Concerning locally homotopy negligible sets and characterization of $l_2$-manifolds

by

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Abstract. Let $A \subseteq X$ be a set such that for every open subset $U$ of $X$ the inclusion $U \setminus A \rightarrow U$ is a weak homotopy equivalence. The following two facts are shown: (A) If $X$ is an ANR(3R)-space, then so is $X \setminus A$; (B) if $A$ is closed in $X$, $X$ is complete-metrizable and $X \setminus A$ is an $l_2$-manifold, then so is $X$. We apply (B) to prove that if $X$ is a separable complete ANR(3R) without isolated points, then the space of paths in $X$ forms an $l_2$-manifold.

Initially the paper was intended to present the proofs of the following two facts, which had been announced, or employed, in [31] and [32]:

(A) If $X$ is a complete separable ANR(3R)-space and $A$ is a countable union of $Z$-sets in $X$, then $X \setminus A \in$ ANR(3R), and

(B) If $X$ is as above, $A$ is a $Z$-set in $X$ and $X \setminus A$ is an $l_2$-manifold, then $X$ is also an $l_2$-manifold.

By a $Z$-set in $X$ we mean here any closed set $A \subseteq X$ with the property that every map $f : [0, 1]^n \rightarrow X$ is a uniform limit of $X \setminus A$-valued maps.

If we use results of infinite-dimensional topology, (A) has a very short proof: by [31], the space $X \times l_2$ is an $l_2$-manifold which clearly contains $A \times l_2$ as a countable union of $Z$-sets; thus, by [2], $X \times X \setminus A \times l_2$ is homeomorphic to $X \times l_2$ and hence $(X \setminus A) \times l_2$ and $X \setminus A$ are ANR(3R)'s. However, the assumption of (A) seems to be too restrictive: for instance, (A) does not include the fact that if $A$ is any subset of the boundary of the square $[0, 1]^2$ (and need not be of type $G_\delta$), then $[0, 1]^2 \setminus A$ is an ANR(3R). (See Fox [15]). Therefore we prove here in § 3 a result more general than (A), namely

(A') If $X \in$ ANR(3R) and $A \subseteq X$ is locally homotopy negligible in $X$ (i.e., for every open set $U \subseteq X$ the inclusion $U \setminus A \rightarrow U$ is a weak homotopy equivalence), then $X \setminus A \in$ ANR(3R).

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Since the properties of non-closed locally homotopy negligible sets have never been explicitly formulated, we devote a section of the paper to presenting the basic facts concerning such sets (see § 2). We note that most of these facts and also of the methods used in their proofs are similar to those of Anderson [14], Ells and Kuiper [11] and Henderson [19] (see also Eilenberg and Wilder [12], Smale [29], Haver [16]); however, several technical changes have to be made if one wants to dispense with the assumption that \( A \) is closed and \( X \) is an ANR(39). The material of § 2 allows us to strengthen the results of [4], [22] and [23] on cell-like mappings of metric spaces (see the Appendix); also, we hope that the study of non-closed locally homotopy negligible sets in concrete spaces can be used to prove that these spaces are ANR(39)'s or infinite-dimensional manifolds.

The result (B), stated before, is proved in § 5 and then applied to show that, for \( Y \) a complete separable connected ANR(39) and \( K \) a polyhedron, the space of maps \( K \rightarrow Y \) is an \( 1 \)-manifold.

In the paper we discuss also an elementary characterization of ANR(39)'s which is used in the proof of (A) (see § 1). Let us note that (A) can also be established by using a characterization of Dowker and Hanner [10]; nevertheless the result of § 1 seems to be of independent interest (for instance, it unifies earlier results of Wojdyslawski [34], Dugundji [9], Himmelberg [20] and others).

Note: By \( f \) we denote the interval \([0,1]\), by \( N \) the set of integers; continuous functions are called "maps". A homotypy \( h: X \rightarrow Y \) is often denoted by \( (h) = h(x,t) \). All spaces are assumed to be normal, and if \( X \) is a metrizable space then \( g \) usually denotes a metric which induces the topology of \( X \).

For \( k \in N \cup \{0 \} \cup \{\infty\} \) and \( i \leq n+1 \) means "not if \( n \neq \infty \) and \( i \neq \infty \) if \( n = \infty \)."

§ 1. A characterization of ANR(39)'s. If \( \mathfrak{R} \) and \( \mathfrak{B} \) are families of subsets of \( X \) then by \( \mathfrak{R} \) we denote the family of all sets \( A \in \mathfrak{R} \) which refine \( \mathfrak{B} \).

Suppose that \( X \) and \( Z \) are spaces, \( \mathfrak{A} \) is a family of subsets of \( X \) and that to certain sets \( A \in \mathfrak{A} \) we have assigned a map \( f_A \) from a non-empty set \( \text{dom}(f_A) \subset Z \).

Given \( \mathfrak{R} \subset \text{cov}(X) \), we say that \( ((f_A), Z) \) is a \( \mathfrak{B} \)-fine admissible approximation to \( \mathfrak{A} \) if there is a \( \mathfrak{B} \subset \text{cov}(X) \) such that the following conditions are satisfied:

(i) if \( A \in \mathfrak{R} \) then \( f_A \) is defined, \( \mathfrak{A} \cup \text{im}(f_A) \) refines \( \mathfrak{B} \) and \( F_A = \text{dom}(f_A) \)

is a homotopy trivial subset of \( Z \);

(ii) if \( \mathfrak{A} \subset \mathfrak{R} \) and \( A \subset B \) then \( f_A \subset f_B \). 

We sometimes say that \( ((f_A), Z) \) is a continuous approximation to \( \mathfrak{A} \) if it is \( \mathfrak{B} \)-fine for all \( \mathfrak{B} \subset \text{cov}(X) \).

If \( Z = X \) and each \( f_A \) is an inclusion, then we say that the approximation is trivial; trivial approximations will be denoted by \( \{F_A\} \), where \( F_A = \text{dom}(f_A) \subset X \).

The aim of this section is to prove the following:

1.1. Theorem. The following conditions are equivalent for a metric space \( X \):

(a) \( X \in \text{ANR}(39) \);

(b) there exists a space \( E \) such that \( X \times E \) has an open basis with all finite intersections of its members being homotopy trivial,

(c) there exist continuous admissible approximations to the family of all finite subsets of \( X \);

(d) there exist arbitrarily fine admissible approximations to the family of all finite subsets of \( X \times (0,1) \).

For simplicity the family of all finite subsets of \( X \) will be denoted by \( \mathcal{F}(X) \) and the family of all subsets of \( X \) by \( \mathcal{F}(X) \).

Remark. The implication (c)\(\Rightarrow\)(a) of 1.1 generalizes earlier results of Dugundji [9] and Himmelberg [20] stating that metric spaces which admit "nice" equiconnecting functions are ANR(39)'s; see also Milnor [24]. In fact, if \( \lambda \) is an equiconnecting function on \( X \), then, letting \( A_1 = A \) and inductively

\[ A_{i+1} = \{\lambda(x,y,t) : x \in A_i, y \in A_i, t \in I\} \]

and \( F_A = \bigcup A_n \), we get a trivial approximation to \( \mathcal{F}(X) \) which is continuous in the situations considered in [9] and [20]. (Note that \( F_A \) is contractible whenever \( A \) is defined.)

