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Concerning locally homotopy negligible sets and characterization of l_2-manifolds *

by

H. Toruńczyk (Warszawa)

Abstract. Let $A \subset X$ be a set such that for every open subset U of X the inclusion $U \setminus A \to U$ is a weak homotopy equivalence. The following two facts are shown: (A) If X is an ANR($\mathfrak{M}$)-space, then so is $X \setminus A$; (B) if A is closed in X, X is complete-metrizable and $X \setminus A$ is an l_2-manifold, then so is X. We apply (B) to prove that if X is a separable complete ANR($\mathfrak{M}$) without isolated points, then the space of paths in X forms an l_2-manifold.

Initially the paper was intended to present the proofs of the following two facts, which had been announced, or employed, in [31] and [32]:

(A) If X is a complete separable ANR($\mathfrak{M}$)-space and A is a countable union of Z-sets in X, then $X \setminus A \in$ ANR($\mathfrak{M}$), and

(B) If X is as above, A is a Z-set in X and $X \setminus A$ is an l_2-manifold, then X is also an l_2-manifold.

By a Z-set in X we mean here any closed set $A \subset X$ with the property that every map $f\colon [0,1]^\infty \to X$ is a uniform limit of $X \setminus A$-valued maps.

If we use results of infinite-dimensional topology, (A) has a very short proof: by [31], the space $X \times l_2$ is an l_2-manifold which clearly contains $A \times l_2$ as a countable union of Z-sets; thus, by [2], $X \times l_2 \setminus A \times l_2$ is homeomorphic to $X \times l_2$ and hence $(X \setminus A) \times l_2$ and $X \setminus A$ are ANR($\mathfrak{M}$)'s. However, the assumption of (A) seems to be too restrictive: for instance, (A) does not include the fact that if A is any subset of the boundary of the square $[0,1]^2$ (A need not be of type G_δ), then $[0,1]^2 \setminus A$ is an ANR($\mathfrak{M}$). (See Fox [13].) Therefore we prove here in § 3 a result more general than (A), namely

(A′) If $X \in$ ANR($\mathfrak{M}$) and $A \subset X$ is locally homotopy negligible in X (i.e., for every open set $U \subset X$ the inclusion $U \setminus A \to U$ is a weak homotopy equivalence), then $X \setminus A \in$ ANR($\mathfrak{M}$).

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Since the properties of *non-closed* locally homotopy negligible sets have never been explicitly formulated, we devote a section of the paper to presenting the basic facts concerning such sets (see § 2). We note that most of these facts and also of the methods used in their proofs are similar to those of Anderson [14], Eells and Kuiper [11] and Henderson [19] (see also Eilenberg and Wilder [12], Smale [29], Haver [16]); however, several technical changes have to be made if one wants to dispense with the assumption that A is closed and X is an ANR($\mathfrak{M}$). The material of § 2 allows us to strengthen the results of [4], [22] and [23] on cell-like mappings of metric spaces (see the Appendix); also, we hope that the study of non-closed locally homotopy negligible sets in concrete spaces can be used to prove that these spaces are ANR($\mathfrak{M}$)'s or infinite-dimensional manifolds.

The result (B), stated before, is proved in § 5 and then applied to show that, for Y a complete separable connected ANR($\mathfrak{M}$) and K a polyhedron, the space of maps $K\to Y$ is an l_2-manifold.

In the paper we discuss also an elementary characterization of ANR($\mathfrak{M}$)'s which is used in the proof of (A′). (See § 1). Let us note that (A′) can also be established by using a characterization of Dowker and Hanner [10]; nevertheless the result of § 1 seems to be of independent interest (for instance, it unifies earlier results of Wojdysławski [34], Dugundji [9], Himmelberg [20] and others).

Notation. By I we denote the interval $[0, 1]$, by N the set of integers; continuous functions are called "maps". A homotopy $h: X\times I\to Y$ is often denoted by $(h_t): X\to Y$, where $h_t(x) = h(x, t)$. All spaces are assumed to be normal, and if X is a metrizable space then ϱ usually denotes a metric which induces the topology of X. $p_X: X\times Y\to Y$ denotes the natural projection. By $\operatorname{cov}(X)$ we denote the family of all open coverings of X. If K is an (abstract) simplicial complex, then $|K|$ denotes its standard geometric realization, endowed with the CW-topology. By i, k, n we denote elements of $N\cup\{0\}\cup\{\infty\}$ and $i<n+1$ means "$i\leqslant n$ if $n\neq\infty$ and $i\neq\infty$ if $n=\infty$".

§ 1. A characterization of ANR($\mathfrak{M}$)'s. If $\mathscr{A}$ and $\mathfrak{B}$ are families of subsets of X then by $\mathscr{A}_{\mathfrak{B}}$ we denote the family of all sets $A\in\mathscr{A}$ which refine $\mathfrak{B}$.

Suppose that X and Z are spaces, $\mathscr{A}$ is a family of subsets of X and that to certain sets $A\in\mathscr{A}$ we have assigned a map f_A from a non-empty set $\operatorname{dom}(f_A)\subset Z$ into X. Given $\mathfrak{U}\in\operatorname{cov}(X)$, we say that $(\{f_A\}, Z)$ is a $\mathfrak{U}$-*fine admissible approximation to* $\mathscr{A}$ if there is a $\mathfrak{B}\in\operatorname{cov}(X)$ such that the following conditions are satisfied:

(i) if $A\in\mathscr{A}_{\mathfrak{B}}$ then f_A is defined, $A\cup\operatorname{im}(f_A)$ refines $\mathfrak{U}$ and $F_A = \operatorname{dom}(f_A)$ is a homotopy trivial subset of Z;

(ii) if $A, B\in\mathscr{A}_{\mathfrak{B}}$ and $A\subset B$ then f_B is an extension of f_A.

We sometimes say that $(\{f_A: A\in\mathscr{A}_{\mathfrak{B}}\}, Z)$ forms an approximation. An approximation $(\{f_A\}, Z)$ will be said to be *continuous* if it is $\mathfrak{U}$-fine for all $\mathfrak{U}\in\operatorname{cov}(X)$. If $Z = X$ and each f_A is an inclusion, then we say that the approximation is *trivial*; trivial approximations will be denoted by $\{F_A\}$, where $F_A = \operatorname{dom}(f_A)\subset X$.

The aim of this section is to prove the following:

1.1. THEOREM. *The following conditions are equivalent for a metric space* X:

(a) $X\in\operatorname{ANR}(\mathfrak{M})$,

(b) *there exists a space E such that $X\times E$ has an open basis with all finite intersections of its members being homotopy trivial,*

(c) *there exist continuous admissible approximations to the family of all finite subsets of* X,

(d) *there exist arbitrarily fine admissible approximations to the family of all finite subsets of* $X\times(0, 1]$.

For simplicity the family of all finite subsets of X will be denoted by $\mathscr{F}(X)$ and the family of all subsets of X by $\mathscr{S}(X)$.

Remark. The implication (c)⇒(a) of 1.1 generalizes earlier results of Dugundji [8] and Himmelberg [20] stating that metric spaces which admit "nice" equiconnecting functions are ANR($\mathfrak{M}$)'s; see also Milnor [24]. In fact, if λ is an equiconnecting function on X, then, letting $A_1 = A$ and inductively

$$A_{n+1} = \{\lambda(x, y, t): x\in A, y\in A_n, t\in I\}$$

and $F_A = \bigcup_n A_n$, we get a trivial approximation to $\mathscr{S}(X)$ which is continuous in the situations considered in [8] and [20]. (Note that F_A is contractible whenever it is defined.)

Remark. Admissible approximations to $\mathscr{F} = \mathscr{F}(X)$ can be obtained as follows. Let $\mathfrak{U}\in\operatorname{cov}(X)$, let K denote the simplicial complex of all $\{x_1, \ldots, x_n\}\in\mathscr{F}_{\mathfrak{U}}$ and suppose that there is given a map $f: |K|\to X$. Then, letting $Z = |K|$ and $f_\sigma = f_{|\sigma|}$ for $\sigma\in K = \mathscr{F}_{\mathfrak{U}}$, we get an approximation to $\mathscr{F}$ which is continuous if for each $x\in X$ and a neighbourhood U of x there is a neighbourhood $V\subset U$ of x such that $f(|\sigma|)\subset U$ for all $\sigma = \{x_1, \ldots, x_n\}\subset V$.

In particular, the "convex structures" of [26] yield continuous approximations of this type and therefore 1.1 generalizes the results stating that spaces which admit convex (or similar) structures are ANR($\mathfrak{M}$)'s (see Himmelberg [20] and Wojdysławski [34]).

