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Hilbert cube modulo an arc

by

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Abstract. Let Q denote the Hilbert cube and let $\alpha, \beta \subset Q$ be arcs. Adapting methods of Bing–Andrews–Curtis–Kwun–Bryant we prove that $Q/\alpha \times I$ and $Q/\alpha \times Q/\beta$ are homeomorphic with Q, where I is a closed interval and Q/α is a space obtained from Q by shrinking α to a point. The same method applies equally well to the case when arcs are replaced with finite-dimensional cells or their intersections.

1. Introduction. We use Q to represent the Hilbert cube (the countable-infinite product of closed intervals). A closed subset $X \subset Q$ is called a *Z-set* if for any non-empty homotopically trivial open set $U \subset Q$, $U - X$ is also non-empty and homotopically trivial. This concept was introduced by R. D. Anderson in [1] and in the infinite-dimensional topology plays a role analogous to a role of tameness conditions in the finite-dimensional topology. Chapman [7] showed that a Z-set $X \subset Q$ has a trivial shape if and only if the space Q/X, obtained from Q by shrinking X to a point, is homeomorphic to Q (in notation, $Q/X \cong Q$). If X is of a trivial shape but not a Z-set, then Q/X may fail to be locally like Q at the point $\tilde{X} = p(X)$, where $p\colon Q \to Q/X$ is a natural projection. Indeed, Wong [14] constructed a copy of the Cantor set with non-simply connected complement in Q. By a standard technique we can pass an arc α through it such that $Q - \alpha$ is also not simply connected. If Q/α were locally Q at the point $\tilde{\alpha}$, then Q/α being a contractible Q-manifold would be homeomorphic to Q [8]. But in Q the complement of every point is simply connected.

The problem SC 1 in [2] asks (in analogy with a similar result for Euclidean spaces established earlier by Andrews and Curtis [3]) whether for any arc $\alpha \subset Q$ multiplying Q/α by the unit interval $I = [0, 1]$ gives the Hilbert cube. In Section 2 of this note we will present a detailed proof, adapting techniques from [3] to the Hilbert cube case, of the following theorem that confirmes this conjecture.

THEOREM 1. *For any arc $\alpha \subset Q$, $(Q/\alpha) \times I$ is homeomorphic with Q.*

Next, in Section 3, we first prove that $A \times B$ is a Z-set in $Q \times Q$ whenever A and B are finite-dimensional closed subsets of Q and then, following Kwun's method [10], establish

THEOREM 2. *Let $\alpha, \beta \subset Q$ be arbitrary arcs. Then $(Q/\alpha) \times (Q/\beta)$ is homeomorphic with Q.*

Finally, the procedures in the proofs of both theorems can be easily extended, as was suggested in [5], to the case of shrinking a finite-dimensional cell in Q and, even further, intersections of such cells in Q (see Corollary 2.11 for precise statement).

Remark. Both Bryant and Chapman claimed proofs of Theorem 1 essentially along the lines presented here.

2. Proof of Theorem 1. Since $\alpha\times 0$ and $\alpha\times 1$ are Z-sets of trivial shape in $Q\times I$, there is a map $F\colon Q\times I\to Q\times I$ of $Q\times I$ onto itself such that $F(\alpha\times 0)$ and $F(\alpha\times 1)$ are distinct points of $Q\times I$ while $F|\ (Q\times I)-(\alpha\times 0\cup\alpha\times 1)$ is a homeomorphism onto $Q\times I-F(\alpha\times 0\cup\alpha\times 1)$. Pick, inductively, a sequence $\varepsilon_1, \varepsilon_2, \ldots$ of positive real numbers satisfying:

(γ) $3\varepsilon_i<\frac{1}{2}(\varepsilon_{i-1})<1$, and

($\gamma\gamma$) if $x\in N_{\varepsilon_i}(\alpha)\times I$ has I-coordinate less than $3\varepsilon_i$ or bigger than $1-3\varepsilon_i$ then $F(x)$ is within ε_{i-1} of either $p=F(\alpha\times 0)$ or $q=F(\alpha\times 1)$, respectively, for each $i>0$.

