1-list" such that the set of the ranks of the elements in the set is co-final with the set of ranks of all elements thus far on the s_1-list*,

(f) *if $\mathscr{D}_N$ is finite, then the least ordinal with greater rank than the set of elements thus far on the s_1-list is the same number, and if $\mathscr{D}_N$ is infinite, it is infinite*,

then, in the $(N+1)$-st *stage, a set of elements and designations can be added to the s_1-list such that there is an ordinal $\mathscr{D}_{N+1}$ in r such that $\mathscr{D}_{N+1} > \mathscr{D}_N$ and properties* (a)-(f) *still hold with respect to $N+1$ instead of N, and with respect to the enlarged s_1-list, and either* (1) *at least one element of some type that appeared fewer than* 10 *times* *with rank less than $\mathscr{D}_N$ appears with rank equal or greater than $\mathscr{D}_N$ but less than $\mathscr{D}_{N+1}$, or (2) $\mathscr{D}_{N+1}$ is the smallest ordinal in r which is greater than the rank of any element of r_1.*

(Note. We need the specification in (c) that the s_1-list has at least $\min\{\alpha, 10\}$ elements of each type, rather than at least $\min\{\alpha, 1\}$, to insure that it possesses enough variety to create all the type of elements out of it that we may have to match what appears in r_1 with some rank β such that $\mathscr{D}_N \leqslant \beta < \mathscr{D}_{N+1}$. And we need the specification (e) to insure that there are subsets of the set of those elements of the s_1-list which appear in the first N stages which have greater rank than any element in the first N stages and which can be added to the s_1-list in accordance with prior designations.)

Proof. Let α be the smallest ordinal in r such that $\alpha > \mathscr{D}_N$ and there is at least one element of a type which appears fewer than 10 times in r_1 with rank less than $\mathscr{D}_N$ and which appears with rank α, if there is such an ordinal. Then $\mathscr{D}_{N+1} = \alpha+1$. If there is no such ordinal, then let $\mathscr{D}_{N+1}$ be the first ordinal in r such that all elements of r_1 have ranks less than $\mathscr{D}_{N+1}$. In either case, we can add a set of elements and designations to the s_1-list which may be finite or denumerable but which is definable by a formula with finitely many constants in s and therefore is such that properties (a) and (b) still hold with respect to $N+1$ instead of N, and with respect to the enlarged s_1-list, as well as properties (c)-(f). This can be verified by considering all the possible cases, but is quite clear if considered carefully. So we omit what would be a very long verification. Q.E.D.

Reference

[1] A. Ehrenfeucht, *An application of games to the completeness problem for formalized theories*, Fund. Math. 49 (1961), pp. 129–141.

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Les ensembles de niveau et la monotonie d'une fonction

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Résumé. Soient X un espace topologique et R l'espace des nombres réels. Étant donnée une fonction $f\colon X\to R$, désignons par Y_c l'ensemble $\{y \in R\colon f^{-1}(y)$ est connexe$\}$, par S_c l'ensemble $f^{-1}(Y_c)$ et par $\bar{S}_c$ la fermeture de l'ensemble S_c. Dans cette note je démontre:

Théorème 1. *Supposons que X soit un espace topologique de Hausdorff connexe et localement connexe. Si une fonction $f\colon X\to R$ est connexe et relativement propre, la fonction partielle $f/\bar{S}_c$ est faiblement monotone*; et

Théorème 2. *Il existe un espace métrique, séparable, connexe et localement connexe et une fonction $f\colon X\to R$ qui est continue, n'est monotone dans aucun ensemble ouvert et non vide de l'espace X et telle que l'ensemble $\{x \in X\colon x$ est un point limite de l'ensemble $f^{-1}(f(x))$ le long de tout arc simple dans X d'extrémité $x\}$ n'est pas résiduel dans l'espace X.*

Le Théorème 1 donne une réponse partielle au Problème 3.11 de [2] et le Théorème 2 donne la réponse au Problème 5.10 de [2].

Soient X un espace topologique et R l'espace des nombres réels. Étant donnée une fonction $f\colon X\to R$, désignons par Y_c l'ensemble $\{y\in R\colon f^{-1}(y)$ est connexe$\}$ et par S_c l'ensemble $f^{-1}(Y_c)$.

Définition. Une fonction $f\colon X\to R$ est dite

(a) *connexe* lorsque $f(A)$ est un ensemble connexe pour tout ensemble connexe $A\subset X$,

(b) *monotone* lorsque $f^{-1}(A)$ est connexe pour tout ensemble connexe $A\subset R$,

(c) *faiblement monotone* lorsque $f^{-1}(y)$ est un ensemble connexe pour tout point $y\in R$,

(d) *relativement propre* lorsque $f^{-1}(A)$ est un ensemble relativement compact pour tout ensemble compact $A\subset R$.

Remarque 1 ([2], Prop. 3.7). Soit $f\colon X\to R$ une fonction connexe définie sur un espace topologique de Hausdorff qui est connexe et localement connexe. Pour que la fonction f soit monotone, il faut et il suffit qu'elle soit faiblement monotone.

Dans le travail [2] Garg a posé les deux questions suivantes:

Problème 1 ([2], Probl. 3.11). Une fonction connexe $f\colon R\to R$ est monotone au sens ordinaire par rapport à la fermeture $\bar{S}_c$ de l'ensemble S_c ([1], Th. 2). Une fonction connexe $f\colon X\to R$ est-elle monotone ou faiblement monotone par rappor