Remark. Admissible approximations to \( \mathcal{F} = \mathcal{F}(X) \) can be obtained as follows. Let \( \mathfrak{B} \subset \text{cov}(X) \), then \( K \) denote the simplicial complex of all \( \{x_1, \ldots, x_n\} \subset \mathfrak{B} \) and suppose that there is a given map \( f: |K| \rightarrow X \). Then, letting \( Z = |K| \) and \( f = f_0 \) for \( \sigma \in F = \mathfrak{B} \), we get an approximation to \( \mathcal{F} \) which is continuous if for each \( x \in X \) and a neighbourhood \( U \) of \( x \) there is a neighbourhood \( V \subset U \) of \( x \) such that \( f([\sigma]) \subset U \) for all \( \sigma = \{x_1, \ldots, x_n\} \subset V \).

In particular, "the convex structures" of [26] yield continuous approximations of this type and therefore 1.1 generalizes the results stating that spaces which admit convex (or similar) structures are ANR(39)'s (see Himmelberg [20] and Wojdyslawski [34]).

In the proof of 1.1 we need the following lemmas:

1.2. Lemma. Let \( Y \) be a metric space and \( Y_0 \) its dense subset. If there are arbitrarily fine admissible approximations to \( \mathcal{F}(Y_0) \), then there are also arbitrarily fine admissible approximations to \( \mathcal{F}(Y) \).

Proof. Fix \( \mathfrak{B} \subset \text{cov}(Y) \), let \( \mathfrak{B}_0 \subset \text{cov}(Y_0) \) be a star-refinement of \( \mathfrak{B} \) and let \( ((f_A), Z) \) be a \( \mathfrak{B} \)-fine admissible approximation to \( \mathcal{F} = \mathcal{F}(Y) \). We assume without loss of generality that \( \mathfrak{B} \) refines \( \mathfrak{B}_0 \). Let \( \mathfrak{B} \subset \text{cov}(Y) \) be a locally finite star-refinement of \( \mathfrak{B} \) and let \( \mathfrak{B} \subset \text{cov}(Y) \)(B) be a refinement of \( \mathfrak{B} \) such that each element of \( \mathfrak{B} \) intersects only finitely many elements of \( \mathfrak{B} \). For each \( W \in \mathfrak{B} \) pick an \( y_W \in X_0 \cap W \) and, given \( S \subset \mathcal{F}_n \), let \( S = f_S \), where

\[ S = \{y_W : W \in \mathfrak{B} \text{ and } W \cap S \neq \emptyset\}. \]
It is easy to see that \( \{(b_y; s \in \mathcal{F}_y), Z\} \) is a \( \mathcal{U} \)-fine approximation to the family \( \mathcal{F} \).

1.3. Lemma. Let \((Y, \mathcal{B})\) be a metric space such that there exist arbitrarily fine admissible approximations to \( \mathcal{F} = \mathcal{F}(Y) \). Then, given \( a: Y \rightarrow (0, \infty) \), there are simplicial complexes \( K \) and maps \( f: Y \rightarrow |K| \) and \( g: |K| \rightarrow Y \) such that \( g(\mathcal{F}(Y)) \subset \alpha(\mathcal{F}) \) for all \( y \in Y \).

Proof. Replacing, if necessary, \( \mathcal{B} \) by \( \mathcal{B} \), we may assume that \( |\mathcal{F}(Y)| \leq \alpha(\mathcal{F}) \). Let \( Y = \left\{ \{y_x, \hat{z}(x)\} \mid y \in Y \right\} \) be the cover of \( Y \) by open balls and let \( \{(j_x; s \in \mathcal{F}_y), Z\} \) be a \( \mathcal{U} \)-fine admissible approximation to \( \mathcal{F} \), \( \mathcal{B} \) being locally finite. Let \( K \) denote the nerve of \( \mathcal{B} \) and for each \( \sigma = \{V_1, \ldots, V_k\} \in K \) let

\[ I(\sigma) = \left\{ \langle f_{r, \ldots, n}(s), \hat{z} \rangle \mid \begin{array}{c} x \in \text{dom}(f_{r, \ldots, n}(s)) \end{array} \right\} = Y \times Z. \]

Clearly, \( I(\sigma) \) is an anti-monotone function from \( K \) to the non-empty homotopy trivial subsets of \( Y \times Z \). Let \( \sigma \) be the subdivision map for each \( \sigma \in K \) we denote by \( \sigma \) its barycentric subet; \( \sigma \) is then a vertex of \( K \).

Sublemma. There is a map \( \eta: |K'| \rightarrow Y \times Z \) such that

\[ \eta(\{r_1, \ldots, r_n\}) = I(\sigma). \]

Proof. For each vertex \( \delta \) of \( K' \) choose a point \( \hat{y}_\delta(\delta) \in I(\sigma) \). Let \((L, \tilde{\mathcal{B}})\) be a maximal pair (under the natural ordering) such that \( L \) is a subcomplex of \( K' \) containing all vertices of \( K' \) and \( \mathcal{B} \). Assume the contrary and let \( \sigma \in K \setminus L \) be maximal in minimal dimension. Then \( |\mathcal{B}| \in \{r_1, \ldots, r_n\} \), whence \( \eta(\delta) \in \mathcal{B} \) is well defined. Representing \( r \) as \( \{r_1, \ldots, r_n\} \), where \( \sigma r_1 \subset \sigma r_2 \subset \ldots \subset \sigma r_n \), we infer from (1) and the anti-monotony of \( I \) that \( \eta(\mathcal{B}) \in I(\sigma) \). Clearly, \( L \) is homotopy trivial, we may extend \( \eta(\mathcal{B}) \) to a \( g: |\mathcal{B}| \rightarrow \eta(\mathcal{B}) \), which is impossible; thus \( L = K' \) and \( g \) is as required.

Proof of 1.3 (continued). Let \( g = g = \hat{y} \). Given \( y \in Y \), let \( \{V_1, \ldots, V_n\} \in \mathcal{B} \). Observe that, by (1), we have

\[ g(\{V_1, \ldots, V_n\}) = \bigcup_{i=1}^n I(F_i(\hat{z}(y))). \]

Since \( \text{im}(f_\hat{y}) \cup V_i \) defines \( \mathcal{U} \) for \( i = 1, 2, \ldots, n \), we infer that \( g(\{V_1, \ldots, V_n\}) \) is contained in the star of \( y \) in \( \mathcal{U} \). Therefore, if \( f: Y \rightarrow |K| \) is induced by a partition of unity \( \langle \mathcal{A}_y; s \in \mathcal{W} \rangle \) with each \( \mathcal{A}_y \) vanishing outside \( V \), then \( g(\mathcal{F}(Y)) \subset \alpha(\mathcal{F}) \) for all \( y \in Y \).

This easily yields \( g(\mathcal{F}(Y)) \subset \alpha(\mathcal{F}) \) for all \( y \in Y \).

(1) \( \hat{y} \) denotes the boundary of \( \mathcal{F} \).

The following lemma is actually a special case of a theorem of Dowker and Hanner (see [10], p. 105); however, we include a short proof of it, which will be used later.

1.4. Lemma. Let \((X, \mathcal{B})\) be a metric space and assume that there are simplicial complexes \( K \) and maps \( f: X \rightarrow [0, 1] \) and \( g: |K| \rightarrow X \) such that \( g(\mathcal{F}(X)) \subset \alpha(\mathcal{F}) \) for all \((x, \alpha) \in X \times [0, 1] \). Then \( X \in \mathcal{ANR}(\mathcal{B}) \).