Here, for any $\varepsilon>0$, $N_\varepsilon(\alpha)$ denotes a closed ε-neighborhood of α in Q relative a fixed metric d on Q and $Q\times I$ is given the product metric.

LEMMA 2.1. *Let* $\alpha=\bigcap_{i>0} T_i$, *where each* $T_i\subset N_{\varepsilon_i}(\alpha)$ *is a closed neighborhood of* α *in* Q *and* $T_{i+1}\subset T_i$. *Suppose that for each positive integer* i *and numbers* $\eta>0$ *and* $0<\varepsilon<1$ *there is an integer* N *and an isotopy* μ_t $(0\leqslant t\leqslant 1)$ *of* $Q\times I$ *onto itself such that*

(a) $\mu_0=\mathrm{id}$ *(identity)*,

(b) $\mu_t|\ Q\times I-T_i\times[\frac{1}{2}\varepsilon, 1-\frac{1}{2}\varepsilon]=\mathrm{id}$,

(c) μ_t *changes* I-*coordinate less than* η, *and*

(d) $\operatorname{diam}\mu_1(T_N\times w)<\eta$ *for all* $w\in[\varepsilon, 1-\varepsilon]$.

Then $(Q/\alpha)\times I\cong Q$.

Proof. The proof of this lemma is very similar to the proof of Theorem 1 in [3] and whenever details are omitted they can be found in [3] or [12].

We will prove that the quotient $(Q\times I)/G$, where G is the upper semicontinuous decomposition of $Q\times I$ with only nondegenerate elements sets $F(\alpha\times t)$, $0<t<1$, is homeomorphic to $Q\times I$ by constructing a pseudo-isotopy $f_t\colon Q\times I\to Q\times I$ such that $f_0=\mathrm{id}$ and f_1 takes each element of G into a distinct point of $Q\times I$. The pseudo-isotopy f_t will be, for $0\leqslant t<1$, the obvious extension of $F\circ h_t\circ F^{-1}|\ Q\times I-\{p, q\}$, where $h_t\colon Q\times I\to Q\times I$ keeps $\alpha\times 0$ and $\alpha\times 1$ fixed at any time. Even though $\lim_{t\to 1} h_t$ is discontinuous, $f_1=\lim_{t\to 1} f_t$ will make a required shrinking of elements of G.

We will define a monotone increasing sequence $n_1=1, n_2, n_3, \ldots$ and a sequence of isotopies h_t^i $((i-1)/i\leqslant t\leqslant i/(i+1))$ of $Q\times I$ onto itself such that

(1) $h_0^1=\mathrm{id}$,

(2) $h_{i/(i+1)}^i=h_{i/(i-1)}^{i+1}$,

(3) $h_{(i-1)/i}^{i-1}|\ Q\times I-T_{n_i}\times[\frac{1}{2}\varepsilon_i, 1-\frac{1}{2}\varepsilon_i]=h_t^i|\ Q\times I-T_{n_i}\times[\frac{1}{2}\varepsilon_i, 1-\frac{1}{2}\varepsilon_i]$,

(4) $\operatorname{diam}h_{i/(i+1)}^i(T_{n_{i+1}}\times w)<\eta(F, \varepsilon_i)$ for all $w\in[\varepsilon_i, 1-\varepsilon_i]$,

(5) no point moves more than $2\varepsilon_{i-2}$ during $f_t=F\circ h_t^i\circ F^{-1}$,

(6) $h_{i/(i+1)}^i(Q\times w)\subset h_{(i-1)/i}^i(Q\times[w-\varepsilon_i, w+\varepsilon_i])$ for every $w\in I$, and

(7) the I-coordinate of $h_{i/(i+1)}^i(x, w)$ is $\leqslant 3\varepsilon_i$ or $\geqslant 1-3\varepsilon_i$ whenever $w\leqslant 2\varepsilon_i$ or $w\geqslant 1-2\varepsilon_i$, respectively.