In the proof of 1.1 we need the following lemmas:

1.2. LEMMA. *Let Y be a metric space and Y_0 its dense subset. If there are arbitrarily fine admissible approximations to $\mathscr{F}(Y_0)$, then there are also arbitrarily fine admissible approximations to $\mathscr{S} = \mathscr{S}(Y)$.*

Proof. Fix $\mathfrak{U}\in\operatorname{cov}(Y)$, let $\mathfrak{U}_1\in\operatorname{cov}(Y)$ be a star-refinement of $\mathfrak{U}$ and let $(\{f_A: A\in\mathscr{F}_{\mathfrak{B}}\}, Z)$ be a $\mathfrak{U}_1$-fine admissible approximation to $\mathscr{F} = \mathscr{F}(Y_0)$. We assume without loss of generality that $\mathfrak{B}$ refines $\mathfrak{U}_1$. Let $\mathfrak{W}\in\operatorname{cov}(Y)$ be a locally finite star-refinement of $\mathfrak{B}$ and let $\mathfrak{N}\in\operatorname{cov}(Y)$ be a refinement of $\mathfrak{W}$ such that each element of $\mathfrak{N}$ intersects only finitely many elements of $\mathfrak{W}$. For each $W\in\mathfrak{W}$ pick an $y_W\in Y_0\cap W$ and, given $S\in\mathscr{S}_{\mathfrak{N}}$, let $g_S = f_{\hat{S}}$, where

$$\hat{S} = \{y_W: W\in\mathfrak{W} \text{ and } W\cap S\neq\emptyset\}.$$

It is easy to see that $(\{g_S\colon S\in\mathcal{S}_{\mathfrak{P}}\}, Z)$ is a $\mathfrak{U}$-fine approximation to the family $\mathcal{S}$.

1.3. Lemma. *Let (Y, ϱ) be a metric space such that there exist arbitrarily fine admissible approximations to $\mathcal{S} = \mathcal{S}(Y)$. Then, given $\alpha\colon Y\to(0, \infty)$, there are a simplicial complex K and maps $f\colon Y\to|K|$ and $g\colon |K|\to Y$ such that $\varrho(gf(y), y)<\alpha(y)$ for all $y\in Y$.*

Proof. Replacing, if necessary, ϱ by $\tilde{\varrho}(y_1, y_2) = \varrho(y_1, y_2)+|\alpha(y_1)-\alpha(y_2)|$, we may assume that $|\alpha(y_1)-\alpha(y_2)|\leqslant\varrho(y_1, y_2)$ for $y_1, y_2\in Y$. Let

$$\mathfrak{U} = \{B_\varrho(y, \tfrac{1}{4}\alpha(y))\colon y\in Y\}$$

be the cover of Y by open balls and let $(\{f_S\colon S\in\mathcal{S}_{\mathfrak{V}}\}, Z)$ be a $\mathfrak{U}$-fine admissible approximation to $\mathcal{S}$, $\mathfrak{V}$ being locally finite. Let K denote the nerve of $\mathfrak{V}$ and for each $\sigma = \{V_1, \ldots, V_n\}\in K$ let

$$I(\sigma) = \{(f_{V_1\cap\ldots\cap V_n}(z), z)\colon z\in\operatorname{dom}(f_{V_1\cap\ldots\cap V_n})\}\subset Y\times Z .$$

Clearly, I is an anti-monotone function from K to the non-empty homotopy trivial subsets of $Y\times Z$ (i.e., if $\sigma_1\subset\sigma_2\in K$ then $I(\sigma_1)\supset I(\sigma_2)$).

Now let K' denote the barycentric subdivision of K and let $i\colon |K|\to|K'|$ be the subdivision map. For each $\sigma\in K$ we denote by $\hat{\sigma}$ its barycenter; $\hat{\sigma}$ is then a vertex of K'.

Sublemma. *There is a map $\tilde{g}\colon |K'|\to Y\times Z$ such that*

$$(*)\qquad \tilde{g}(|\{\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_n\}|)\subset I(\sigma_1) \quad \textit{for all } \sigma_1\subset\sigma_2 \ldots \subset\sigma_n\in K.$$

Proof. For each vertex $\hat{\sigma}$ of K' choose a point $\tilde{g}_0(\hat{\sigma})\in I(\sigma)$. Let $(L, \tilde{g})$ be a maximal pair (under the natural ordering) such that L is a subcomplex of K' containing all vertices of K' and $\tilde{g}$ extends $\tilde{g}_0$ and satisfies $(*)$; we shall show that $L = K'$. Assume the contrary and let $s\in K'\setminus L$ be a simplex of minimal dimension. Then $\dim(s)\geqslant 1$ and [1] $|\dot{s}|\subset|L|$, whence $\tilde{g}\,||\dot{s}|\colon |\dot{s}|\to Y\times Z$ is well defined. Representing s as $\{\hat{\sigma}_1, \ldots, \hat{\sigma}_n\}$, where $\sigma_1\subset\sigma_2 \ldots \subset\sigma_n\in K$, we infer from $(*)$ and the anti-monotony of I that $\tilde{g}(|\dot{s}|)\subset I(\sigma_1)$. Since the set $I(\sigma_1)$ is homotopy trivial, we may extend $\tilde{g}\,||\dot{s}|$ to a $g_s\colon |s|\to I(\sigma_1)$. Clearly $(L\cup\{s\}, \tilde{g}\cup g_s)$ exceeds $(L, \tilde{g})$, which is impossible; thus $L = K'$ and $\tilde{g}$ is as required.

Proof of 1.3 (continued). Let $g = p_Y\circ\tilde{g}\circ i$. Given $y\in Y$, let $\{V_1, \ldots, V_n\} = \{V\in\mathfrak{V}\colon y\in V\}$. Observe that, by $(*)$, we have

$$g(|\{V_1, \ldots, V_n\}|)\subset p_Y\big(\bigcup_{i\leqslant n} I(\{V_i\})\big) = \bigcup_{i\leqslant n}\operatorname{im}(f_{V_i}) .$$

Since $\operatorname{im}(f_{V_i})\cup V_i$ refines $\mathfrak{U}$ for $i = 1, 2, \ldots, n$, we infer that $g(|\{V_1, \ldots, V_n\}|)$ is contained in the star of y in $\mathfrak{U}$. Therefore, if $f\colon Y\to|K|$ is induced by a partition of unity $\{\lambda_V\colon V\in\mathfrak{V}\}$ with each λ_V vanishing outside V, then $gf(y)\in\operatorname{st}(y, \mathfrak{U})$ for all $y\in Y$. This easily yields $\varrho(gf(y), y)<\alpha(y)$ for all $y\in Y$.

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[1] $\dot{s}$ denotes the boundary of s.

The following lemma is actually a special case of a theorem of Dowker and Hanner (see [10], p. 105); however, we include a short proof of it, which will be used later.

1.4 Lemma. *Let (X, ϱ) be a metric space and assume that there are a simplicial complex K and maps $f\colon X\times(0, 1]\to|K|$ and $g\colon |K|\to X$ such that $\varrho(gf(x, t), x)<t$ for all $(x, t)\in X\times(0, 1]$. Then $X\in\mathrm{ANR}(\mathfrak{M})$.*

Proof. Let $h\colon A\to X$ be a map of a closed set A of a metric space (B, ϱ_B); we shall construct a neighbourhood extension of h.

For this purpose let $u = h\times\mathrm{id}\colon A\times(0, 1]\to X\times(0, 1]$. Since simplicial complexes are neighbourhood extensors for metric spaces, $f\circ u$ admits an extension $v\colon U\to|K|$, where $U\subset B\times(0, 1]$ is an open set containing $A\times(0, 1]$ (see [21], p. 105). Without loss of generality we may assume that U is contained in the set $\{(b, t)\in U\colon$ there is an $a\in A$ with $\varrho_B(a, b)<t$ and $\varrho(gv(b, t), h(a))<t\}$, which, by our assumptions, is a neighbourhood of $A\times(0, 1]$. Let $\lambda\colon B\to[0, 1]$ be such that $\lambda|A = 0$ and $\{(b, \lambda(b))\colon b\in B\setminus A\}\subset U\cup(B\setminus A)\times\{1\}$. ($\lambda$ can easily be constructed by using Tietze's theorem and the fact that for each $a\in(0, 1]$ there is a closed neighbourhood W of A in B with $W\times[a, 1]\subset U$). We let $V = \{b\in B\colon \lambda(b)<1\}$ and define $\bar{h}\colon V\to X$ by

$$\bar{h}(b) = \begin{cases} h(b) & \text{if } b\in A, \\ gv(b, \lambda(b)) & \text{if } b\in V\setminus A. \end{cases}$$

It is easily seen that $\bar{h}$ is continuous.

Now we complete the proof of 1.1. To show that (a)⇒(c) consider X as a closed subset of a convex set Z in a normed linear space ([21], p. 81) and for sufficiently small sets $A\subset X$ let $F_A = \operatorname{conv}(A)\subset Z$ and $f_A = r|F_A$, where r is a neighbourhood retraction onto X. (c)⇒(d) is a consequence of the fact that any pair of continuous admissible approximations to $\mathcal{F}(X)$ and to $\mathcal{F}(Y)$ induces a continuous product approximation to $\mathcal{F}(X\times Y)$. Further, (d)⇒(a) by Lemmas 1.2-1.4, and (a)⇒(b) by a result of [31] stating that if $X\in\mathrm{ANR}(\mathfrak{M})$, then there exists a normed linear space E with $X\times E$ homeomorphic to an open subset of E. Finally, (b)⇒(a) follows from the implication (c)⇒(a) and the following fact applied to $Y = X\times E$.