Proof. Let \( h: A \rightarrow X \) be a map of a closed set \( A \) of a metric space \( (X, \mathcal{B}) \); we shall construct a neighbourhood extension of \( h \).

For this purpose let \( u = h \times id: A \times [0, 1] \rightarrow X \times [0, 1] \). Since simplicial complexes are neighbourhood extendors for metric spaces, \( f \times u \) admits an extension \( v: U \rightarrow |K| \), where \( U \subset B \times [0, 1] \) is an open set containing \( A \times [0, 1] \) (see [21], p. 105). Without loss of generality we may assume that \( U \) is contained in the set \( \{0, 1\} \times U \); there is an \( a \in A \) with \( g_{(a, b)}(b) < c \) and \( g_{(a, b)}(b, 1, h(b)) < t \), which, by our assumptions, is a neighbourhood of \( A \times [0, 1] \). Let \( \mathcal{B}: [0, 1] \rightarrow \mathcal{B} \) be such that \( \mathcal{B}(0) = 0 \) and \( \mathcal{B}(1) = \mathcal{B}(b, b) \subset U \cup (U \setminus \mathcal{B}(0)) \times \{1\} \). It can easily be constructed by using Klee's theorem and the fact that for each \( b \in [0, 1] \) there is a closed neighbourhood \( V \) of \( A \) in \( B \) with \( W \times [a_1, 1] = V \). We let \( v = (b \in B: \mathcal{B}(b) < 1) \) and define \( h: A \rightarrow X \) by

\[ h(b) = \begin{cases} h(b), & b \in A, \\ g_{(b, \mathcal{B}(b))}(b, \mathcal{B}(b)), & b \in B \setminus \mathcal{B}(1). \end{cases} \]

It is easily seen that \( h \) is continuous.

Now we complete the proof of 1.1. To show that \( (a) \Rightarrow (c) \) consider \( X \) as a closed subset of a convex set \( Z \) in a normed linear space \( (Z, \mathcal{B}) \), p. 81) and for sufficiently small sets \( A \subset X \) let \( F_A = \mathcal{conv}(A) = Z \) and \( f_A = r \circ F_A \), where \( r \) is a neighbourhood retraction onto \( X \). \( (c) \Rightarrow (d) \) is a consequence of the fact that any pair of continuous admissible approximations to \( \mathcal{F}(X) \) and to \( \mathcal{F}(Y) \) induces a continuous product approximation to \( \mathcal{F}(X \times Y) \). Further, \( (d) \Rightarrow (a) \) by Lemmas 1.2.1-4, and \( (a) \Rightarrow (b) \) by a result of [31] stating that if \( X \in \mathcal{ANR}(\mathcal{B}) \), then there exists a normed linear space \( E \) with \( X \subset E \) homeomorphic to an open subset of \( E \). Finally, \( (b) \Rightarrow (a) \) follows from the implication \( (c) \Rightarrow (a) \) and the following fact applied to \( Y = X \times E \).

Sublemma. If \( Y \) has a base \( \mathcal{B} \) with homotopy trivial intersections, then there exist trivial continuous approximations to \( \mathcal{F}(Y) \).

Proof. For each \( n \in \mathbb{N} \) let \( \mathcal{B}_n \in \mathcal{B}(X) \) be a locally finite refinement of \( \mathcal{B} \) with \( \text{diam}_{\mathcal{F}_n}(U) \leq 1/n \) for all \( U \in \mathcal{B}_n \). Let \( \mathcal{B} = \bigcup \mathcal{B}_n \), let \( U \rightarrow V \) be a function of \( B \) into \( H \) such that \( V \subset U \), for all \( V \in \mathcal{B} \), and for sufficiently small \( A \subset Y \) let

\[ F_A = \bigcup_{V \in \mathcal{B}(A)} U, \quad \mathcal{B}(A) = \{V \in \mathcal{B} : A \subset V \}. \]

It is easy to see that \( \mathcal{B}(A) \) is finite for all \( A \subset Y \) and \( F_A \) is a continuous trivial approximation to \( \mathcal{F}(Y) \).
1.5. Corollary. Let $X$ be a separable complete metric space which is $I^{-}$-stable (i.e., $X \times [0, 1] \cong X$). Suppose that there exist arbitrarily fine admissible approximations to $\mathcal{F}(X)$, where $X \subset X$ is a dense subset. Then $X$ is an $I^{-}$-manifold.

Proof. By a theorem of Klee we have $I^{-}(\mathbb{N}) \subseteq (\mathbb{Q})$ (see [31]) and therefore $X \times [0, 1] \cong X \times [0, 1]$ and $X \times [0, 1] \cong X$. Hence, by 1.1 and 1.2, $X \in ANR^{\mathbb{N}}$ and thus, by [31], $X \times [0, 1]$ is an $I^{-}$-manifold. Since $X \times [0, 1] \cong X$, then result follows.

Clearly, the conditions of 1.5 are also necessary for a connected space $X$ to be an $I^{-}$-manifold (recall that each separable $I^{-}$-manifold is $I^{-}$-stable and is homeomorphic to an open subset of $I^{-}$, see [3] and [31]).

§ 2. Locally homotopy negligible sets.

2.1. Definition. A set $A \subset X$ will be said to be locally $n$-negligible if, given $x \in X$, $k \geq n+1$ and a neighbourhood $U$ of $x$, there is a neighbourhood $V \subset U$ of $x$ such that for each $f: (I^k, \partial I^k) \to (V, \partial V)$ there is a homotopy $(h_t): (I^k, \partial I^k) \to (U, \partial U)$ with $h_0 = f$ and $h_1(I^k) \subset U \setminus A$. Locally $\infty$-negligible sets will also be called locally homotopy negligible (briefly: l. h. negligible).

The aim of this section is to discuss certain properties of l. h. negligible sets; we formulate the corresponding results for locally $n$-negligible sets with $n < \infty$ only if their proofs require no extra work.

2.2. Remark. Let $A$ be a locally $n$-negligible set in $X$. Then

(a) For every space $E$, $A \times E$ is locally $n$-negligible in $X \times E$.

(b) For every open set $U \subset X$, $U \cap A$ is locally $n$-negligible in $U$.

2.3. Theorem. Let $A \subset X$, where $X$ is normal. The following conditions are equivalent:

(a) $A$ is locally $n$-negligible in $X$.

(b) Given $x \in X$, $n \geq 0$, a pseudometric $q$ on $X$ and a map $f: (K, |K|) \to (X, X \setminus A)$, where $|K| \to X$ is a finite simplicial pair with $\dim(|K|) < n + 1$, there is a homotopy $(h_t): |K| \to X$ such that $h_t = f, h_1(|K|) \subset X \setminus A, h_1(x) = f(x)$ for $(x, t) \in |K| \times I$, and $q(h_t(x), f(x)) < \varepsilon$ for $(x, t) \in |K| \times I$.

(c) Given a simplicial pair $(K, \partial K)$ with $\dim(K) \leq n$, a pseudometric $q$ on $X$ and maps $\alpha, \beta: |K| \to [0, \infty)$ and $f: |K| \times \{0\} \cup [0, \infty) \times I \to X$ with $q(f(x, t), f(x, 0)) < \alpha(x)$ and $f(x, 1) \notin \partial A$ for all $(x, t) \in |K| \times I$, such that $f(x, t) \in X$ which extends $f$ and satisfies $q(f(x, t), f(x, 0)) < \varepsilon(x)$ and $f(x, 1) \notin \partial A$ for all $(x, t) \in |K| \times I$.