The number $\eta(F, \varepsilon_i)$ in (4) is determined according to the following definition.

DEFINITION 2.2. Let $f\colon X\to Y$ be a map between compact metric spaces and $\varepsilon>0$ a given number. Define $\eta(f, \varepsilon)$ to be

$$\sup\{\eta>0|\ d(x, x')<\eta \text{ in } X \text{ implies } d(f(x), f(x'))<\varepsilon \text{ in } Y\}.$$

The existence of h_t^1 $(0\leqslant t\leqslant\frac{1}{2})$ follows from the assumptions in the lemma.

We proceed, inductively, to define h_t^i and n_{i+1}. By (4), the uniform continuity of $h_{(i-1)/i}^{i-1}$ and the relation $2\varepsilon_{i-2}>\varepsilon_{i-1}$, there is $\gamma>0$ with the property: $\operatorname{diam}h_{(i-1)/i}^{i-1}(T_{n_i}\times[a, b])<\eta(F, 2\varepsilon_{i-2})$ whenever $a, b\in[\varepsilon_{i-1}, 1-\varepsilon_{i-1}]$ satisfy $|a-b|<\gamma$. Also, there is an isotopy μ_t $((i-1)/i\leqslant t\leqslant i/(i+1))$ of $Q\times I$ and an integer $n_{i+1}>n_i$ such that

(i) $\mu_{(i-1)/i}=\mathrm{id}$,

(ii) $\mu_t|\ Q\times I-T_{n_i}\times[\frac{1}{2}\varepsilon_i, 1-\frac{1}{2}\varepsilon_i]=\mathrm{id}$,

(iii) μ_t changes I-coordinate less than $\mathrm{Min}(\gamma, \varepsilon_i)$, and

(iv) $\operatorname{diam}\mu_{i/(i+1)}(T_{n_{i+1}}\times w)<\eta(h_{(i-1)/i}^{i-1}, \eta(F, \varepsilon_i))$ for $w\in[\varepsilon_i, 1-\varepsilon_i]$.

Now, define $h_t^i=h_{(i-1)/i}^{i-1}\circ\mu_t$. Then (1) and (2) are clearly satisfied, (3) follows from (ii), (4) from (iv), and (5) holds because if for $x\in F(T_{n_i}\times I)-\{p, q\}$ I-coordinates of both $F^{-1}(x)$ and $\mu_t\circ F^{-1}(x)$ are in $[\varepsilon_{i-1}, 1-\varepsilon_{i-1}]$ then, since they are by (iii) at most γ apart, the way γ was choosen gives

$$d(F\circ h_{(i-1)/i}^{i-1}\circ F^{-1}(x), F\circ h_{(i-1)/i}^{i-1}\circ\mu_t\circ F^{-1}(x))<2\varepsilon_{i-2}$$

and, on the other hand, if I-coordinate of at least one of points $F^{-1}(x)$ and $\mu_t\circ F^{-1}(x)$ is in, say, $[0, \varepsilon_{i-1}]$ then by (iii) both are in $[0, \varepsilon_{i-1}+\varepsilon_i]\subset[0, 2\varepsilon_{i-1}]$ so that condition (7) for $h_{(i-1)/i}^{i-1}$ implies that I-coordinates of $h_{(i-1)/i}^{i-1}\circ F^{-1}(x)$ and $h_{(i-1)/i}^{i-1}\circ\mu_t\circ F^{-1}(x)$ are in $[0, 3\varepsilon_{i-1}]$; the requirement ($\gamma\gamma$) forces $F\circ h_{(i-1)/i}^{i-1}\circ\mu_t\circ F^{-1}(x)$ and $F\circ h_{(i-1)/i}^{i-1}\circ F^{-1}(x)$ to be within ε_{i-2} of p, i.e., at most $2\varepsilon_{i-2}$ apart. Finally, (6) is a consequence of (iii), and (7) follows from the fact that $h_{(i-1)/i}^{i-1}$ is the identity outside $Q\times[\frac{1}{2}\varepsilon_{i-1}, 1-\frac{1}{2}\varepsilon_{i-1}]\subset Q\times[3\varepsilon_i, 1-3\varepsilon_i]$ and (iii) since given $(x, w)\in Q\times I$ with, say, $w\in[0, 2\varepsilon_i]$ then,