Sublemma. *If Y has a base $\mathfrak{U}$ with homotopy trivial intersections, then there exist trivial continuous approximations to $\mathcal{S}(Y)$.*

Proof. For each $n\in N$ let $\mathfrak{V}_n\in\operatorname{cov}(X)$ be a locally finite refinement of $\mathfrak{U}$ with $\operatorname{diam}_\varrho V<1/n$ for all $V\in\mathfrak{V}_n$. Let $\mathfrak{V} = \bigcup_n\mathfrak{V}_n$, let $V\mapsto U_V$ be a function of $\mathfrak{V}$ into $\mathfrak{U}$ such that $V\subset U_V$ for all $V\in\mathfrak{V}$, and for sufficiently small $A\subset Y$ let

$$F_A = \bigcup_{V\in\mathfrak{V}(A)} U_V, \quad \text{where} \quad \mathfrak{V}(A) = \{V\in\mathfrak{V}\colon A\subset V\} .$$

It is easy to see that $\mathfrak{V}(A)$ is finite for all $A\subset Y$ and F_A is a continuous trivial approximation to $\mathcal{S}(Y)$.

1.5. COROLLARY. *Let X be a separable complete metric space which is l_2-stable (i.e., $X\times l_2 \cong X$). Suppose further that there exist arbitrarily fine admissible approximations to $\mathscr{F}(X_0)$, where $X_0 \subset X$ is a dense set. Then X is an l_2-manifold.*

Proof. By a theorem of Klee we have $l_2\times(0,1]\cong l_2$ (see [31]) and therefore $X\times l_2\times(0,1]\cong X\times l_2$ and $X\times(0,1]\cong X$. Hence, by 1.1 and 1.2, $X\in \mathrm{ANR}(\mathfrak{M})$ and thus, by [31], $X\times l_2$ is an l_2-manifold. Since $X\times l_2\cong X$, then result follows.

Clearly, the conditions of 1.5 are also necessary for a connected space X to be an l_2-manifold (recall that each separable l_2-manifold is l_2-stable and is homeomorphic to an open subset of l_2, see [3] and [31]).

§ 2. Locally homotopy negligible sets.

2.1. DEFINITION. A set $A\subset X$ will be said to be *locally n-negligible* if, given $x\in X$, $k<n+1$ and a neighbourhood U of x, there is a neighbourhood $V\subset U$ of x such that for each $f\colon (I^k,\partial I^k)\to(V, V\setminus A)$ there is a homotopy $(h_t)\colon (I^k,\partial I^k)\to(U, U\setminus A)$ with $h_0=f$ and $h_1(I^k)\subset U\setminus A$. Locally ∞-negligible sets will also be called *locally homotopy negligible* (briefly: *l. h. negligible*).

The aim of this section is to discuss certain properties of l. h. negligible sets; we formulate the corresponding results for locally n-negligible sets with $n<\infty$ only if their proofs require no extra work.

2.2. Remark. Let A be a locally n-negligible set in X. Then

(a) For every space E, $A\times E$ is locally n-negligible in $X\times E$.

(b) For every open set $U\subset X$, $U\cap A$ is locally n-negligible in U.

2.3. THEOREM. *Let $A\subset X$, where X is normal. The following conditions are equivalent:*

(a) *A is locally n-negligible in X.*

(b) *Given $\varepsilon>0$, a pseudometric ϱ on X and a map $f\colon (|K|,|L|)\to(X, X\setminus A)$, where (K,L) is a finite simplicial pair with $\dim(K)<n+1$, there is a homotopy $(h_t)\colon |K|\to X$ such that $h_0=f$, $h_1(|K|)\subset X\setminus A$, $h_t(x)=f(x)$ for $(x,t)\in|L|\times I$, and $\varrho(h_t(x),f(x))<\varepsilon$ for $(x,t)\in|K|\times I$.*

(c) *Given: a simplicial pair (K,L) with $\dim(K)\leqslant n$, a pseudometric ϱ on X and maps $\alpha\colon |K|\to(0,\infty)$ and $f\colon |K|\times\{0\}\cup|L|\times I\to X$ with $\varrho(f(x,t),f(x,0))<\alpha(x)$ and $f(x,1)\notin A$ for all $(x,t)\in|L|\times I$, there is an $\bar f\colon |K|\times I\to X$ which extends f and satisfies $\varrho(\bar f(x,t),\bar f(x,0))<\alpha(x)$ and $\bar f(x,1)\notin A$ for all $(x,t)\in|K|\times I$.*

(d) *For each open $U\subset X$ and $i<n+1$ the relative homotopy group $\pi_i(U, U\setminus A)$ vanishes.*

(e) *Each $x\in X$ has a basis $\mathfrak{U}_x$ of open neighbourhoods with $\pi_i(U, U\setminus A)=0$ for all $U\in\mathfrak{U}_x$ and $i<n+1$.*

Proof. (a)$\Rightarrow$(b). Let (b_p) denote the condition obtained from (b) with "$\dim(K)<n+1$" replaced by "$\dim(K)\leqslant p$"; we shall show that (a)$\Rightarrow$$(\mathrm{b}_p)$ for $0\leqslant p<n+1$. Assume that (a)$\Rightarrow$$(\mathrm{b}_{p-1})$ has been established (evidently (a)$\Rightarrow$$(\mathrm{b}_0)$) and let K, L, f and be as in (b_p). Given $\varepsilon\in(0,\frac12)$, cover the compact set $f(|K|)$ by

open sets $V_1,\ldots,V_k$ such that for each $g\colon (I^p,\partial I^p)\to(V_i, V_i\setminus A)$, where $1\leqslant i\leqslant k$, there is a homotopy $h=(h_t)\colon (I^p,\partial I^p)\to(X, X\setminus A)$ with $h_0=g$, $h_1(I^p)\cap A=\emptyset$ and $\operatorname{diam}_\varrho \operatorname{im}(h)<\varepsilon$. Let a subdivision (K',L') of (K,L) be so fine that $\{f(|\sigma|)\colon \sigma\in K'\}$ refines $\{V_1,\ldots,V_k\}$ (we identify $|K'|$ with $|K|$), and for each $\sigma\in K'$ let $\lambda_\sigma\colon X\to[0,1]$ be a map that is 1 on $f(|\sigma|)$ and 0 outside a $V^\sigma\in\{V_1,\ldots,V_k\}$. Write

$$d(x,y)=\sum_{\sigma\in K'}|\lambda_\sigma(x)-\lambda_\sigma(y)|+\varrho(x,y)\quad\text{ for }\quad x,y\in X,$$

and let M be the union of L' and the $p-1$ skeleton of K'. By (b_{p-1}), there is a homotopy $(\bar f_t)\colon |M|\to X$ such that $\bar f_0=f\,|\,|M|$, $\bar f_1(|M|)\cap A=\emptyset$, $\bar f_t|\,|L|=f\,|\,|L|$ and $d(\bar f_t(x),f(x))<\varepsilon$ for all $(x,t)\in|M|\times I$. For each $\sigma\in K'\setminus M$ denote $T_\sigma=|\dot\sigma|\times I\cup|\sigma|\times\{0\}$ and let $\bar f^\sigma$ be the map induced by $\bar f$ on $|\dot\sigma|\times[0,1]$ and by f on $|\sigma|\times\{0\}$. Then $(T_\sigma, |\sigma|\times\{1\})\cong(I^p,\partial I^p)$ and $\bar f^\sigma(T_\sigma)\subset V_\sigma$ for all $\sigma\in K'\setminus M$ and therefore, by our construction, there are homotopies $(g_t^\sigma)\colon (T_\sigma, |\dot\sigma|\times\{1\})\to(X, X\setminus A)$ such that $g_0^\sigma=\bar f^\sigma$, $g_1^\sigma(T_\sigma)\cap A=\emptyset$ and $\operatorname{diam}_\varrho\operatorname{im}(g^\sigma)<\varepsilon$ for each $\sigma\in K'\setminus M$. Then the g^σ's induce maps $h^\sigma\colon |\sigma|\times I\to X$ such that $h^\sigma|T_\sigma=\bar f^\sigma$, $h^\sigma(|\sigma|\times\{1\})\subset X\setminus A$ and $\operatorname{diam}_\varrho\operatorname{im}(h^\sigma)<\varepsilon$ (we take $h^\sigma=f^\sigma u^\sigma$, where u^σ is a homeomorphism of $|\sigma|\times I$ onto $T_\sigma\times I$ such that $u^\sigma(x)=(x,0)$ for $x\in T_\sigma\subset|\sigma|\times I$ and $u^\sigma(|\sigma|\times\{1\})=T_\sigma\times\{1\}\cup|\dot\sigma|\times\{1\}\times I$). We let

$$h_t(x)=\begin{cases}h^\sigma(x,t) & \text{if}\quad x\in|\sigma| \text{ and } \sigma\in K'\setminus M,\\ \bar f_t(x) & \text{if}\quad x\in|M|\,.\end{cases}$$

(b)$\Rightarrow$(c). By the Kuratowski–Zorn lemma it suffices to consider the case where $K=\sigma$ is a simplex and $L=\dot\sigma$. Assume that $|\sigma|$ is embedded in a euclidean space and for each $A\subset|\sigma|$ denote by λA the image of A under the λ-homothethy with respect to the barycenter 0 of $|\sigma|$. Let $\varepsilon>0$ satisfy $\varepsilon<\min\{\alpha(x)\colon x\in|\sigma|\}$ and $\varepsilon<\min\{\alpha(x)-\varrho(f(x,t),f(x,0))\colon (x,t)\in|\dot\sigma|\times I\}$. Set $T=|\sigma|\times\{0\}\cup|\dot\sigma|\times I$; by (b) there is an ε-homotopy $w\colon T\times I\to X$ such that $w_1(T)\cap A=\emptyset$, $w_0=f$ and $w_t(x)=f(x,1)$ for $x\in|\dot\sigma|\times\{1\}$. Now, for each $x\in|\sigma|\setminus\{0\}$ let

$$A(x)=\{(\lambda x,0)\colon \lambda\in[1,\mu]\}\cup\{(\mu x,t)\colon t\in I\}\,,$$

where $\mu\geqslant1$ is chosen so that $\mu x\in|\dot\sigma|$. Then the inequality

$$\sup\{\varrho(f(y),f(x,0))\colon y\in A(x)\}<\alpha(x)-\varepsilon \tag{1}$$

holds for all $x\in|\dot\sigma|$ and therefore, by compactness, there is a $\lambda\in(0,1)$ such that (1) holds for all $x\in|\sigma|\setminus\lambda|\sigma|$. Let $(u_t)\colon |\sigma|\to T\times I$ be a homotopy such that:

(i) $u_t(x)=((x,t),0)$ if $(x,t)\in|\dot\sigma|\times I\cup|\sigma|\times\{0\}$,

(ii) $u_t(x)=((x,0),t)$ if $(x,t)\in\lambda|\sigma|\times I$;

(iii) $u_1(|\sigma|)\subset T\times\{1\}\cup|\dot\sigma|\times\{1\}\times I$; and

(iv) $p_T u_t(x)\in A(x)$ if $(x,t)\in(|\sigma|\setminus\lambda|\sigma|)\times I$.

