Stronger topologies preserving the class of continuous functions

by

Helena Nonas (Łódź)

Abstract. The work includes the detailed discussion of the conditions concerning two topological spaces (X, T), (X, T_0), $T_0 \subset T$, under which the following equivalence holds: the class of real functions continuous with respect to T is equal to the class of real functions continuous with respect to T_0 if and only if every real function continuous with respect to T is a bounded function.

The sufficient condition for this equivalence is pseudo-compactness of (X, T_0) and the fact that every one-point set is a G_δ-set in (X, T_0).

The necessary condition is an impossibility of representation of (X, T_0) in the form of the sum of two disjoint non-empty sets X_1, X_2 such that $X_1 \notin T_0$ and subspaces $(X_1, T_0|X_1)$ and $(X_2, T_0|X_2)$ are *-compact (*-compactness means that every continuous real function is bounded).

For a Tichonov countably compact space (X, T_0) with the topology induced by the order the necessary and sufficient condition of the above equivalence is the condition that every one-point set is a G_δ in (X, T_0).

As known, if (X, T_0) and (X, T) are two topological spaces and topology T is stronger than T_0, then every continuous function on (X, T_0) is continuous also on (X, T). In particular, if $C(X, T)$ denotes the set of real-valued functions continuous on (X, T), then $C(X, T_0) \subset C(X, T)$. In some cases, even if the topology T is essentially stronger than T_0, it happens that $C(X, T_0) = C(X, T)$.

E. Kocela in [2] showed that if (X, T_0) is a compact metrizable space and $T \supset T_0$, the conditions

(i) $$C(X, T_0) = C(X, T),$$

(ii) $$C(X, T) \subset B(X),$$

where $B(X)$ stands for the set of functions bounded on X, are equivalent.

The aim of this paper is to analyse conditions concerning the space (X, T_0) at which

(1) $$\text{(i)} \Leftrightarrow \text{(ii)}.$$

Condition (ii) concerning the set $C(X, T)$ is equivalent to that of $*$-compactness of the space (X, T) introduced in [2]. One can prove without difficulty that a metric space is compact if and only if it is $*$-compact.

THEOREM 1. *Let a topological space* (X, T_0) *be given such that* $X = X_1 \cup X_2$, *where* X_1, X_2 *are disjoint, nonempty,* $*$*-compact spaces endowed with topologies induced by* T_0 *and* $X_1 \notin T_0$. *Let* T *be a topology generated by* $T_0 \cup \{X_1\} \cup \{X_2\}$. *Then* (X, T) *is a* $*$*-compact space and*

$$C(X, T) \neq C(X, T_0).$$

Proof. By the definition of the topology T we have

$$T = \{E = (A \cap X_1) \cup (B \cap X_2):\ A, B \in T_0\}.$$

Observe that in spite of the essential extension of the initial topology T_0, topologies induced by T_0 and T in subsets X_1 and X_2 are identical.

To show that (X, T) is a $*$-compact space, we shall consider an arbitrary real function f which is T-continuous, i.e. $f \in C(X, T)$. The reduced functions $f|X_1$ and $f|X_2$ are continuous with respect to suitable topologies induced by T_0. By Theorem 4 in [2], they are bounded. Consequently, f is bounded. Thus we have shown that (X, T) is a $*$-compact space.

To show that $C(X, T) \neq C(X, T_0)$, it is enough to find a T-continuous function which is not T_0-continuous. For example, the function

$$g(x) = \begin{cases} 1 & \text{for } x \in X_1, \\ 0 & \text{for } x \notin X_1, \end{cases}$$

satisfies the condition. This completes the proof.

COROLLARY 1. *Let* (X, T_0) *be a topological countably compact space* (cf. [1]). *Suppose that there exists a point* $x_0 \in X$ *which is an accumulation point of the set* X *but is not an accumulation point of any of the countable subsets of* X. *Let* T *be a topology generated by* $T_0 \cup \{x_0\}$.