$$h_{i/(i+1)}^i(x, w)=h_{(i-1)/i}^{i-1}(\mu_{i/(i+1)}(x, w))=\mu_{i/(i+1)}(x, w)\in Q\times[0, 3\varepsilon_i]. \ ■$$

To complete the proof of Theorem 1 it remains to construct isotopies μ_t from the hypothesis in Lemma 2.1.

Consider Q as a countable infinite product $\prod_{i>0} J_i$, where $J_i=[-1, 1]$ for each $i>0$. Since $\alpha\times\frac{1}{2}$ is a Z-set in $Q\times I$ [7, Corollary 2.4] by the homeomorphism extension theorem of [1], there is a homeomorphism $\varphi\colon Q\times I\to Q\times I$ such that

$$\varphi(\{0\}\times[\tfrac{1}{4},\tfrac{3}{4}]) = \alpha\times\tfrac{1}{2},$$

$$\varphi(\{-1\}\times\prod_{i>1} J_i\times I) = Q\times 0$$

and

$$\varphi(\{1\}\times\prod_{i>1} J_i\times I) = Q\times 1 .$$

Let V_j denote a closed neighborhood

$$[-1/j, 1/j]^j\times\prod_{i>j} J_i \quad \text{of } 0 = (0,0,\ldots)\in Q$$

and let

$$P_j = V_j\times[\tfrac{1}{4}-(1/2j), \tfrac{3}{4}+(1/2j)]$$

for each $j>3$. We can represent P_j as the union of Hilbert cubes $P_j^1, P_j^2, \ldots, P_j^{j-1}$ where each P_j^k is a product of V_j with a subinterval of $[\frac{1}{4}-(1/2j), \frac{3}{4}+(1/2j)]$ of lenght $(j+2)/2j(j-1)$. Define $Q_j^k = \varphi(P_j^k)$, $Q_j = \varphi(P_j)$, $R_j = Q_j\cap(Q\times\frac{1}{2})$, and $R_j^k = Q_j^k\cap(Q\times\frac{1}{2})$. As in [3, p. 2] we can choose a subsequence $\{P_i\}$ of $\{P_j\}$ such that

(i) diam $Q_i^k<\mathrm{Min}(1/i, \eta(\varphi, \varepsilon_i))$.

(ii) For each i and each k, there is an s such that

$$Q_{i+1}^k\subset(R_i^s\cup R_i^{s+1})\times I.$$

(iii) For each i and each s, there is a k such that

$$Q_{i+1}^k\subset R_i^s\times I,$$

and if $m\leqslant k$ then,

$$Q_{i+1}^m\subset(R_i^1\cup R_i^2\cup\ldots\cup R_i^s)\times I.$$

Let $T_i = R_{2i}$. Then $\{T_i\}$ will be the sequence of neighborhoods of α for which we will construct isotopies required by Lemma 2.1.

LEMMA 2.3. *Given a positive integer k and real numbers $\varepsilon>0$ and $0<a<b<1$, there exists a Hilbert cube E such that*

$$T_{k+1}\times[a,b]\subset\mathrm{int}E\subset E\subset T_k\times[a-\varepsilon, b+\varepsilon].$$

Proof. The proof of this lemma is identical with the proof of Theorem 2 in [3]. Last two conditions on φ guarantee that $\varphi_1(P_{2k+1})$ does not have points with I-coordinates 0, 1 so that the homeomorphism analogous to φ_2 in [3] can be constructed. ■

COROLLARY 2.4. *Given T_k, any integer $m>2$, and a sequence of real numbers $0<a_1<a_2<\ldots<a_{m-2}<b_{m-2}<\ldots<b_2<1$, there is a sequence $E_1, E_2, \ldots, E_{m-2}$ of Hilbert cubes such that*

$$T_k\times[a_1,b_1]\supset E_1\supset T_{k+1}\times[a_2,b_2]\supset\ldots\supset E_{m-2}\supset T_{k+m-2}\times[a_{m-2}, b_{m-2}].$$

Remark 2.5. The note on p. 4 of [3] also holds in our situation.