Then $\bar f\colon |\sigma|\times I\to X$ defined by $\bar f(x,t)=w(u_t(x))$ is the required extension of f.

The implications (c)$\Rightarrow$(b) and (d)$\Rightarrow$(e)$\Rightarrow$(a) are evident. To prove that (b)$\Rightarrow$(d), fix $f\colon (I^k,\partial I^k)\to(U, U\setminus A)$, where $U\subset X$ is open and $k<n+1$. Let $\lambda\colon X\to I$ be a function that is 0 on $X\setminus U$ and 1 on $f(I^k)$ and let $(h_t)\colon I^k\to X$ be a homotopy such that

$h_0 = f$, $h_1(I^k) \cap A = \emptyset$, $h_t(x) = f(x)$ for $x \in \partial I^k$ and $|\lambda f_t(x) - \lambda f(x)| < \frac{1}{2}$ for $x \in I^k$ (all $t \in I$). Then $h_t(I^k) \subset U$ for all $t \in I$ and hence f is trivial in $\pi_k(U, U \setminus A)$.

If (X, ϱ) is a metric space and h: $M \times I \to X$ and α: $M \times I \to [0, \infty)$ are maps, then we shall say that h is an *α-homotopy* if $\varrho(h_t(x), h_0(x)) \leqslant \alpha(x, t)$ for all $(x, t) \in M \times I$.

2.4. THEOREM. *Let A be an l. h. negligible set in a metric space (X, ϱ) and let f: $M \to X$ be a map of an* ANR($\mathfrak{M}$)*-space M. Then, given α: $M \times [0, 1] \to [0, \infty)$ with $\alpha(x, t) > 0$ for $(x, t) \in f^{-1}(A) \times (0, 1]$, there is an α-homotopy (h_t): $M \to X$ such that $h_0 = f$ and $h_t(M) \subset X \setminus A$ for $t \in (0, 1]$.*

We first consider a special case of 2.4.

SUBLEMMA. *Let X, M, A and f be as above and let γ: $M \to (0, \infty)$. Then there exists a g: $M \to X \setminus A$ such that $\varrho(g(x), f(x)) < 4\gamma(x)$ for $x \in M$.*

Proof. Let $\mathfrak{U} \in \operatorname{cov}(M)$ be so fine that $\operatorname{diam}_\varrho f(U) < \sup\{\gamma(x)\colon x \in U\} < 2\inf\{\gamma(x)\colon x \in U\}$ for all $U \in \mathfrak{U}$, and let a simplicial complex K and maps u_1: $M \to |K|$, u_2: $|K| \to M$ be such that for each $x \in M$ there is a $U \in \mathfrak{U}$ with $\{u_2 u_1(x), x\} \subset U$ ([11], p. 138). By 2.3 there exists a g_0: $|K| \to X \setminus A$ such that $\varrho(g_0(y), fu_2(y)) < \gamma u_2(y)$ for all $y \in |K|$. We let $g = g_0 u_1$.

Proof of 2.4. Let $X' = X \times (0, 1]$, $A' = A \times (0, 1]$, $M' = \alpha^{-1}(0, \infty)$ and let f': $M' \to X'$ be defined by $f'(x, t) = (f(x), t)$. By 2.2 and the sublemma, there exists a g: $M' \to X' \setminus A'$ such that

$$\varrho'(g(x, t), f'(x, t)) < \min(t, \alpha(x, t)) \quad \text{for} \quad (x, t) \in M',$$

where $\varrho'((x, t), (y, s)) = \varrho(x, y) + |t - s|$. We let $h_t(x) = p_X g(x, t)$ if $(x, t) \in M'$ and $h_t(x) = f(x)$ if $(x, t) \in M \times \{0\} \cup \alpha^{-1}(0)$.

2.5. Remark. Assume that X, M and f are as in 2.4 and that A is locally n-negligible in X. If $\dim(M) \leqslant n-1$, then the assertion of 2.4 still holds. If $\dim(M) \leqslant n$, then for every β: $X \to [0, \infty)$ with $\beta | A > 0$ there is a homotopy (h_t): $M \to X$ such that $h_0 = f$, $h_1(M) \subset X \setminus A$ and $\varrho(h_t(x), f(x)) < \beta(f(x))$ for all $(x, t) \in M \times I$. (We apply the proof of 3.4 and the fact that if $M_1 \in$ ANR($\mathfrak{M}$) is of covering dimension n, then there are an n-dimensional simplicial complex K and maps $M_1 \xrightarrow{u_1} |K| \xrightarrow{u_2} M_1$ such that $u_2 u_1$ is homotopic to the identity by means of a small homotopy).

2.6. COROLLARY. *If A is an l. h. negligible set in a metric space X and A' is a subset of A, then A' is also l. h. negligible in X.*

Proof. Let an open set $U \subset X$ and f: $(I^n, \partial I^n) \to (U, U \setminus A')$ be given, and let $\varepsilon = \varrho(f(I^n), X \setminus U)$. By 2.4, there exists an ε-homotopy (h_t): $I^n \to X$ such that $h_0 = f$ and $h_t(I^n) \cap A = \emptyset$ for $t \in (0, 1]$. Then (h_t): $(I^n, \partial I^n) \to (U, U \setminus A')$ satisfies the condition in 2.1.

2.7. COROLLARY. *Let $A_1, A_2, \ldots$ be closed l. h. negligible sets in X. If X is complete-metrizable, then $A = \bigcup_i A_i$ is l. h. negligible in X.*

Proof. Fix $n \geqslant 0$ and consider the space Y of all maps of $I^n \times (0, 1]$ into X, equipped with the "fine topology" generated by all sets

$$V(g, \alpha) = \{h \in Y\colon \varrho(h(x), g(x)) < \alpha(x)\},$$

where ϱ is a fixed complete metric on X, $g \in Y$ and α is a map from $I^n \times (0, 1]$ into $(0, \infty)$.

By 2.4, all the sets $Y_n = \{g \in Y\colon \operatorname{im}(g) \cap A_n = \emptyset\}$ are dense and open in Y. Moreover, it is easy to verify that Y has the Baire property (cf. [30]) and therefore $Y_\infty = \bigcap_n Y_n$ is dense in Y. Thus for each f: $I^n \to X$ there is an $h \in Y_\infty$ with $\varrho(h(x, t), f(x)) < \varepsilon t$ for all $(x, t) \in I^n \times (0, 1]$; this easily completes the proof.

We conclude this section by giving a condition for a set $A \subset X$ to be locally n-negligible. Following [12], we say that $B \subset X$ is LC^n rel. X at a point $x \in X$ if, given $k < n+2$ and a neighbourhood U of x, there is a neighbourhood $V \subset U$ of x such that each f: $\partial I^k \to B \cap V$ extends to an $\bar{f}$: $I^k \to B \cap U$.

2.8. THEOREM (compare [12]). *Let X be a metric space and let $A \subset X$ be a set such that $X \setminus A$ is dense in X and is* LC^n *rel. X at each point of $\bar{A}$. If $n < \infty$, then A is locally n-negligible in X and each map f: $I^{n+1} \to X$ can be approximated by maps f': $I^{n+1} \to X \setminus A$ which coincide with f on an arbitrary given compact subset of $f^{-1}(X \setminus \bar{A})$.*

Proof. Let f: $K \to X$ be a fixed map of a compact polyhedron K. Writing $L = f(K) \cap \bar{A}$, we let for any map g: $Z \to X$ of a compact space Z

$$\delta(g) = \operatorname{diam}_\varrho g(Z) + \sup\{\varrho(g(z), L)\colon z \in Z\},$$

and we say that g is λ-small if $\delta(g) < \lambda$. By a standard compactness argument there exist a $\lambda_0 > 0$ and a function ε: $(0, \lambda_0] \to (0, \infty)$ with $\lim_{\lambda \to 0} \varepsilon(\lambda) = 0$ and such that each λ-small g: $\partial I^k \to X \setminus A$ admits an $\varepsilon(\lambda)$-small extension $\bar{g}$: $I^k \to X \setminus A$ $(k = 0, 1, \ldots, n+1)$. Without loss of generality we can assume that $\lambda_0 > 3$ and that ε is non-decreasing.