Then (X, T) *is a* $*$*-compact space and*

$$C(X, T) \neq C(X, T_0).$$

Proof. To show that all the assumptions of Theorem 1 are satisfied, we put $X_1 = \{x_0\}$ and $X_2 = X \setminus \{x_0\}$. Obviously, X_1 is a $*$-compact space. The space X_2 endowed with topology induced by T_0 is countably compact since it is formed from the set X, by deleting the point x_0 which is not an accumulation point of any sequence. Thus X_2 is a $*$-compact space, too. $X_1 = \{x_0\} \notin T_0$ because x_0 is a T_0-accumulation point of the set X.

EXAMPLE 1. Let X be the set of all ordinal numbers not greater than Ω. Let T_0 denote a topology whose base is composed of the following family of sets:

$$B = \{(y, x]\} \cup \{0\},$$

where $(y, x] = \{z:\ y < z \leqslant x\}$, $y < x \leqslant \Omega$, and 0 stands for the ordinal type of an empty set.

Then (X, T) is a compact space (cf. [2]). Ω has properties of the point x_0 from Corollary 1. Therefore (X, T) with topology generated by $T_0 \cup \{\Omega\}$ is a $*$-compact space and $C(X, T) \neq C(X, T_0)$.

COROLLARY 2. *Let* (X, T_0) *be a countably compact space. Let* X_0 *denote an open but not closed set is* (X, T_0) *such that every accumulation point of the set* X_0 *which does not belong to* X_0 *is not an accumulation point of any sequence from* X_0. *Let* T *be a topology generated by* $T_0 \cup \{X \setminus X_0\}$. *Then* (X, T) *is a* $*$*-compact space and*

$$C(X, T) \neq C(X, T_0).$$

Proof. To show that the assumptions of Theorem 1 are satisfied, we put $X_2 = X_0$ and $X_1 = X \setminus X_0$. X_1 endowed with topology induced by T_0 is a countably compact space since X_1 is a closed subset of the set X. Thus it is also $*$-compact.

By the definition of the set X_0 it is clear that the space X_0 with topology induced by T_0 is countably compact and thus $*$-compact.

EXAMPLE 2. Let $A = \{\alpha:\ 0 \leqslant \alpha \leqslant \Omega\}$ be the set of ordinal numbers not greater than Ω and let $B = \{x:\ 0 < x \leqslant 1\}$ be the set of real numbers from the interval $(0, 1]$.

Let us put

$$X = A \cup B.$$

Let T_0 denote a topology whose base is composed of the following family of sets:

$\{0\}$, where 0 denotes the ordinal type of an empty set,

$\{\alpha:\ \beta_1 < \alpha \leqslant \beta_2\}$ for $\beta_1, \beta_2 < \Omega$;

$\{\alpha:\ \beta < \alpha \leqslant \Omega\} \cup \{x:\ 0 < x < b\}$ for $\beta < \Omega$, $b \in (0, 1]$,

$\{x:\ a < x < b\}$ for $a, b \in (0, 1]$,

$\{x:\ a < x \leqslant 1\}$ for $a \in (0, 1)$.

It is easily seen that (X, T_0) is a compact space. Ω is the only accumulation point of the open set $\{\alpha:\ 0 \leqslant \alpha < \Omega\}$ which does not belong to this set and such that no sequence of ordinal numbers less than Ω tends to Ω. Thus, by Corollary 2, (X, T) with topology T generated by $T_0 \cup (X \setminus \{\alpha:\ 0 \leqslant \alpha < \Omega\})$ is a $*$-compact space and $C(X, T) \neq C(X, T_0)$.

The result of E. Kocela, given at the beginning of this paper, means that both compactness and metrizability of the space (X, T_0) constitute for $T \supset T_0$ the sufficient condition for the equivalence

$$(1) \qquad (C(X, T) = C(X, T_0)) \Leftrightarrow ((X, T) \text{ is a } *\text{-compact.})$$

Examples 1 and 2 show that compactness alone of the space (X, T_0) is not sufficient. The condition of E. Kocela may, however, be strengthened if one substitutes compactness and metrizability by certain weaker assumptions.

LEMMA 1. *Let* (X, T_0) *be a Tichonov space* (cf. [2]) *containing at least two points and* $x_0 \in X$.