For any set $\pi = \{0<a_0<a_1<\ldots<a_{m-3}<b_{m-3}<\ldots<b_0<1\}$ of real numbers put $L_i^\pi = [a_i, a_{i+1}]\cup[b_{i+1}, b_i]$, $0\leqslant i\leqslant m-4$, and $L_{m-3}^\pi = [a_{m-3}, b_{m-3}]$. Also, $J_i^\pi = [a_i, b_i]$, $0\leqslant i\leqslant m-3$.

LEMMA 2.6. *Let $\{T_i\}$ be a sequence of neighborhoods of α constructed above, and let $T_k\in\{T_i\}$, with $C_1, \ldots, C_m$ being the chain of R_{2k}^p's in T_k. Then there is an isotopy μ_t on $Q\times I$ starting with the identity and ending with a homeomorphism h of $Q\times I$ onto itself such that*

$$\mu_t = \mathrm{id} \text{ outside of } T_k\times J_0^\pi,$$
$$h = \mathrm{id} \text{ on } (C_3\cup C_4\cup\ldots\cup C_m)\times L_0^\pi,$$
$$h = \mathrm{id} \text{ on } (C_4\cup C_5\cup\ldots\cup C_m)\times L_1^\pi,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h = \mathrm{id} \text{ on } \qquad C_m\times L_{m-3}^\pi,$$

and

$$h(\{T_k\cap(C_1\cup C_2)\}\times L_0^\pi)\subset(C_1\cup C_2)\times J_0^\pi,$$
$$h(\{T_{k+1}\cap(C_1\cup C_2\cup C_3)\}\times L_1^\pi)\subset(C_2\cup C_3)\times J_0^\pi,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h(\{T_{k+m-3}\cap(C_1\cup\ldots\cup C_{m-1})\}\times L_{m-3}^\pi)\subset(C_{m-2}\cup C_{m-1})\times J_0^\pi.$$

Proof. Once again, the proof is almost identical with the proof of Theorem 3 in [3] except that the role of Lemma 2 there in our situation plays

LEMMA 2.7. *Let r be an arbitrary positive integer, $A = I^r\times Q\times I$, and $A_2 = I^r\times Q\times[\frac{1}{2}, 1]$. Let $B\subset(\mathrm{int}I^r)\times Q\times(\mathrm{int}I)\cup I^r\times Q\times 1$ be a closed subset. Then there is an isotopy $\gamma\colon A\times I\to A$ such that γ_0 is the identity, $\gamma_t|$ $(\mathrm{Bd}I^r\times Q\times I\cup I^r\times Q\times\{0,1\}$ is the identity, and $\gamma_1(B)\subset A_2$.*

Proof. The isotopy γ_t $(t\in I)$ can be realized as $\mathrm{id}_Q\times\Delta_t$, where Δ_t is an isotopy on $I^r\times I$ constructed using Lemma 2 in [3] such that $\Delta_0 = \mathrm{id}$, Δ_t fixes boundary points of $I^r\times I$ for every t, and $\Delta_1(\pi(B))\subset I^r\times[\frac{1}{2}, 1]$, where $\pi(B)$ is a projection of B onto the factor $I^r\times I$. ■

LEMMA 2.8. *Let $T_k\in\{T_i\}$, $\eta>0$, and $0<\varepsilon<1$ be given. Then there is an integer N and a homeomorphism $\varphi\colon Q\times I\to Q\times I$ such that*

(1) $\varphi|\ Q\times I-T_k\times[\frac{1}{2}\varepsilon, 1-\frac{1}{2}\varepsilon] = \mathrm{id}$,

(2) *φ changes I-coordinate less than η, and*

(3) $\mathrm{diam}\,\varphi(T_N\times w)<\eta$ *for all* $w\in[\varepsilon, 1-\varepsilon]$.