CLAIM (A). *If $\dim(K) \leqslant n+1$ then, for every $\mu \in (0, 1]$, there exists a g: $K \to X \setminus A$ such that $\hat{\varrho}(f, g) < \varepsilon(3\mu) + 3\mu$ and $g(x) = f(x)$ if $\varrho(f(x), L) > \mu$* (²).

Proof. We use induction on $\dim(K)$. Suppose that (A) holds true if $\dim(K) \leqslant p$ (it does hold if $\dim(K) = 0$) and assume $\dim(K) = p+1 \leqslant n+1$. Let T be a triangulation of K such that $\operatorname{diam}_\varrho f(|\sigma|) < \mu$ for any simplex $\sigma \in T$ and let S denote the p-skeleton of T. Let g_0: $|S| \to X \setminus A$ be a map such that $\hat{\varrho}(g_0, f||S|) < \mu$ and such that $g_0(x) = f(x)$ if x lies in a simplex of T which is disjoint from $f^{-1}(\bar{A})$. Now, let $\sigma \in T$ be any $(p+1)$-simplex. If $|\sigma| \cap f^{-1}(A) \neq \emptyset$, then $g_0||\dot{\sigma}|$ is 3μ-small and therefore it admits an $\varepsilon(3\mu)$-small extension g^σ: $|\sigma| \to X \setminus A$. If $|\sigma| \cap f^{-1}(\bar{A}) = \emptyset$, then put $g^\sigma = f||\sigma|$. Clearly, g_0 and the g^σ's induce the required g: $K \to X \setminus A$.

CLAIM (B). *If $\dim(K) \leqslant n$, then there is a homotopy (h_t): $K \to X$ with $h_0 = f$ and $h_t(K) \subset X \setminus A$ for $t \in (0, 1]$.*

(²) By $\hat{\varrho}$ we denote the sup-metric induced by ϱ.

Proof. Define $\varepsilon_0(\mu) = \varepsilon(3\mu)$ and inductively $\varepsilon_{j+1}(\mu) = 2\varepsilon_j(\mu)+3\mu$; we may assume that all $\varepsilon(\mu), \varepsilon_1(\mu), \ldots, \varepsilon_n(\mu)$ are defined for $\mu \leqslant 1$. Let T_i, $i \geqslant 1$, be triangulations of K such that T_{i+1} is a subdivision of T_i and $\operatorname{diam}_\varrho f(|\sigma|) < 2^{-i}$ for all $\sigma \in T_i$ and $i \geqslant 1$, and for each $i \geqslant 1$ let $g_i\colon K \to X \setminus A$ be a map such that $\hat{\varrho}(g_i, f) < 2^{-i}$ and $g_i(x) = f(x)$ if x lies in a simplex of T_i disjoint from $f^{-1}(\bar{A})$. We shall find the required homotopy in such a way that $h_t = g_i$ if $t = 2^{-i+1}$, $i = 1, 2, \ldots$ To make this possible it suffices to construct for each $i \geqslant 1$ an $h^i\colon K \times I \to X \setminus A$ with $h_0^i = g_i$, $h_1^i = g_{i+1}$ and $\hat{\rho}(h_t^i, f) < \varepsilon_n(2^{-i}) + 2^{-i}$ for $t \in I$.

To this end, fix i and let v be a vertex of T_i. If $v \in f^{-1}(\bar{A})$ then $g_i(v)$ and $g_{i+1}(v)$ can be joined by an $\varepsilon(3\cdot 2^{-i})$-small path lying in $X \setminus A$, and if $v \notin f^{-1}(\bar{A})$ then this path can be taken as constant. Proceeding in this way with all vertices v of T_i, we get an $h^{i,0}\colon |T_i^0| \times I \cup K \times \{0, 1\} \to X$ with $h^{i,0}\,|K \times \{0\} = g_i$ and $h^{i,0}\,|K \times \{1\} = g_{i+1}$ (by T_i^k we denote the k-skeleton of T_i). Now let $\sigma \in T_i^1$. If $|\sigma| \cap f^{-1}(A) \neq \emptyset$, then $h^{i,0} \| |\dot{\sigma}| \times I \cup |\sigma| \times \{0, 1\}$ is an $2\varepsilon_0(2^{-i}) + 3\cdot 2^{-i}$-small map of a 1-sphere and therefore it can be extended to an $\varepsilon_1(2^{-i})$-small map of $|\sigma| \times I$ into $X \setminus A$; if $|\sigma| \cap f^{-1}(\bar{A}) = \emptyset$, then this extension can be taken to be constant on all intervals $\{x\} \times I$, $x \in |\sigma|$. In this way one gets an $h^{i,1}\colon |T_i^1| \times I \cup K \times \{0, 1\} \to X$ which extends $h^{i,0}$ and has the property that $h^{i,1} \| |\sigma| \times I$ is $\varepsilon_1(2^{-i})$-small for all $\sigma \in T_i^1$. Inductively, we get maps $h^{i,j}\colon |T_i^j| \times I \cup K \times \{0, 1\} \to X$, $j = 1, 2, \ldots, n$, such that $h^{i,j+1}$ extends $h^{i,j}$ and $h^{i,j} \| |\sigma| \times I$ is $\varepsilon_j(2^{-i})$-small for all $\sigma \in T_i^j$. We let $h^i = h^{i,n}$.

Clearly (A) and (B) imply the assertion of 2.8.

2.9. Remark. If $Y \subset X$ is a dense set which is uniformly LC^∞ in a metric of X, then Y is LC^∞ rel. X at each $x \in X$ and, hence, $X \setminus Y$ is l. h. negligible in X. (A version of this remark was made by Eilenberg and Wilder [12] and various forms of it were applied by Haver [16], [17] in a study of function spaces.)

§ 3. Locally homotopy negligible sets in ANR($\mathfrak{M}$)'s and LC^∞-spaces.

3.1. Theorem. *Let $X \in \mathrm{ANR}(\mathfrak{M})$ and let A be a locally homotopy negligible set in X. Then $X \setminus A \in \mathrm{ANR}(\mathfrak{M})$.*

Proof. By 1.1, there exists a space E such that $X \times E$ has an open basis (say $\mathfrak{U}$) with homotopy trivial intersections. Then $A \times E$ is l. h. negligible in $X \times E$ and therefore the basis $\{U \setminus A \times E\colon U \in \mathfrak{U}\}$ of $(X \setminus A) \times E$ has homotopy trivial intersections. Hence $(X \setminus A) \times E$ and $X \setminus A$ are ANR($\mathfrak{M}$)'s (we use 1.1 again).

3.2. Proposition. *Let $X \in \mathrm{ANR}(\mathfrak{M})$ and let A be a locally n-negligible set in X. If $\dim(X) \leqslant n$ then A is l.h. negligible in X.*

Proof. Fix $f\colon (I^k, \partial I^k) \to (X, X \setminus A)$ and $\varepsilon > 0$. By 2.5, there exists an ε-homotopy $(h_t)\colon X \to X$ such that $h_1(X) \subset X \setminus A$ and $h_t(x) = x$ if $(x, t) \in X \times \{0\} \cup f(\partial I^k) \times I$. Hence $(h_t f)\colon (I^k, \partial I^k) \to (X, X \setminus A)$ is an ε-homotopy with $h_0 f = f$ and $h_1 f(I^k) \subset X \setminus A$; thus A is l. h. negligible in X.

For $0 \leqslant k \leqslant \infty$ let us say that A is a *Z_k-set* in X if each map $f\colon I^k \to X$ can be approximated by maps into $X \setminus A$. It is easy to see that A is a Z_∞-set in X iff it is a Z_k-set for all $k \in N$; closed Z_∞-sets in X will be called *Z-sets*.

3.3. Corollary. *Let $X \in LC^n$ be a metric space. The following conditions on a closed set $A \subset X$ are equivalent:*

(a) *$X \setminus A$ is dense in X and is LC^{n-1} rel. X at each $x \in A$;*

(b) *A is a Z_n-set in X;*

(c) *A is locally n-negligible in X.*

Proof. (a)⇒(b) follows from 2.8 and (c)⇒(a) is trivial. Finally, (b)⇒(c) by the well-known properties of LC^n-spaces (see [21], p. 160).

§ 4. Enlarging an ANR($\mathfrak{M}$) — open questions and remarks.

Let X be a locally contractible metric space and A its l. h. negligible subset. By 3.2, $X \in \mathrm{ANR}(\mathfrak{M}) \Rightarrow X \setminus A \in \mathrm{ANR}(\mathfrak{M})$. We do not know whether the converse implication is true (³).