The following two conditions are equivalent:

(a) *one-element set* $\{x_0\}$ *is of the type* G_δ,

(b) *there exists a T_0-continuous function φ such that*

$$0<\varphi(x)<\varphi(x_0) = \tfrac{1}{2}\pi \quad \textit{for} \quad x \neq x_0 .$$

Proof. It is obvious that if a function φ defined by (b) exists, then $\{x_0\}$ is a set of the type G_δ.

We shall show that converse is also true. Assume that $\{x_0\}$ is a set of the type G_δ. Consequently, there is a sequence of sets $\{G_n\}$ such that

$$G_n \in T_0, \quad G_n \supset G_{n+1} \quad \text{for} \quad n = 1, 2, \ldots$$

and

$$\bigcap_{n=1}^{\infty} G_n = \{x_0\} .$$

Then

$$X \setminus \{x_0\} = \bigcup_{n=1}^{\infty} F_n ,$$

where $X \setminus F_n \in T_0$, $F_n \subset F_{n+1}$ for $n = 1, 2, \ldots$ Without loss of generality we may assume $F_1 \neq \emptyset$.

By the assumption (X, T_0) is a Tichonov space. Thus for every natural n there is a T_0-continuous function f_n such, that

$$f_n(x_0) = \frac{\pi}{2} \cdot \frac{1}{2^n}$$

and

$$f_n(x) = 0 \text{ for } x \in F_n \quad \text{and} \quad 0 \leqslant f_n(x) \leqslant \frac{\pi}{2} \cdot \frac{1}{2^n} \text{ for every } x \in X.$$

Furthermore, put

$$\varphi(x) = \sum_{n=1}^{\infty} f_n(x) .$$

The function φ being a limit of a uniformly convergent sequence of T_0-continuous functions is T_0-continuous and

$$\varphi(x_0) = \sum_{n=1}^{\infty} f_n(x_0) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\pi}{2} .$$

Let now x be an arbitrary element of the set X different from x_0. Then there exists a set F_k such that $x \in F_k$, and consequently $f_k(x) = 0$. Hence

$$\varphi(x) = \sum_{n=1}^{\infty} f_n(x) < \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\pi}{2} = \varphi(x_0) .$$

Thus the function φ defined above has all properties listed in the condition (b). This completes the proof of the lemma.

THEOREM 2. *Let (X, T_0) be a pseudocompact space* (cf. [2]), *in which every one-element set is of the type G_δ and $T \supset T_0$. Then the equivalence* (1) *holds.*

Proof. First let $C(X, T_0) = C(X, T)$. To show that (X, T) is a $*$-compact space we consider an arbitrary T-continuous function f. By the assumption, every T-continuous function is T_0-continuous and thus bounded, since the space (X, T_0) is pseudocompact. Hence f is bounded and, by Theorem 4 in [1], (X, T) is $*$-compact.

Let now (X, T) be a $*$-compact space. Assume that $C(X, T) \neq C(X, T_0)$. Therefore, there exists a T-continuous function f which is not T_0-continuous. Let x_0 be a T_0-discontinuity point of the function f. Then there exists a number $\varepsilon_0 > 0$ such, that in every T_0-neighbourhood of the point x_0 there are points for which $|f(x) - f(x_0)| \geqslant \varepsilon_0$. Let us put

$$f_1(x) = \frac{f(x) - f(x_0)}{\varepsilon_0} \cdot \frac{\pi}{2} .$$

Then f_1 is a T-continuous function.

Let us put further

$$f_2(x) = \min\big(f_1(x), \varphi(x)\big)$$

and

$$f_3(x) = \max\big(f_2(x), -\varphi(x)\big) ,$$

where φ is the function constructed in Lemma, i.e., T_0-continuous and such that at the point x_0 it attains the maximal value $\frac{1}{2}\pi$ and at all other points distinct from x_0 it takes positive values less than $\frac{1}{2}\pi$. Obviously f_2 and f_3 are T-continuous functions and for each x

$$|f_3(x)| < \tfrac{1}{2}\pi .$$

It is easy to verify that

$$\sup_{x \in X} |f_3(x)| = \tfrac{1}{2}\pi .$$

Finally, let us put

$$f_4(x) = \operatorname{tg}\big(f_3(x)\big) .$$

Then f_4 is a T-continuous, unbounded function what contradicts the assumption of $*$-compactness of the space (X, T). This contradiction results from the supposition that $C(X, T) \neq C(X, T_0)$. Thus at those two topologies the classes of continuous functions coincide. This completes the proof.