(d) For each open $U \subset X$ and $n < \infty$ the relative homotopy group $\pi_n(U, \partial U) \to X$ vanishes.

(e) Each $x \in X$ has a basis $\mathcal{U}_x$ of open neighbourhoods with $\pi_n(U, \partial U) = 0$ for all $U \in \mathcal{U}_x$ and $n < \infty + 1$.

Proof. (a) $\Rightarrow$ (b). Let $(b_n)$ denote the condition obtained from (b) with "$\dim(|K|) < n + 1" \ replaced by "$\dim(|K|) < n"; \ then \ we \ shall \ show \ that \ (a) \Rightarrow (b_n)$ for $0 \leq n < \infty + 1$. Assume that $(a) \Rightarrow (b_n)$ has been established (evidently $(a) \Rightarrow (b_0)$) and let $K, \partial K, f$ and be as in $(b_n)$. Given $x \in (0, \mathbb{N})$, cover the compact set $f(|K|)$ by open sets $V_1, \ldots, V_k$ such that for each $i \leq k$, there is a homotopy $h_i: (I^k, \partial I^k) \to (X, X \setminus A)$ with $h_0 = f, h_1(I^k) \cap A = \emptyset$ and $\text{diam}_m(h_i(x)) < \varepsilon$.

Let a subdivision $(K', \partial K')$ of $(K, \partial K)$ be so fine that $\{f(x) : x \in K'\}$ refines $\{V_1, \ldots, V_k\}$ (we identify $K'$ with $[0, \infty)$, and for each $x \in K'$ let $\lambda_x: [0, \infty) \to [0, 1]$ be a map that is 1 on $f(x)$ and 0 outside a finite $\mathbb{N}$-sequence $\{V_1, \ldots, V_k\}$. Write

$$d(x, y) = \sum_{x \in K'} |\lambda_x(s) - \lambda_y(s)| + q(x, y)$$

for $x, y \in X$,

and let $M$ be the union of $U$ and the $\mathbb{N}$-sequence $\{V_1, \ldots, V_k\}$. By $(b_n)_1$, there is a homotopy $(f^M): |M| \to X$ such that $f_0 = f, f_1(|M|) \cap A = \emptyset$ and $\text{diam}_m(f^M(x)) < \varepsilon$ for all $(x, t) \in |M| \times I$.

Then for each $x \in K' \cap A$ let $\text{diam}_m(f^M(x)) < \varepsilon$ and $\text{diam}_m(f^M(x)) < \varepsilon$ for each $x \in K' \cap A$. Then the $\mathbb{N}$-sequence induces maps $f^M: |M| \times I \to X$ such that $h_0(T_x) = f^M, h_0(T_x) = f^M$ and $h_1(T_x) \to X$ such that $h_1(T_x) = f^M$ for $(x, 0) \in X \times I$ onto $T_x \times I$ such that $h_1(x) = (x, 0)$ for $x \in T_x \subset |M| \times I$ and $h_1(x) = (x, 0)$ for $x \in |M| \times I$. We let

$$h(x) = \begin{cases} h^M(x, t) & \text{if } x \in |M|, \\ f^M(x) & \text{if } x \in X \setminus |M|. \end{cases}$$

(b) $\Rightarrow$ (a). By the Kuratowski-Zorn lemma it suffices to consider the case where $K = \sigma$ is a simplex and $\partial K = \partial \sigma$. Assume that $\sigma$ is embedded in a euclidean space and for each $A \subset X$ denote by $\mathcal{L}(\sigma)$ the image of $A$ under the $\lambda$-homotopy with respect to the barycenter 0 of $\sigma$. Let $x \in \mathcal{L}(\sigma)$ satisfy $\varepsilon = \min \{d(x) : x \in |A|\}$ and $\varepsilon = \min \{d(x) - d(f(x), f(x, 0)) : (x, t) \in |\sigma| \times I\}$. Set $T = \{x \times [0, \infty) \cup |\sigma| \times I\}$ and by (b) there is an $\varepsilon$-homotopy $w: T \times I \to X$ such that $w(T) \cap A = \emptyset$, $w_0 = f$ and $w_1(x) = f(x, 1)$ for $x \in |\sigma| \times I$. Now, for each $x \in |\sigma| \times I$ let

$$A(x) = \{(x, 0) : x \in |\sigma| \times I\} \cup \{(x, t) : x \in |\sigma| \times I\},$$

where $\mu \geq 1$ is chosen so that $\mu x \in |\sigma|$. Then the inequality

$$\sup \{q(f(x, y), f(x, 0)) : y \in A(x) \} < \varepsilon$$

holds for all $x \in |\sigma| \times I$ and therefore, by compactness, there is a $x \in \{0, 1\} \setminus \{x\}$ such that (1) holds for all $x \in |\sigma| \times I$.

Let $u_k: [0, \infty) \to |\sigma| \times I$ be such a homotopy:

(i) $u_k(x) = (x, t), 0 \leq x \leq I \times \emptyset \cup |\sigma| \times \{0\}$,

(ii) $u_k(x) = (x, 0), 0 \leq x \leq |\sigma| \times \{1\}$,

(iii) $u_k(x) = (x, t), 0 \leq x \leq |\sigma| \times \{1\}$,

(iv) $p \circ u_k(x) \in A(x)$ if $x \in \{0\} \setminus \{x\} \times 1$.

Then $f: |\sigma| \times I \to X$ is defined by $f(x, t) = w_k(x))$ is the required extension of $f$. The implications (b) $\Rightarrow$ (a) and (a) $\Rightarrow$ (b) are evident. To prove that (b) $\Rightarrow$ (a), fix $f: (I^k, \partial I^k) \to (U, \partial U)$ where $U \subset X$ is open and $\mathbb{N}$-sequence 1. Let $\lambda: X \times I$ be a function that is 0 on $X \setminus U$ and 1 on $f(I)$ and let $(b_n): I^k \to X$ be a homotopy such that
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Proof. Fix \(\mathfrak{p} > 0\) and consider the space \(Y\) of all maps of \(I^* \times [0, 1] \) into \(X\), equipped with the "fine topology" generated by all sets

\[
V(g, \alpha) = \{ h \in Y : g(h(x), g(x)) < \alpha(x) \},
\]

where \(g\) is a fixed complete metric on \(X, g \in Y\) and \(\alpha\) is a map from \(I^* \times [0, 1] \) into \((0, \infty)\).

By 2.4, all the sets \(Y_\alpha = \{ g \in Y : \text{im}(g) \cap A_\alpha = \emptyset \}\) are dense and open in \(Y\). Moreover, it is easy to verify that \(Y\) has the Baire property (cf. [30]) and therefore \(Y_\alpha = \bigcap Y_\alpha\) is dense in \(Y\). For each \(f: I^* \to X\) there is a \(h \in Y_\alpha\) with

\[
g(h(x), f(x)) < \mathfrak{p}\text{ for all } (x, t) \in I^* \times [0, 1]\]

this easily completes the proof.

We conclude this section by giving a condition for a set \(A \subset X\) to be locally \(n\)-negligible. Following [12], we say that \(B \subset L\) is \(L^*\text{-rel.} X\) at a point \(x \in L\) if, given \(k < n+2\) and a neighbourhood \(V \subset U\) of \(x\), there is a neighbourhood \(V \cap U\) of \(x\) such that each \(f: \mathbb{R}^k \to \mathbb{R} \) extending to \(A\) at \(x\).