In this connection let ut show:

4.1. Proposition. *Let X be a metric space and A its ANR($\mathfrak{M}$)-subset. Then, A may be enlarged to an ANR($\mathfrak{M}$)-set $\tilde{A} \subset X$ which is of type G_δ in X and has the property that $\tilde{A} \setminus A$ is l. h. negligible in $\tilde{A}$.*

Proof. By well-known properties of ANR($\mathfrak{M}$)'s there is a $\mathfrak{U} \in \operatorname{cov}(A \times (0, 1])$ and a map $g\colon |K| \to A$, where K is the nerve of $\mathfrak{U}$, such that if $f\colon A \times (0, 1] \to |K|$ is any canonical map, then $\varrho(gf(x, t), x) < t$ for all $(x, t) \in A \times (0, 1]$ (see [21], p. 138 or use the proof of 1.1). Let $\mathfrak{V}$ be a family of open subsets of $X \times (0, 1]$ such that $\mathfrak{U} = \{V \cap A \times (0, 1]\colon V \in \mathfrak{V}\}$, let L be the nerve of $\mathfrak{V}$ and V the union of all elements of $\mathfrak{V}$, and let $f\colon V \to |L|$ be a canonical map. Identifying K with a subcomplex of L, we infer that $C = f^{-1}(|K|)$ is a relatively closed subset of V and therefore the set $B = \{(x, t) \in C\colon \varrho(gf(x, t), x) < t\}$ is of type G_δ in $X \times (0, 1]$ and contains $A \times (0, 1]$. Since $(0, 1]$ is σ-compact, $\tilde{A} = X \setminus p_X(X \times (0, 1] \setminus B)$ is a G_δ-subset of X containing A. By 1.4, $\tilde{A} \in \mathrm{ANR}(\mathfrak{M})$. The following sublemma shows that $\tilde{A} \setminus A$ is l. h. negligible in $\tilde{A}$.

Sublemma. *Every set $C \subset X \setminus \operatorname{im}(g)$, is l. h. negligible in X.*

Proof. Let $h\colon I^n \to X$ be given. Identify I^n with $I^n \times \{0\} \subset I^n \times I$. By 1.4 there are $\varepsilon > 0$ and $\bar{h}\colon I^n \times [0, \varepsilon] \to X$ such that $\bar{h}\,|\,I^n = h$; moreover, the formula given in the proof of 1.4 yields $\bar{h}(x, t) \in \operatorname{im}(g)$ for $t \in (0, \varepsilon]$. Therefore there is a homotopy $(u_t)\colon I^n \to X$ such that $u_0 = h$ and $u_t(I^n) \cap C = \emptyset$ for $t > 0$; this concludes the proof.

4.2 Remark. The set $\tilde{A}$ of 4.1 is in no way unique: e.g., if $B \supset A$ is any G_δ-subset of $\tilde{A}$, then B also satisfies the assertion of 4.1 (see 2.6 and § 3).

4.3. Remark. Let X be a compact PL-manifold, let H denote its homeomorphism group with compact-open topology and let P be the subgroup of H consisting of PL-maps. It was shown by Haver [15], [17], that $P \in \mathrm{ANR}(\mathfrak{M})$ and the closure G of P is an open subgroup of H. Let $G_0 \supset P$ be an ANR($\mathfrak{M}$)-extension of P to a G_δ-subset of G; since P is uniformly locally contractible (see [17]), we infer, by 2.9 and 2.6, that $G \setminus G_0$ is l. h. negligible in G. Thus $G \times l_2$ contains an l_2-manifold

(³) Added in proof. It is not, without assuming X to be locally contractible, as is shown by Taylor's example (BAMS 81, p. 629) combined with 6.1 and 6.3.

(namely $G_0 \times l_2$, see [31]) with an l. h. negligible complement. Since H is a union of open cosets of G and since $H \times l_2 \cong H$ (Geoghegan [14]), H also contains an l_2-manifold with an l. h. negligible complement. It is however an open question if H is an ANR($\mathfrak{M}$).

4.4. Remark. Similarly, it follows from [15] and 4.1 that if E is any separable complete linear metric space then $E \setminus K \in \mathrm{AR}(\mathfrak{M})$ for some l.h. negligible F_σ-set K; it is though unknown if $E \in \mathrm{AR}(\mathfrak{M})$.

§ 5. Enlarging a manifold. In this section we show that if a complete ANR($\mathfrak{M}$)-space X contains an l_2-manifold whose complement is a Z-set in X, then X is necessarily an l_2-manifold. We start with:

5.1. Proposition. *Let E denote the Hilbert cube or a locally convex linear metric space such that $E \cong E^\infty$ or $E \cong \sum E = \{(x_i) \in E^\infty\colon x_i = 0$ for almost all $i\}$ and let A be a Z-set in a metric space X. If $X \times E$ and $X \setminus A$ are E-manifolds, then $X \cong X \times E$ and X is an E-manifold.*

The proof is divided into 3 steps and involves an idea of Cutler (see [7] and also [33], where some special cases of 5.1 are established).

1^0 *If M is an E-manifold and K is a Z-set in M, then there is a homotopy $(f_t)\colon M \to M$ such that $f_0 = \mathrm{id}$, $f_t(M) \subset \operatorname{int} f_s(M)$ if $0 < s < t \leqslant 1$, $\bigcup_{t>0} f_t(M) = M \setminus K$ and $(x, t) \mapsto (f_t(x), t)$ is a closed embedding of $M \times I$ into itself.*

Proof. Under our assumptions there is a homeomorphism $h\colon M \xrightarrow{\text{onto}} M \times I$ such that $h(K) \subset M \times \{0\}$ (see [30]). Let ϱ be any product metric on $M \times I$; then for each $t \in I$ the formula

$$\alpha_t(x) = \inf\{s \in I\colon \varrho((x, s), h(K)) \geqslant t\}$$

defines a continuous function on M. We let $f_t = h^{-1} g_t h$, where $g_t(x, s) = (x, s)$ if $s \geqslant \alpha_t(x)$ and $g_t(x, s) = (x, \tfrac{1}{2}\alpha_t(x) + \tfrac{1}{2}s)$ otherwise.

Given spaces Z and F and a closed set $L \subset Z$, we denote by $(Z \times F)_L$ the space $(Z \setminus L) \times F \cup L$ equipped with the topology generated by open subsets of $(Z \setminus L) \times F$ and by sets of the form $U \cap L \cup (U \setminus L) \times F$, where $U \subset Z$ is open. CF denotes $(I \times F)_{\{0\}}$, the cone over F.

2^0 *Under the assumptions of 5.1, the spaces $X \times CE$ and $(X \times CE)_A$ are homeomorphic.*

Proof. Set $M = X \times E$ and $K = A \times E$ and let $(f_t)\colon M \to M$ be the homotopy from 1^0. Define $h\colon X \times CE \to (X \times CE)_A$ by the formula

$$h(x, y) = \begin{cases} (x, y) & \text{if } y = 0, \\ \left(p_X f_t(x, e), \left(p_E f_t(x, e), \dfrac{t}{\beta f_t(x, e)}\right)\right) & \text{if } y = (t, e) \text{ and } t > 0, \end{cases}$$

where $\beta(x, e) = \sup\{s \in I\colon (x, e) \in f_s(M)\}$. It is a matter of routine but tedious verification to show that h is a homeomorphism of $X \times CE$ onto $(X \times CE)_A$.

Proof of 5.1. It is known that E and CE are homeomorphic (see [18] and [33]), and therefore $X \times E \cong (X \times E)_A$. Let ϱ be any metric for X. Since $X \setminus A$ is an E-manifold, there is a homeomorphism $g\colon (X \setminus A) \times E \xrightarrow{\text{onto}} X \setminus A$ such that $\varrho(g(z), p_X(z)) < \varrho(p_X(z), A)$ for all $z \in (X \setminus A) \times E$ (see [28]). Extending g by identity over A, we get a homeomorphism of $(X \times E)_A$ onto X. Thus $X \times E \cong X$.

Combining 5.1 with the results of [32], we get

5.2. Theorem. *Let X be an ANR($\mathfrak{M}$)-space, let A be a Z-set in X and assume that $X \setminus A$ is a manifold modelled on a space E. In any of the following cases X is also an E-manifold:*

(a) E is an infinite-dimensional Hilbert space and X is complete;

(b) E is a locally convex linear metric space with $E \cong \sum E$ and X admits a closed embedding into E.

For a discussion of certain special cases in which the condition (b) is satisfied see [31], § 1.

In the remaining part of this section we apply 5.2 to show that certain function spaces are l_2-manifolds. If X is a space and A is a compactum, then $C(A, X)$ denotes the space of maps of A into X (compact-open topology), for $x \in X$ we denote by $\hat{x}$ the constant map with value x, and we let $\hat{X} = \{\hat{x}\colon x \in X\}$. $C((A, A_0), (X, X_0))$ has the usual meaning. We need two lemmas leading to the fact that if $X \in \mathrm{ANR}(\mathfrak{M})$ has no isolated points, then one can continuously assign to each $x \in X$ a non-constant path starting from x.