Proof. Choose $N'\geqslant k$ so large that $\mathrm{diam}\,R_{2N'}^p<\frac{1}{16}\eta\sqrt{2}$. Then $T_{N'}$ has $m = 2N'-1$ chambers $C_1 = R_{2N'}^1, \ldots, C_m = R_{2N'}^m$. Pick $s(m-2)$ points $a_0^i<\ldots<a_{m-3}^i$ $(1\leqslant i\leqslant s)$ in I such that $a_0^1 = \frac{1}{2}\varepsilon$, $a_{m-3}^s = 1-\frac{1}{2}\varepsilon$, $a_{m-3}^1<\varepsilon$, $a_0^s>1-\varepsilon$, $a_{m-3}^i<a_0^{i+1}$ for every $i = 0, \ldots, s-1$, and a distance between any two consecutive a_j^i's is less than $\eta\sqrt{2}/8(2m-5)$. Put $N = N'+m-3$. A homeomorphism φ is the union of homeomorphisms $h_1, \ldots, h_{s-1}$ where h_i is a homeomorphism given by Lemma 2.6 with $\pi = \{a_0^i, \ldots, a_{m-3}^i, a_0^{i+1}, \ldots, a_{m-3}^{i+1}\}$ for every $i = 1, \ldots, s-1$, and for i odd pushing is done toward C_m while for i even toward C_1. ■

It is clear that above φ can be obtained as the end of an isotopy satisfying assumptions of Lemma 2.1. This completes the proof of Theorem 1.

Remark 2.9. Without any additional effort adapting a technique in [5], for the case when the considered k-cell is flat (i.e., the case I in [5]), word by word in a way explained above for an arc, we can get

THEOREM 2.10. *Let $\beta\colon I^k\to Q$ be an embedding of the k-cell ($k\geqslant 0$) I^k into the Hilbert cube Q. Then $[Q/\beta(I^k)]\times I$ is homeomorphic with Q.*

COROLLARY 2.11. *Let $A\subset Q$ be a decreasing intersection of finite-dimensional topological cells (of possibly varying dimensions). Then $(Q/A)\times I$ and Q are homeomorphic.*

Proof. This follows immediately from the corollaries in [6]. ■

3. Proof of Theorem 2. Throughout this section $P=\prod_{i>0} I_i$, $P_2=\prod_{i>1} I_i$, $Q=\prod_{i>0} J_i$, and $Q_2=\prod_{i>1} J_i$, where $I_i=J_i=[0,1]$ for each i, are Hilbert cubes and $\alpha\subset P$ and $\beta\subset Q$ are arbitrary arcs. By the Homeomorphism Extension Theorem [1] there is no loss of generality to assume that no point of α and β has its first coordinate smaller than 2γ or larger than $1-2\gamma$, for some $\gamma>0$.

In order to apply isotopies from Section 2 we must show that shrinking arcs "on the ends of $P\times Q$" gives a Hilbert cube.

Let $f'\colon P\times Q\to X'$ be the quotient map of $P\times Q$ onto the decomposition space X' of the upper semicontinuous decomposition whose only non-degenerate elements are arcs $\alpha\times\{(0,q)\}$, $\alpha\times\{(1,q)\}$, $\{(0,s)\}\times\beta$, and $\{(1,s)\}\times\beta$, where $q\in Q_2$ and $s\in P_2$.