2.8. Theorem (compare [12]). Let \(X\) be a metric space and \(A \subset X\) be a set such that \(X \\setminus A\) is dense in \(X\) and \(L^*\text{-rel.} X\) at each point of \(A\). If \(n < \infty\), then \(A\) is locally \(n\)-negligible in \(X\) and each map \(f: I^* \to X\) can be approximated by maps \(f^\varepsilon: I^* \to I^* \times A\) which coincide with \(f\) on an arbitrary given compact subset \(K\). (Compare [12].)

Proof. Let \(f: K \to X\) be a fixed map of a compact polyhedron \(K\). Writing \(L = f(K) \cap A\), let for any map \(g: Z \to X\) of a compact subset \(Z\)

\[
\delta(g) = \text{diam}_Z(g(Z)) + \|g(x) - \delta_l(x)\| = \|Z\|,
\]

and we say that \(g\) is \(L\)-small if \(\delta(g) < \beta\). By a standard compactness argument there exist \(\delta_0 > 0\) and a function \(\varepsilon: (0, \delta_0] \to (0, \infty)\) with \(\lim \varepsilon(\delta) = 0\) and such that each \(L\)-small \(g: \mathbb{R}^k \to \mathbb{R}\) extends \(g\) on an \(L\)-small extension \(\tilde{g}: \mathbb{R} \subset \mathbb{R}^k \) (\(k = 0, 1, \ldots, n+1\)). Without loss of generality we can assume \(\delta_0 > 3\) and that \(\varepsilon\) is non-decreasing.

Claim (A). If \(\text{dim}(K) \leq n + 1\) then, for every \(\mu \in [0, 1]\), there exists a \(g: K \setminus X \setminus A\) such that

\[
\delta(g) < \varepsilon(3\mu) + \mu \quad \text{and} \quad g(x) = f(x) \quad \text{if} \quad f(x) \neq g(x) \quad \text{if} \quad f(x) \setminus Z > 1\).
\]

Proof. We use induction on \(\text{dim}(K)\). Suppose that (A) holds true if \(\text{dim}(K) < \rho\) (it holds if \(\text{dim}(K) = 0\) and assume \(\text{dim}(K) = p + 1 > n+1\). Let \(T\) be a triangulation of \(K\) such that \(\text{diam}_{\mathbb{R}^k}(\text{int}(K)) < \mu\) for any simplex \(\sigma \in T\) and \(\delta\) denote the \(p\)-skeleton of \(T\). Let \(g_0: |S| \to X\) be a map such that \(g_0(f_0) \cap |S| < \mu\) and \(\text{diam}_{\mathbb{R}^k}(\text{int}(K)) < \mu\) if \(f_0\) lies in a simplex of \(T\) which is disjoint from \(f_0\). Now, let \(\sigma \in T\) be an \((p+1)\)-simplex. If \(|\sigma| \cap f^{-1}(\mathbb{R}) \neq \emptyset\), then \(g_0|\sigma| = 3\mu\text{-small}\) and therefore it admits an \(3\mu\text{-small}\) extension \(g_0^\varepsilon: |\sigma| \to X\). If \(|\sigma| \cap f^{-1}(\mathbb{R}) = \emptyset\), then put \(g^\varepsilon = f^\varepsilon|\sigma|\). Clearly, \(g_0\) and the \(g^\varepsilon\)'s induce the required \(g: K \setminus X \setminus A\).

Claim (B). If \(\text{dim}(K) \leq n\), then there is a homotopy \(h_0: K \to X\) with \(h_0 = f\) and \(h_0(K) \subset X \setminus A\) for \(t \in (0, 1)\).

(1) By \(\delta\) we denote the sup-metric induced by \(g\).
3.3. COROLLARY. Let \( X \in \text{LC}^c \) be a metric space. The following conditions on a closed set \( A \subseteq X \) are equivalent:

(a) \( X \setminus A \) is dense in \( X \) and is \( \text{LC}^{-1} \) rel. \( X \) at each \( x \in A \);

(b) \( A \) is a \( Z_n \)-set in \( X \);

(c) \( A \) is locally \( n \)-negligible in \( X \).

Proof. (a) \( \Rightarrow \) (b) follows from 2.8 and (c) \( \Rightarrow \) (a) is trivial. Finally, (b) \( \Rightarrow \) (c) by the well-known properties of \( \text{LC}^c \)-spaces (see [21], p. 160).

§ 4. Enlarging an ANR(\( \mathbb{R}^n \))-open questions and remarks. Let \( X \) be a locally contractible metric space and \( A \) its l. h. negligible subset. By 3.2, \( X \in \text{ANR}(\mathbb{R}^n) \) \( \Rightarrow X \setminus A \in \text{ANR}(\mathbb{R}^n) \). We do not know whether the converse implication is true (\( \ast \)).

In this connection let ut show:

4.1. PROPOSITION. Let \( X \) be a metric space and \( A \) its ANR(\( \mathbb{R}^n \))-subset. Then, \( A \) can be enlarged to an ANR(\( \mathbb{R}^n \))-set \( \bar{A} \subseteq X \) which is of type \( G_T \) in \( X \) and has the property that \( \text{A} \) \( \bar{A} \) in \( \mathbb{R}^n \).

Proof. By well-known properties of ANR(\( \mathbb{R}^n \))'s there is a \( U \subseteq \text{cov}(A \times 0 \times 1) \) and a map \( g : [X] \to A \), where \( K \) is the nerve of \( H \), such that if \( j : A \times 0 \times 1 \to [X] \) is any canonical map, then \( g(f(x, t, \cdot), \cdot, \cdot) \in \text{clos}(\mathbb{R}^n) \subseteq [X] \) for all \( x, t \in A \times 0 \times 1 \) (see [21], p. 158 or use the proof of 1.1). Let \( B \) be a family of open subsets of \( X \times 0 \times 1 \) such that \( A = \bigcup_{B \in \mathbb{B}} V \) \( \mathbb{B} \) is a family of \( V \subseteq X \times \mathbb{R}^n \). Let \( B \) be the nerve of \( \mathbb{B} \) and let \( f \) be a canonical map. Identifying \( K \) with a subcomplex of \( L \), we infer that \( C = f^{-1}(K) \) is a relatively closed subset of \( V \) and therefore the set \( B = \{ (x, t) : C \subseteq \text{clos}(\mathbb{R}^n) \} \) is of type \( G_T \) in \( X \times 0 \times 1 \) and contains \( A \subseteq 0 \times 1 \) (since \( 0 \subseteq 0 \times 1 \) is compact, \( A = X \setminus \text{clos}(\mathbb{R}^n) \subseteq X \times 0 \times 1 \)).

4.2. REMARK. Let \( X \) be a compact PL-manifold, let \( H \) denote its homomorphic extension of compact-open topology and let \( P \) be the subgroup of \( H \) of PL-maps. It was shown by Haver [15, 17], that \( P \) is ANR(\( \mathbb{R}^n \)) and the closure \( G \) of \( P \) is an open subgroup of \( H \). Let \( G_\alpha \) be any ANR(\( \mathbb{R}^n \))-extension of \( P \) to a \( G_\alpha \)-subset of \( G \); since \( P \) is uniformly locally contractible (see [17]), we infer, by 2.9 and 2.6, that \( G \setminus G_\alpha \) is l. h. negligible in \( G \). Thus \( G \setminus G_\alpha \) contains an \( l_2 \)-manifold

\( (\ast) \) Added in proof. It is not, without assuming \( X \) to be locally contractible, as is shown by Taylor's example (BAMS 81, p. 629) combined with 6.1 and 6.2.
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(remind G2×I2, see [31]) with an i.h. negligible complement. Since H is a union of open cosets of G and since H×I2 ⊆ H (Geoghegan [14]), H also contains an I2-manifold with an i.h. negligible complement. It is however an open question if H is an ANR (98).