5.3. Lemma. *Let $Y \in \mathrm{ANR}(\mathfrak{M})$, let $A_0 \subset A$ be compacta and let $y_0 \in Y$. If neither $\{y_0\}$ nor A_0 are open, then the singleton $\{\hat{y}_0\}$ is a Z-set in*

$$S = C((A, A_0), (Y, y_0)).$$

Proof. Since every $f \in S$ factorizes through a map of $(A/A_0, [A])$ into (Y, y_0), we may assume that $A_0 = \{a_0\}$ is a one-point set. Consider A as a (nowhere-dense) subset of l_2 and let $(a_n) \in (A \setminus A_0)^\infty$, $(z_n) \in (l_2 \setminus A)^\infty$ and $(y_n) \in (Y \setminus \{y_0\})^\infty$ be sequences such that $\lim z_n = \lim a_n = a_0$ and $\lim y_n = y_0$. Given $f\colon A \times I^\infty \to Y$ with $f(\{a_0\} \times I^\infty) \subset \{y_0\}$, extend f to $f_1\colon (A \cup \{z_n\colon n \in N\}) \times I^\infty \to Y$ by letting $f_1(\{z_n\} \times I^\infty) = \{y_n\}$, $n \in N$, and extend f_1 to an $\bar{f}\colon U \times I^\infty \to Y$ where $U \supset A \cup \{z_n\colon n \in N\}$ is open in l_2. Let (g_n) be a sequence of mappings $g_n\colon A \to U$ such that $\lim g_n = \mathrm{id}$ and moreover $g_n(a_0) = a_0$ and $g_n(a_n) = z_n$ for all sufficiently big n's. The maps $f_n\colon A \times I^\infty \to Y$ defined by

$$f_n(a, q) = \bar{f}(g_n(a), q), \quad (a, q) \in A \times I^\infty,\ n \in N$$

converge to f and have the property that, for each $q \in I^\infty$, the map $a \mapsto f_n(a, q)$ belongs to $S \setminus \{\hat{y}_0\}$. Since $f\colon A \times I^\infty \to Y$ was induced by an arbitrary map of I^∞ into S, the result follows.

5.4. Lemma. *Let Y be an ANR($\mathfrak{M}$)-space without isolated points and let $\varepsilon > 0$. There is a $v\colon Y \to C(I, Y) \setminus \hat{Y}$ such that $v(y)(0) = y$ and $\hat{\varrho}(v(y), \hat{y}) < \varepsilon$ for all $y \in Y$ ($\hat{\varrho}$ denotes here the sup-metric induced by ϱ).*

Proof. $C(I, Y)$ is an ANR($\mathfrak{M}$)-space and therefore, by 2.4, 3.3 and elementary properties of ANR($\mathfrak{M}$)'s, it suffices to show that $C(I, Y)\setminus\hat{Y}$ is LC$^\infty$ rel. $C(I, Y)$ at each point $\hat{y} \in \hat{Y}$.

To this end let us fix $k \in N$, $\hat{y}_0 \in \hat{Y}$ and $\varepsilon_0 > 0$; we shall find a $\delta > 0$ such that, under the notation $S = C(I, Y)$ and $J = [-1, 1]$, we have

(*) Each $f\colon \partial J^k \to S\setminus\hat{Y}$ with $\sup\{\hat{\varrho}(f(x), \hat{y}_0)\colon x \in \partial J^k\} < \delta$ extends to an $\bar{f}\colon J^k \to S\setminus\hat{Y}$ with $\sup\{\hat{\varrho}(\bar{f}(x), \hat{y}_0)\colon x \in J^k\} < \varepsilon_0$.

First observe that, by 5.3 and 3.2, there is a $\delta_0 > 0$ such that each $g\colon \partial J^k \to C(([0,2], 2), (Y, y_0))\setminus\{\hat{y}_0\}$ with $\hat{\varrho}(g(x), \hat{y}_0) < 2\delta_0$ for $x \in \partial J^k$ admits an extension $g\colon J^k \to C(([0,2], 2), (Y, y_0))\setminus\{\hat{y}_0\}$ with $\hat{\varrho}(g(x), \hat{y}_0) < \varepsilon_0$ for $x \in J^k$. Since $Y \in$ ANR($\mathfrak{M}$), there exists further a $\delta > 0$ such that the δ-ball of Y centred at y_0 can be deformed to y_0 inside the δ_0-ball centred at y_0. We shall show that δ satisfies (*). Indeed, if f is as in (*), then there exists a $w\colon J^k \to Y$ with $w(x) = f(x)(1)$ for $x \in \partial J^k$, $w(0) = y_0$, and $\varrho(w(x), y_0) < \delta_0$ for $x \in J^k$. Letting

$$g(x)(t) = \begin{cases} f(x)(t) & \text{if } t \in [0, 1], \\ w((2-t)x) & \text{if } t \in [1, 2], \end{cases}$$

we get a $g\colon \partial J^k \to C(([0, 2], 2), (Y, y_0))$ with $\hat{\varrho}(g(x), \hat{y}_0) < 2\delta_0$ for $x \in \partial J^k$. Since $\hat{y}_0 \notin \operatorname{im}(g)$, g admits an extension $\bar{g}\colon J^k \to C(([0, 2], 2), (Y, y_0))\setminus\{\hat{y}_0\}$ with $\hat{\varrho}(\bar{g}(x), \hat{y}_0) < \varepsilon$ for all $x \in J^k$. If we let $h_r(t) = (-2r+3)t$, $t \in I$, then

$$\bar{f}(rx) = \begin{cases} g(x) \circ h_r & \text{if } x \in \partial J^k,\ r \in [\tfrac{1}{2}, 1], \\ \bar{g}(2rx) \circ h_{1/2} & \text{if } x \in J^k,\ r \in [0, \tfrac{1}{2}], \end{cases}$$

defines the extension required in (*).

5.5. Theorem. *Let X and $X_1, \ldots, X_n \subset X$ be separable complete* ANR($\mathfrak{M}$)*'s, let A be a compactum and $A_1, \ldots, A_n$ its disjoint closed subsets, and let U be an open subset of X whose boundary is compact and collared in $\bar{U}$. If either $U \cap (A_1 \cup \ldots \cup A_n) = \emptyset$ and X has no isolated points or $U \subset A_1$ and X_1 has no isolated points, then the space $S = \{f \in C(A, X)\colon f(A_i) \subset X_i$ for $i = 1, 2, \ldots, n\}$ is an l_2-manifold.*

Proof. Let $K = \{f \in S\colon f$ is constant on $U\}$. It is known that $S\setminus K$ is an l_2-manifold and S is a complete separable ANR($\mathfrak{M}$)-space (see [31], § 4). Therefore it remains to show that K is a Z-set in S.

To this end fix $\varepsilon > 0$ and $f\colon I^\infty \times A \to X$ such that $f_q = f(q, \cdot) \in S$ for all $q \in I^\infty$. By assumption there exist a compactum C in U and a homotopy $(u_t)\colon A \to A \times \{0\} \cup C \times I$ such that $u_t(a) = (a, 0)$ if $a \notin U$ or $t = 0$ and $u_t(A) = A \times \{0\} \cup C \times [0, t]$ for all $t \in I$. Define $\bar{f}\colon I^\infty \times (A \times \{0\} \cup C \times I) \to X$ by

$$\bar{f}(q, z) = \begin{cases} f(q, z) & \text{if } q \in I^\infty,\ z \in A = A \times \{0\}, \\ v(f(q, c))(t) & \text{if } q \in I^\infty,\ z = (c, t) \in C \times (0, 1], \end{cases}$$

where v satisfies 5.4 with $Y = X$ if $U \cap (A_1 \cup \ldots \cup A_n) = \emptyset$ and with $Y = X_1$ if $U \subset A$. Choose $\delta > 0$ such that $\hat{\rho}(\bar{f}(u_\delta \times \mathrm{id}), f) < \varepsilon$ and define $g\colon I^\infty \times A \to X$ by

$$g(q, a) = \begin{cases} \bar{f}(q, u_\delta(a)) & \text{if } u_\delta(a) \in A \times \{0\}, \\ \bar{f}(q, (c, t/\delta)) & \text{if } u_\delta(a) = (c, t) \in C \times I. \end{cases}$$

One easily verifies that $g_q = g(q, \cdot) \in S\setminus K$ and $\hat{\rho}(g_q, f_q) < 2\varepsilon$ for all $q \in I^\infty$. This shows that K is a Z_∞-set in Y.

5.6. Corollary. *Let X and $X_1, \ldots, X_n \subset X$ be complete separable* ANR($\mathfrak{M}$)*'s, where X has no isolated points. If A is a connected compact finite-dimensional manifold (with or without boundary), then for any closed mutually disjoint proper subsets $A_1, \ldots, A_n$ of A the space $\{f \in C(A, X)\colon f(A_i) \subset X_i$ for $i = 1, 2, \ldots, n\}$ forms an l_2-manifold. In particular, the space of paths from X_1 to X_2 and the space of closed curves starting from X_1 are l_2-manifolds.*

Appendix. Locally homotopy negligible sets and UV$^\infty$-maps. We shall show here how the properties of l. h. negligible sets are related to the results of Armentrout–Price, Kozlowski and Lacher on cell-like mappings of metric spaces.

All spaces are assumed to be metrizable. If $f\colon X \to Y$ is a map, then by the mapping cylinder of f we mean the space $Z_f = X \times [0, 1) \cup Y \times \{1\}$ equipped with the topology generated by open subsets of $X \times [0, 1)$ and by sets $f^{-1}(U) \times (t, 1) \cup U \times \{1\}$, where $t > 0$ and $U \subset Y$ is open. Note that Z_f is metrizable: if we consider X and Y as bounded subsets of normed spaces E and F respectively, then

$$Z_f \cong \{(x - tx, t, tf(x))\colon t \in I,\ x \in X\} \cup \{0\} \times \{1\} \times Y \subset E \times I \times F.$$

We identify X with $X \times \{0\}$, Y with $Y \times \{1\}$, and we denote by $p\colon Z_f \to Y$ and $q\colon Z_f\setminus Y \to X$ the collapse and projection, respectively.