LEMMA 3.1. *The space X' is homeomorphic with Q.*

Proof. Clearly, the union of all non-degenerate point inverses of f' is a Z-set in $P\times Q$. It follows easily from West's theorem [13] that X' is homeomorphic to Q provided X' is an AR. To establish this later property for X' we need J. H. C. Whitehead's theorem (see Theorem (9.1) on p. 116 in [4]) in order to get X' is an ANR, the fact that onto maps between ANR's with point inverses of trivial shape are homotopy equivalences [11], and that a contractible ANR is an AR. ■

We claim that $\alpha\times\beta$ is a Z-set in $P\times Q$. This follows from the more general Lemma 3.2.

LEMMA 3.2. *Let $A, B\subset Q$ be finite dimensional closed subsets of Q. Then $A\times B$ is a Z-set in $Q\times Q$.*

Proof. As in [9] by an open cube in Q we mean a basis element of the product topology, i.e., a product of relatively open subintervals of $[0,1]$ such that only finitely many (maybe none) are different from the whole interval.

Take two open cubes $U\subset P$ and $V\subset Q$. Then $U\times V-A\times B=(U-A)\times V\cup U\times(V-B)$ and $(U-A)\times V\cap U\times(V-B)=(U-A)\times(V-B)$ is arcwise connected. Consequently, by the trivial part of van Kampen's theorem, the fundamental group of $U\times V-A\times B$ is generated by loops contained in $(U-A)\times V$ or $U\times(V-B)$. Since both inclusions $(U-A)\times V\to U\times V-A\times B$ and $U\times(V-B)\to U\times V-A\times B$

are homotopic to a constant map, we infer that $U\times V-A\times B$ is 1-connected. Thus $P\times Q-A\times B$ is 1-$\overline{\text{ULC}}$ in the sense of Kroonenberg and $A\times B$ is a Z-set in $P\times Q$ [9]. ■

Let X be a space obtained from X' by shrinking $f'(\alpha\times\beta)$ to a point and let $f\colon P\times Q\to X$ be a natural projection. As a consequence of Lemmas 3.1 and 3.2, $X\cong Q$.

The rest of the proof is very similar to [10].

The product $P/\alpha\times Q/\beta$ is obtained from X by shrinking each of the arcs $f(\alpha\times y), f(x\times\beta)$, where $x\in P-\alpha$, $y\in Q-\beta$ and $x_1, y_1\neq 0$ or 1, to a point. We shall show that such shrinking may be achieved by a pseudo-isotopy of X. Then it follows that $P/\alpha\times Q/\beta\cong X\cong Q$.

In order to apply the method of Section 2, we need to separate these arcs $f(\alpha\times y)$, $f(x\times\beta)$ into two groups. Let $X_1=f(\alpha\times(Q-\beta))$ and $X_2=((P-\alpha)\times\beta)$. We wish to find two convenient disjoint open sets U_1 and U_2 of X such that $X_1\subset U_1$ and $X_2\subset U_2$. Then we will shrink arcs in X_i without disturbing points outside U_i ($i=1,2$).

Consider the relation $P/\alpha\times I\cong P$. Let $T_1^1\supset T_2^1\supset\ldots$ be a sequence of closed neighborhoods of α in P missing $\{0,1\}\times P_2$ constructed in Section 2. Let $T_1^2\supset T_2^2\supset\ldots$ be a similar sequence corresponding to $Q/\beta\times I\cong Q\times I$.

Let

$$U_1=\bigcup f((\operatorname{int}T_i^1)\times(Q-T_i^2)),$$
$$U_2=\bigcup f((P-T_i^1)\times(\operatorname{int}T_i^2)).$$

Next we show that there is a pseudo-isotopy h_t' of X which is the identity outside U_1 and shrinks the arcs in X_1. This, combined with an analogous pseudo-isotopy shrinking the arcs in X_2, will complete the proof.

As in Lemma 2.1, the following lemma provides us with the desired pseudo-isotopy.