4.4. Remark. Similarly, it follows from [15] and 4.1 that if E is any separable complete linear metric space then E×K ⊆ AR(98) for some i.h. negligible Fσ-set K; it is though unknown if E ⊆ AR(98).

§ 5. Enlarging a manifold. In this section we show that if a complete ANR(98)-space X contains an I2-manifold whose complement is a Z-set in X, then X is necessarily an I2-manifold. We start with:

5.1. Proposition. Let E denote the Hilbert cube or a locally convex linear metric space such that E ⊆ E∞ or E ⊆ ∑E ⊆ {x ∈ E∞ : x = 0 for almost all i}; and let A be a Z-set in a metric space X. If X×E and X×A are E-manifolds, then X×E and X×A are E-manifolds.

The proof is divided into 3 steps and involves an idea of Cutler [see (7) and also [33], where some special cases of 5.1 are established].

1° If M is an E-manifold and K is a Z-set in M, then there is a homotopy (f0 : M → M such that f0 = id, f1(M) = int f0(M) if 0 < s < r ≤ 1, \( \bigcup_{t>0} f_t(M) = M×K \) and \( (x,i) \mapsto (f_t(x),i) \) is a closed embedding of M×I into itself.

Proof. Under our assumptions there is a homomorphism h : M → M×I such that h(K) = M×{0} (see [30]). Let \( g_0 \) be any product metric on M×I; then for each \( \epsilon > 0 \) the formula

\[ d_\epsilon(x,y) = \inf \{ t ≥ 0 : g_\epsilon(x,s), h(K)(y,t) ≥ t \} \]

defines a continuous function on M. We let \( f_t = h^{-1} g_\epsilon h \), where \( g_\epsilon(x,s) = (x,s) \) if \( s ≥ 2t \) and \( g_\epsilon(x,s) = (x,t) \) otherwise.

Given spaces Z and F and a closed set L⊂Z, we denote by \( (Z×L)×F \cup L \) equipped with the topology generated by open subsets of \( (Z×L)×F \) and by sets of the form \( U×L \cup (U×Z)×F \), where \( U⊂Z \) is open. \( CF \) denotes \( (I×F)×I \), the cone over F.

2° Under the assumptions of 5.1, the spaces \( X×CE \) and \( (X×CE)_A \) are homeomorphic.

Proof. Set \( M = X×E \) and \( K = A×E \) and let \( f_0 : M → M \) be the homotopy from 1°. Define \( h : X×CE → (X×CE)_A \) by the formula

\[ h(x,y) = \begin{cases} (x,y) & \text{if } y = 0, \\ \left( p_x f_t(x), \frac{t}{\beta_f(x)}, t \right) & \text{if } y = (t,s) \text{ and } t > 0, \end{cases} \]

where \( \beta_f(x) = \sup \{ s ∈ I : (x,s) ∈ f_t(M) \} \). It is a matter of routine but tedious verification to show that \( h \) is a homeomorphism of \( X×CE \) onto \( (X×CE)_A \).

Proof of 5.1. It is known that \( E \) and \( CE \) are homeomorphic (see [18] and [33]), and therefore \( X×E \) is homeomorphic to \( X×A \). Let \( \epsilon \) be any metric for \( X \). Since \( X×A \) is an E-manifold, there is a homeomorphism \( g : (X×A)×E → X×A \) such that \( g(\epsilon(x),p_\epsilon(x)) = g(p_\epsilon(x),A) \) for all \( x ∈ (X×A)×E \) (see [28]). Extending \( g \) by identity over \( A \), we get a homeomorphism of \( (X×E)_A \) onto \( X \). Thus \( X×E \cong X \).

Combining 5.1 with the results of [32], we get

5.2. Theorem. Let \( X \) be an ANR(98)-space, let \( A \) be a Z-set in \( X \) and assume that \( X×A \) is a manifold modelled on a space \( E \) of any of the following cases:

(a) \( E \) is an infinite-dimensional Hilbert space and \( X \) is complete;
(b) \( E \) is a locally convex linear metric space with \( E \cong \sum E \) and \( X \) admits a closed embedding into \( E \).

For a discussion of certain special cases in which the condition (b) is satisfied see [31], §1.

In the remaining part of this section we apply 5.2 to show that certain function spaces are I2-manifolds. If \( X \) is a space and \( A \) is a compactum, then \( C(A×X) \) denotes the space of maps of \( A \) into \( X \) (compact-open topology), for \( x ∈ X \) we denote by \( \X \) the constant map with value \( x \), and we let \( \X = \{ x ∈ X \} \). \( C(A×A,
\X) \) has the usual meaning. We need two lemmas leading to the fact that if \( X ∈ ANR(98) \) has no isolated points, then one can continuously assign to each \( x ∈ X \) a non-constant path starting from \( x \).

5.3. Lemma. Let \( Y ∈ ANR(98) \), let \( A_0 ⊆ A \) be compact and let \( y_0 ∈ Y \). If neither \( y_0 \) nor \( A_0 \) are open, then the singleton \( \{ y_0 \} \) is a Z-set in \( S = C((A_0×A)/{(A_0×\X)} \).

Proof. Since every \( f ∈ S \) factorizes through a map of \( A(A_0×A) \) into \( (Y,y_0) \), we may assume that \( A_0 = (a_0) \) is a one-point set. Consider \( A \) as a (nowhere-dense) subset of \( I_2 \) and let \( (a_0) ∈ (A×A)_m \), \( (z_0) ∈ (I_2×A)^{m} \) and \( (y_0) ∈ (Y_{y_0})^{m} \) be sequences such that \( \lim_s(z_0) = a_0 \) and \( \lim_s(y_0) = y_0 \). Given \( f : A×I^m → Y \) with \( f((a_0)×I^m) = (y_0) \), extend \( f \) to \( f_1 : A∪(z_0)n×I^m → Y \) by letting \( f_1(z_0)t = (y_0) \), \( n ∈ N \), and extend \( f_1 \) to an \( f : U×I^m → Y \) where \( U⊂A∪\{z_0\}n×N \) is open in \( I_2 \). Let \( (a_n) \) be a sequence of mappings \( g_n : A→U \) such that \( g_n(a_0) = a_0 \) and \( g_n(a_0) = a_n \) for all sufficiently big \( n \). The maps \( f_2 : A×I^m → Y \) defined by

\[ f_2((a_n),(a),q) = g_n(a_0) \]

converge to \( f \) and have the property that, for each \( q ∈ I^m \), the map \( a→f(q)(a, q) \) belongs to \( S_{\{y_0}\} \). Since \( f : A×I^m → Y \) was induced by an arbitrary map of \( I^m \) into \( S \), the result follows.

5.4. Lemma. Let \( Y \) be an ANR(98)-space without isolated points and let \( εₐ → 0 \).

There is \( a : Y → C(I,Y) \) such that \( a(y)(0) = y \) and \( αa(α(y),y) < ε \) for all \( y ∈ Y \).