A map $f\colon X \to Y$ will be said to be UV^n at $y \in Y$ if, given $k < n+1$ and a neighbourhood U of y, there is a neighbourhood $V \subset U$ of y such that each $g\colon \partial I^k \to f^{-1}(V)$ extends to an $g\colon I^k \to f^{-1}(U)$. If f is UV^n at all $y \in Y$, then we say that it is a UV^n-map. Similarly if the projection $X \to X/A$ is UV^n at [A], then we say that A is a UV^n-subset of X.

6.1. Remark. f is a UV^n-mapping iff $Z_f\setminus Y$ is LC^n rel. Z_f at each point of Y. If all the $f^{-1}(y)$'s are compact and f is a surjection, then f is a UV^n-map iff all the $f^{-1}(y)$'s, $y \in Y$, are UV^n-subsets of X.

It is known that compacta of trivial shape are UV$^\infty$-subsets of ANR($\mathfrak{M}$)'s in which they lie (see [5]).

6.2. Proposition (compare [27], [22], [4]). *If $f\colon X \to Y$ is a* UV^n*-map and $f(X)$ is dense in Y, then f induces an isomorphism of the n-th homotopy group.*

Proof. Apply 2.8 and the fact that f induces an isomorphism of the nth homotopy group iff the inclusion $Z_f\setminus Y \to Z_f$ does so.

6.3. Proposition. *Let $f\colon X \to Y$ be a* UV$^\infty$*-map with a dense image and let $M \in$* ANR($\mathfrak{M}$)*. Then, given $u\colon M \to Y$ and $\alpha\colon M \times (0, 1] \to (0, \infty)$, there is* *a* $g\colon M\times(0,1]\to K$ *such that* $\varrho(fg_t(x), u(x))<\alpha(x,t)$ *for* $(x,t)\in M\times(0,1]$. *If, in addition,* $K\subset X$ *is a closed set, U is its neighbourhood and* $v\colon U\to X$ *is any lifting of* $u|U$, *then g may be constructed in such a way that* $g_t|K = v|K$ *for all t.*

Proof. Put on Z_f a metric d in which the collapse $p\colon (Z_f, d)\to(Y,\varrho)$ is a contraction and let $\lambda\colon M\to[0,1]$ satisfy $\lambda|K=1$ and $M\setminus U\subset \operatorname{int}\lambda^{-1}(0)$. Define $w\colon M\to Z_f$ by

$$w(x) = \begin{cases}(v(x),\lambda(x))\in X\times[0,1] & \text{if } \lambda(x)>0,\\ u(x) & \text{if } \lambda(x)=0.\end{cases}$$

Since, by 2.8, Y is l. h. negligible in Z_f, there exists an α-homotopy $(h_t)\colon M\to Z_f$ such that $h_t(M)\subset Z_f\setminus Y$ and $h_t|K=w$ for all $t>0$. We let $g_t = qh_t$.

6.4. Proposition. *Let* $f\colon X\to Y$ *be an* UV^∞*-map of* $\mathrm{ANR}(\mathfrak{M})$*'s and assume that* $f(X)$ *is dense in Y. Then, given* $\alpha\colon Y\times(0,1]\to(0,\infty)$, *there exist* $g\colon Y\times(0,1]\to X$ *and a homotopy* $(h_t)\colon X\to X$ *such that* $h_0=\mathrm{id}$, $h_1 = g_1 f$ *and* $\varrho(fg_t(y),y)<\alpha(y,t)$ *and* $\varrho(fh_t(x), f(x))<\alpha(f(x),t)$ *for all* $t\in(0,1]$, $x\in X$, $y\in Y$.

Proof. Let λ be any increasing homeomorphism of $[-1,2]$ onto $[0,1]$. By 6.3 there is a $g\colon Y\times(0,1]\to X$ such that, for all $(y,t)\in Y\times(0,1]$,

$$\varrho(fg_t(y),y)<\tfrac12\min(\alpha_t(y), t, \alpha_{\lambda(t)}(y), \inf\{\alpha_s(y)\colon s\in\lambda([1,2])\}).$$

Let $M = X\times[-1,2]$, $K = X\times\{-1,2\}$, $U = X\times([-1,0)\cup(1,2])$, and define $u\colon M\to Y$ by

$$u_t = \begin{cases} f & \text{if } t\in[-1,0],\\ fg_t f & \text{if } t\in(0,1],\\ fg_1 f & \text{if } t\in[1,2].\end{cases}$$

Using 6.3 again, construct $\tilde h\colon M\to X$ with $\tilde h_{-1}=\mathrm{id}$, $\tilde h_2 = g_1 f$ and

$$\varrho(f\tilde h_t(x), u_t(x))<\tfrac12\alpha_{\lambda(t)}(f(x)) \quad \text{for} \quad (t,x)\in M = X\times[-1,2].$$

Finally, let $h_t = \tilde h_{\lambda^{-1}(t)}$.

6.5. Remark. Let $f\colon X\to Y$ be a UV^n-map with a dense image and assume that X is an LC^n-space and $\dim(Y)\leqslant n<\infty$. It easily follows from 6.1 and 2.8 that Y is LC^n and therefore $Y\in\mathrm{ANR}(\mathfrak{M})$ by [6], p. 122.

6.6. Remark. Let $f\colon X\to Y$ be an UV^{n-1}-map with a dense image and assume that X and Y are $\mathrm{ANR}(\mathfrak{M})$'s and $\max(\dim(Y), \dim(X)+1)\leqslant n<\infty$. Then, $\dim(Z_f)\leqslant n$ and Z_f is locally contractible, and therefore $Z_f\in\mathrm{ANR}(\mathfrak{M})$ (see [21], p. 168). Hence, by 3.2, Y is l. h. negligible in Z_f and f is actually a UV^∞-map; thus 6.4 applies.

We also observe that if X and Y are locally compact spaces and f is a proper map, then the homotopies $\mathrm{id}\cup(fg_t)_{t>0}$ and (h_t) of 6.4 are proper if α is taken sufficiently small (slightly weaker versions of 6.5 and 6.6 form the theorems of Lacher [23]).

6.7. Corollary. *Let* $f\colon X\to Y$ *be a surjection such that all the* $f^{-1}(y)$*'s,* $y\in Y$, *are compact* UV^n*-subsets of X. If X is an n-dimensional* $\mathrm{ANR}(\mathfrak{M})$*-space and Y is finite-dimensional, then* $Y\in\mathrm{ANR}(\mathfrak{M})$ *and f is a* UV^∞*-map.*

Proof. If $n=\infty$ then the result follows from 6.5. Assume $n<\infty$, fix $y_0\in Y$ and consider the quotient map $\pi\colon X\to X/f^{-1}(y_0) = S_{y_0}$. By 6.5 we have $S_{y_0}\in\mathrm{ANR}(\mathfrak{M})$ and therefore, by 6.6, π is a UV^∞-map. Thus all the $f^{-1}(y)$'s, $y\in Y$, are UV^∞-subsets of X and the assertion follows from 6.1 and 6.5.

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Hilbert cube modulo an arc

by

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Abstract. Let Q denote the Hilbert cube and let $\alpha, \beta \subset Q$ be arcs. Adapting methods of Bing–Andrews–Curtis–Kwun–Bryant we prove that $Q/\alpha \times I$ and $Q/\alpha \times Q/\beta$ are homeomorphic with Q, where I is a closed interval and Q/α is a space obtained from Q by shrinking α to a point. The same method applies equally well to the case when arcs are replaced with finite-dimensional cells or their intersections.

1. Introduction. We use Q to represent the Hilbert cube (the countable-infinite product of closed intervals). A closed subset $X \subset Q$ is called a *Z-set* if for any non-empty homotopically trivial open set $U \subset Q$, $U - X$ is also non-empty and homotopically trivial. This concept was introduced by R. D. Anderson in [1] and in the infinite-dimensional topology plays a role analogous to a role of tameness conditions in the finite-dimensional topology. Chapman [7] showed that a Z-set $X \subset Q$ has a trivial shape if and only if the space Q/X, obtained from Q by shrinking X to a point, is homeomorphic to Q (in notation, $Q/X \cong Q$). If X is of a trivial shape but not a Z-set, then Q/X may fail to be locally like Q at the point $\tilde{X} = p(X)$, where $p\colon Q \to Q/X$ is a natural projection. Indeed, Wong [14] constructed a copy of the Cantor set with non-simply connected complement in Q. By a standard technique we can pass an arc α through it such that $Q - \alpha$ is also not simply connected. If Q/α were locally Q at the point $\tilde{\alpha}$, then Q/α being a contractible Q-manifold would be homeomorphic to Q [8]. But in Q the complement of every point is simply connected.

The problem SC 1 in [2] asks (in analogy with a similar result for Euclidean spaces established earlier by Andrews and Curtis [3]) whether for any arc $\alpha \subset Q$ multiplying Q/α by the unit interval $I = [0, 1]$ gives the Hilbert cube. In Section 2 of this note we will present a detailed proof, adapting techniques from [3] to the Hilbert cube case, of the following theorem that confirmes this conjecture.

THEOREM 1. *For any arc $\alpha \subset Q$, $(Q/\alpha) \times I$ is homeomorphic with Q.*

Next, in Section 3, we first prove that $A \times B$ is a Z-set in $Q \times Q$ whenever A and B are finite-dimensional closed subsets of Q and then, following Kwun's method [10], establish

THEOREM 2. *Let $\alpha, \beta \subset Q$ be arbitrary arcs. Then $(Q/\alpha) \times (Q/\beta)$ is homeomorphic with Q.*