LEMMA 3.3. *For given positive real numbers $\eta>0$, $0<\varepsilon<1$ and an integer N_0, there exist integers $i_0=i_0(N_0)$, $N>N_0$ and an isotopy λ_t, $0\leqslant t\leqslant 1$, of X such that*

(1) $\lambda_0=\mathrm{id}$,

(2) *each λ_t is the identity outside $f([T_{i_0}^1\times(Q-T_{i_0}^2)]\cap[P\times[\tfrac12\varepsilon, 1-\tfrac12\varepsilon]]\times Q_2)$,*

(3) *λ_t does not affect coordinates in Q_2, and*

$|\pi(f^{-1}(\lambda_t(x)))-\pi(f^{-1}(x))|<\eta$ for every $x\in X$, where π is a projection of $P\times Q$ onto J_1, and

(4) *$\operatorname{diam}\lambda_1\circ f(T_N\times y)<\eta$ for all $y\in Q$.*

Proof. Let i_1 be an integer such that $\operatorname{diam} f(T_{i_1}^1\times T_{i_1}^2)<\frac16\eta$. Let i_2 be an integer with the property that any arc $f(\alpha\times y)$ meeting $f(T_{i_2}^1\times T_{i_2}^2)$ lies in the interior of $f(T_{i_1}^1\times T_{i_1}^2)$.

Let $\Delta=\eta(f,\frac16\eta)$ (see Definition 2.2) and let $i_0>\operatorname{Max}(i_2,N_0)$ be an integer such that $T_{i_0}^1$ lies in a $\frac13\Delta$-neighborhood of α in P.

Devide Q_2 into finitely many "rectangular" Hilbert cubes $K_1, \ldots, K_m$ each of diameter $<\frac{1}{3}\Delta$.

Let μ_t be the isotopy of $P\times J_1$ from Lemma 2.1 as constructed in Lemma 2.8 with i_0, $\frac{1}{6}\Delta$, ε replacing i, η, ε, respectively. Let N be the integer determined by Lemma 2.1.

Let, for each $i = 0, \ldots, s-1$, $R_i = T_{i_0}^1\times[a_0^i, a_{m-3}^{i+1}]$, where points $a_j^i \in J_1$ are picked as in the proof of Lemma 2.8 with $k = i_0$ and $\eta = \frac{1}{6}\Delta$. Also, put $R_{-1} = T_{i_0}^1\times[0, \frac{1}{2}\varepsilon]$ and $R_s = T_{i_0}^1\times[1-\frac{1}{2}\varepsilon, 1]$. Now,

$$\bigcup_{i=-1}^{s} R_i = T_{i_0}^1\times J_1\,,$$

and

$$T_{i_0}^1\times Q = \bigcup_{i,j} R_i\times K_j\,.$$

We are ready to define λ_t. λ_t is the identity on

$$f\big((P-T_{i_0}^1)\times Q \cup P\times([0, \tfrac{1}{2}\varepsilon] \cup [1-\tfrac{1}{2}\varepsilon, 1])\times Q_2\big)\,.$$

On $f(T_{i_0}^1\times[\frac{1}{2}\varepsilon, 1-\frac{1}{2}\varepsilon]\times Q_2)$ we define λ_t on each piece $f(R_i\times Q_2)$ $(0\leqslant i\leqslant s-1)$ separately in such a way that $\lambda_t| f\big((\mathrm{Bd}\,R_i)\times Q_2\big) = \mathrm{id}$, where $\mathrm{Bd}\,R_i$ is the boundary of R_i in $P\times J_1$. Then all this λ_t's will match together nicely.

The construction of λ_t on each $f(R_i\times Q_2)$ and the verification that the isotopy of X obtained in this way is the required one is the same as in [10] ■

Now, the pseudo-isotopy h_t that performs promissed shrinking of arcs in X_1 is constructed in a way analogous to the construction of a pseudo-isotopy f_t in the proof of Lemma 2.1. This completes the proof of Theorem 2.

Remark 3.4. Extensions of Theorem 2 similar to Theorem 2.10 and Corollary 2.11 can also be proved with only minor changes in the above procedures (see Remark 2.9).

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