(α denotes here the sup-metric induced by α).
Proof. $C(Y, Y)$ is an ANR(3)-space and therefore, by 2.4, 3.3 and elementary properties of ANR(3)'s, it suffices to show that $C(Y, Y)$ is LC* rel. $\Gamma(Y, Y)$ at each point $y \in Y$.

To this end let us fix $k \in N$, $y_0 \in Y$ and $\varepsilon_0 > 0$; we shall find a $\delta > 0$ such that, under the notation $S = C(Y, Y)$ and $f = \{1, 2, 3\}$, we have

1. $\exists \delta > 0$ such that $\forall y \in S, \exists \delta > 0$ for $x \in \partial S$.

First observe that, by 5.3 and 3.2, there is a $\delta > 0$ such that $g: \partial S \to C([0, 1], 2, (Y, y_0)) \setminus \{y_0\}$ with $g(\partial S(\partial S)) < \delta_0$, where $x \in \partial S$.

Indeed, $g(y)(x) = \frac{f(y)(x)}{f(\partial S(\partial S))^x}$ if $x \in [0, 1]$, $g(\partial S(\partial S)) = \frac{f(y)(x)^2}{f(\partial S(\partial S)))^4}$, and $g(y)(x) = \frac{f(y)(x)^3}{f(\partial S(\partial S)))^6}$. Letting $g(y)(x) = \frac{f(y)(x)}{f(\partial S(\partial S)))^x}$, we get a $g: \partial S \to C([0, 1], 2, (Y, y_0)) \setminus \{y_0\}$ with $g(\partial S(\partial S)) < \delta_0$, where $x \in \partial S$.

5.5. Theorem. Let $X \times X_1, \ldots, X_n \subset X$ be separable complete ANR(3)'s, let $A$ be a compactum and $A_1, \ldots, A_n$ be disjoint closed subsets, and let $U$ be an open subset of $X$ whose boundary is compact and collared in $U$. If either $U \cap A_1, \ldots, A_n = \emptyset$ and $X$ has no isolated points or $U \cap A_1$ and $X_1$ has no isolated points, then the space $S = \{f \in C(A, X): f(A_1) = X_1 \}$ for $i = 1, 2, \ldots, n$ is an $l^n$-manifold.

Proof. Let $K = \{f \in S: f \neq \text{constant on } U\}$. It is known that $\partial S$ is a complete separable ANR(3)-space (see [51], §4). Therefore it remains to show that $K$ is a $Z$-set in $S$.

To this end fix $x \in U$ and $f: \Gamma^n \to X$ such that $f_0 = f_1(x) \in S$ for all $g \in \Gamma^n$. By assumption there exists a compactum $C$ in $U$ and a homotopy $(u_t): A \times [0, 1] \to C \times I$ such that $u_0(a) = (a, 0)$ if $a \notin U$ or $a = b$ and $u_1(A) = A \times [0, 1] \subset C \times [0, 1]$ for all $t \in I$. Define $f: \Gamma^n \times (A \times [0, 1]) \to X$ by $f(g, z) = \{f(g, z) \in \Gamma^n \times A \times [0, 1]:$ $g \in \Gamma^n, z \in A \times [0, 1], f_0 = f_1(x)$, $f_0 = f_1(x)$ for all $t \in I$. $\}$

where $v$ satisfies 5.4 with $Y = X$ if $U \cap (A_1 \cup \ldots \cup A_n) = \emptyset$ and with $Y = X_1$ if $U \cap A_1$. Choose $\delta > 0$ such that $\rho(f(g, z), f(y)) < \varepsilon$ and define $g: \Gamma^n \times A \to X$ by $g(t, z) = \{f_0 = f_1(x)$, $t \in A \times [0, 1])$ for $u_0(a) = (a, 0)$, \}$

One easily verifies that $g_v = g(v, \cdot) \in S \times K$ and $\beta(g_v, f) < 2v$ for all $v \in \Gamma^n$. Thus this shows that $K$ is a $Z$-set in $S$.

5.6. Corollary. Let $X \times X_1, \ldots, X_n \subset X$ be complete separable ANR(3)'s, where $X$ has no isolated points. If $A$ is a connected compact finite-dimensional manifold (with or without boundary), then for any closed mutually disjoint proper subsets $A_1, \ldots, A_n$ of $A$ the space $S = \{f \in C(A, X): f(A_i) \subset X_i \}, f = 1, 2, \ldots, n$, forms an $l^n$-manifold. In particular, the space of paths from $X_1$ to $X_2$ and the space of closed curves starting from $X_1$ are $l^n$-manifolds.

Appendix. Locally homotopy negligible sets and UV*-maps. We shall show here how the properties of i.e. negligible sets are related to the results of Armentrout–Price, Kozlowski and Lacher on cell-like mappings of metric spaces.

All spaces are assumed to be metrizable. If $X \times Y \times Z$ is a map, then by the mapping cylinder of $f$ we mean the space $Z = X \times [0, 1]$ with the topology generated by open subsets of $X \times [0, 1]$ and by sets $f^{-1}(U) \times [0, 1] \cup U \times [0, 1]$, where $U \times [0, 1] \subset X \times Y \times Z$. Note that $Z$ is metrizable; if we consider $X$ and $Y$ as bounded subsets of normed spaces $E$ and $F$, respectively, then $Z \times (X \times Y) \times (X_1 \times Y) \times (X_2 \times Y_2) \times \cdots \times (X_n \times Y_n) \times (X_{n+1} \times Y_{n+1})$.

We identify $X$ with $X_1 \times [0, 1]$, $Y$ with $X_2 \times [0, 1]$, and we denote by $p: X \to Y$ and $q: X \to Y$ the collapse and projection, respectively.

A map $f: X \to Y$ will be said to be UV*-map if, given $x \in X$ and a neighborhood $U$ of $y$, there is a neighborhood $V \subset U$ of $y$ such that each $g: \Gamma^n \to Y$ extends to an $f: \Gamma^n \to f^{-1}(U)$. If $f$ is a UV*-map at all $y \in Y$, then we say that $X$ is a UV*-map. Similarly if the projection $X \to Y$ is UV*-map at $Y$, then we say that $X$ is a UV*-subspace of $Y$.

6.1. Remark. If $f$ is a UV*-map if $Z \times Y \times \Gamma^n$ is LC* rel. $Z$ at each point of $Y$.

If all the $f^{-1}(y)$'s are compact and $f$ is a surjection, then $f$ is a UV*-map if all the $f^{-1}(y)$'s, $y \in Y$, are UV*-subsets of $Y$.

It is known that compacta of trivial shape are UV*-subsets of ANR(3)'s in which they lie (see [5])

6.2. Proposition (compare [27], [22], [4]). If $f: X \to Y$ is a UV*-map and $f(X)$ is dense in $Y$, then $f$ induces an isomorphism of the $n$th homotopy group.

Proof. Apply 2.8 and the fact that $f$ induces an isomorphism of the $n$th homotopy group if the inclusion $Z \times Y \to Z$ does so.

6.3. Proposition. Let $f: X \to Y$ be a UV*-map with a dense image and let $M \in ANR(3)$. Then, given $u: M \to X$ and $\tau: M \times [0, 1] \to (0, \infty), there is
a : M×(0, 1)→K such that \( q(f_{\alpha}(x), u(x)) < \alpha(x, t) \) for \((x, t) \in M \times (0, 1)\). If, in addition, \( K \subset X \) is a closed set, \( U \) is its neighbourhood and \( v : U \rightarrow X \) is any lifting of \( v(U) \), then \( g \) may be constructed in such a way that \( g(1, K) = v\) for all \( t \).