Remark. Theorem 2 is a substantial generalization of E. Kocela's theorem because there exists an example of a compact unmetrizable space, in which every closed set is of the type G_δ. Such example was given by Urysohn in [3] (pp. 936–939).

The example presented below shows that even with the assumption of compactness of the space (X, T_0), the equivalence (1) does not imply that every one-element set of the space (X, T_0) is of the type G_δ.

EXAMPLE 3. Let X be an uncountable set and $x_0 \in X$. Define a topology T_0 as follows:

$$(E \in T_0) \Leftrightarrow \big((x_0 \notin E) \vee (x_0 \in E \wedge \overline{\overline{X \setminus E}} < \aleph_0)\big).$$

It can be easily verified that (X, T_0) is a Hausdorff space and that from any covering of this space with open sets, a finite covering can be chosen. Therefore the space (X, T_0) is compact. The one-element set $\{x_0\}$ is not of the type G_δ because if it were X would have to be at most countable. Let now T be an arbitrary essential extension of the topology T_0. We shall show that (X, T) is not $*$-compact. Since $T \supset T_0$, there exists a set $G \in T \setminus T_0$. Therefore $x_0 \in G$ and $\overline{\overline{X \setminus G}} \geqslant \aleph_0$.

Denote by $F = \{x_n\}$ an arbitrary countable subset of the set $X \setminus G$. F is T-closed. Let us put

$$f(x) = \begin{cases} 0 & \text{for } x \in X \setminus F, \\ n & \text{for } x = x_n,\ n = 1, 2, \ldots \end{cases}$$

The function f is T-continuous and unbounded. Thus indeed, (X, T) is not $*$-compact. Therefore, the only topology $T \supset T_0$, at which (X, T) is $*$-compact is the topology T_0, and hence

$$\big((X, T) \text{ is a } *\text{-compact space}\big) \Leftrightarrow (T = T_0) \Leftrightarrow \big(C(X, T) = C(X, T_0)\big).$$

From the above discussion it is seen that in the general case the condition stating that in a compact space every one-element set is of type G_δ is not necessary and sufficient for the equivalence (1). However, there exist spaces in which Theorem 2 can be conversed.

LEMMA 2. *Let X be a set endowed with the topology determined by order* (cf. [2]). *A one-element set $\{x_0\}$ is of the type G_δ if and only if the following conditions are simultaneously satisfied*:

(A) *If every interval of the form (c, x_0) is non-empty, then x_0 is the accumulation point of the sequence $\{a_n\}$ of the elements preceding x_0.*

(B) *If every interval of the form (x_0, d) is non-empty, then x_0 is the accumulation point of the sequence $\{b_n\}$ of the elements following x_0.*

The easy proof of this lemma is omitted.

THEOREM 3. *Let (X, T_0) be a countably compact space with the topology determined by order and let $x_0 \in X$ be such that $\{x_0\} \notin G_\delta$. Then there exists an extension T of the topology T_0 such that (X, T) is $*$-compact and*

$$C(X, T) \neq C(X, T_0).$$

Proof. From the assumption, $\{x_0\}$ is not of the type G_δ. Thus, by Lemma 2, x_0 is, at least from one, say left-hand, side, an accumulation point of the set X and it is not an accumulation point of any sequence of elements preceding x_0. Let us put $X_0 = \{x: x < x_0\}$. The assumptions of Corollary 2 are satisfied and therefore the

space (X, T) with topology generated by $T_0 \cup (X \setminus X_0)$ is $*$-compact and $C(X, T) \neq C(X, T_0)$. This completes the proof.

The following corollary is the immediate consequence of Theorems 2 and 3.

COROLLARY 3. *Let (X, T_0) be a countably compact, Tichonov space with topology determined by order. Then for the equivalence* (1) *to hold it is necessary and sufficient that every one-element set be of the type G_δ in the topology T_0.*

References

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[3] P. Urysohn, Труды по топологии и другим областям математики Москва–Ленинград 1951, II.

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