Proof. Put on \( Z \), a metric \( d \) in which the collapse \( y : (Z', d') \rightarrow (Y, \rho) \) is a contraction and let \( \lambda : M \rightarrow [0, 1] \) satisfy \( \lambda(K) = 1 \) and \( M \cup U = \lambda^{-1}(0) \). Define \( w : M \rightarrow Z \) by

\[
(\alpha(x), \psi(x)) = \begin{cases} \\
(\alpha(x), \psi(x)) \in X \times (0, 1] & \text{if } \lambda(x) > 0, \\
(\alpha(x), 0) & \text{if } \lambda(x) = 0.
\end{cases}
\]

Since, by 2.8, \( Y \) is l.h. negligible in \( Z \), there exists an \( \alpha \)-homotopy \( h_{\alpha} : M \rightarrow Z \) such that \( h_{\alpha}(M) \subset Z \setminus Y \) and \( h_{\alpha}(K) = w \) for all \( t \). We let \( g \equiv h_{\alpha} \).

6.4. PROPOSITION. Let \( f : X \rightarrow Y \) be an UV\(\alpha\)-map of ANR(08)'s and assume that \( f(X) \) is dense in \( Y \). Then, given \( \varepsilon \) : \( X \times (0, 1) \rightarrow (0, \infty) \), there exist \( g : X \times (0, 1) \rightarrow X \) and a homotopy \( h_{\varepsilon} : X \rightarrow X \) such that \( h_{\varepsilon}(0, 1) = h_{\varepsilon} \) and \( \varepsilon(f_{\alpha}(x), f(x)) < \alpha(x, t) \) for all \((x, t) \in X \times (0, 1] \), \( x \in X \) and \( y \in Y \).

Proof. Let \( \lambda \) be any increasing homeomorphism of \([-1, 0] \) onto \([0, 1] \). By 6.3 there is a \( g : X \times (0, 1) \rightarrow X \) such that, for all \((y, t) \in X \times (0, 1] \),

\[
q(f_{\alpha}(x), u(x)) < \min(\alpha(x, t), t, \chi_{\alpha}(y), \inf(\alpha(y, s) : s \in \lambda([-1, 0])))
\]

Let \( M = X \times [-1, 2], K = X \times \{1, 2\}, U = X \times [-1, 0) \cup (1, 2] \), and define \( w : M \rightarrow Y \) by

\[
u_{\varepsilon} = \begin{cases} f & \text{if } t \in [-1, 0], \\
\varepsilon(f, \varepsilon) & \text{if } t \in (0, 1],
\end{cases}
\]

Using 6.3 again, construct \( h_{\varepsilon} : M \rightarrow X \) with \( h_{\varepsilon}(x) \equiv \lambda(1) \), \( h_{\varepsilon} \equiv g_{1}f \) and

\[
q(f_{\alpha}(x), u(x)) < \chi_{\alpha}(\varepsilon(f(x))) \quad \text{for } (x, t) \in M \times [-1, 2].
\]

Finally, let \( h_{\alpha} = h_{\chi_{\alpha}} \).

6.5. Remark. Let \( f : X \rightarrow Y \) be a UV\(\alpha\)-map with a dense image and assume that \( X \) is an LC\(\alpha\)-space and \( \dim(Y) \leq \alpha < \infty \). It easily follows from 6.1 and 2.8 that \( Y \) is LC\(\alpha\) and therefore \( y \) is ANR(08) by [6], p. 222.

6.6. Remark. Let \( f : X \rightarrow Y \) be an UV\(\alpha\)-map with a dense image and assume that \( X \) and \( Y \) are ANR(08)'s and \( \dim(X), \dim(Y) \leq \alpha < \infty \). Then, \( \dim(Z) \leq \alpha \) and \( Z \) is locally contractible, and therefore \( Z \in \text{ANR}(08) \) (see [21], p. 168). Hence, by 3.2, \( Y \) is l.h. negligible in \( Z \) and \( f \) is actually a UV\(\alpha\)-map; thus 6.4 applies.

We also observe that if \( X \) and \( Y \) are locally compact spaces and \( f \) is a proper map, then the homotopies \( f_{\alpha} \cup \varphi_{\alpha} = (h_{\alpha} \cup f_{\alpha}) \) of 6.4 are proper if \( \alpha \) is taken sufficiently small (slightly weaker versions of 6.5 and 6.6 form the theorems of Lacher [23]).
Hilbert cube modulo an arc

by

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Abstract. Let $Q$ denote the Hilbert cube and let $\alpha, \beta \subset Q$ be arcs. Adapting methods of Bing-Andrews-Curtis-Kwun-Bryant we prove that $Q/\alpha \times I$ and $Q/\beta \times Q/\beta$ are homeomorphic with $Q$, where $I$ is a closed interval and $Q/\alpha$ is a space obtained from $Q$ by shrinking $\alpha$ to a point. The same method applies equally well to the case when arcs are replaced with finite-dimensional cells or their intersections.

I. Introduction. We use $Q$ to represent the Hilbert cube (the countable-infinite product of closed intervals). A closed subset $X \subset Q$ is called a Z-set if for any non-empty, homotopically trivial open set $U \subset Q$, $U - X$ is also non-empty and homotopically trivial. This concept was introduced by R. D. Anderson in [1] and in the infinite-dimensional topology plays a role analogous to a role of tameness conditions in the finite-dimensional topology. Chapman [7] showed that a Z-set $X \subset Q$ has a trivial shape if and only if the space $Q/X$, obtained from $Q$ by shrinking $X$ to a point, is homeomorphic to $Q$ (in notation, $Q/X \simeq Q$). If $X$ is of a trivial shape but not a Z-set, then $Q/X$ may fail to be locally like $Q$ at the point $\hat{X} = p(X)$, where $p: Q \to Q/X$ is a natural projection. Indeed, Wong [14] constructed a copy of the Cantor set with non-simply connected complement in $Q$. By a standard technique we can pass an arc $a$ through it such that $Q - a$ is also not simply connected. If $Q/\alpha$ were locally $Q$ at the point $\hat{\alpha}$, then $Q/\alpha$ being a contractible $Q$-manifold would be homeomorphic to $Q$ [5]. But in $Q$ the complement of every point is simply connected.

The problem SC 1 in [2] asks (in analogy with a similar result for Euclidean spaces established earlier by Andrews and Curtis [3]) whether for any arc $a \subset Q$ multiplying $Q/a$ by the unit interval $I = [0, 1]$ gives the Hilbert cube. In Section 2 of this note we will present a detailed proof, adapting techniques from [3] to the Hilbert cube case, of the following theorem that confirms this conjecture.

**Theorem 1.** For any arc $a \subset Q$, $(Q/a) \times I$ is homeomorphic with $Q$.

Next, in Section 3, we first prove that $A \times B$ is a Z-set in $Q \times Q$ whenever $A$ and $B$ are finite-dimensional closed subsets of $Q$ and then, following Kwun's method [10], establish

**Theorem 2.** Let $\alpha, \beta \subset Q$ be arbitrary arcs. Then $(Q/\alpha) \times (Q/\beta)$ is homeomorphic with $Q$.