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Accepté par la Rédaction le 5. 4. 1976

Shape properties of hyperspaces

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Abstract. Using some ideas from shape theory several results on the hyperspaces of subcontinua are obtained. The hyperspaces of circle-like continua are studied in great detail.

0. Introduction. By a continuum we mean a compact connected metric space. Given a continuum X by $C(X)$ we denote the hyperspace of nonvoid subcontinua of X with the Hausdorff metric $\operatorname{dist}(\cdot,\cdot)$ (see for instance [11] where several facts about $C(X)$ are proved). A map f, i.e., a continuous function, from X into Y defines a map $\hat{f}\colon C(X)\to C(Y)$ given by $\hat{f}(A) = f(A)$, which is called the *map induced by* f. Throughout this paper maps with hats above will always denote the induced maps. By $\hat{X}$ we denote the base of $C(X)$, that is the set of all singletons in $C(X)$. This space is isometric to X and occasionally is identified with X. Continuum X regarded as a point of $C(X)$ is called the *vertex of* $C(X)$. For every two continua $A, B \in C(X)$ such that $A \subset B$ there is a maximal monotone collection of continua between them which forms an arc in $C(X)$. Such a collection will be denoted by AB and called a *segment in* $C(X)$. If A is a singleton and $B = X$, then AB is called a *maximal segment*. A map μ from $C(X)$ into reals R is called a *Whitney map on* $C(X)$ provided the conditions are satisfied:

$$(*)\qquad A\subset B \text{ and } A\neq B \Rightarrow \mu(A)<\mu(B),$$

$$(**)\qquad \mu(\{x\}) = 0 \quad \text{for each } x\in X.$$

Whitney maps always exist [23]. We take the opportunity to show how we can construct many Whitney maps on $C(X)$.

Let $U_1, U_2, \ldots$ be an open base for X and call a pair $\alpha = (U_i, U_j)$ normal if $\overline{U}_i \subset U_j$. For such a pair let f_α denote the Uryshon map from X into the unit interval $I = [0,1]$ sending $\overline{U}_i$ into 0 and $X\setminus U_j$ into 1, and let $\mu_\alpha\colon C(X)\to R$ be given by

$$\mu_\alpha(A) = \operatorname{diam} f_\alpha(A).$$

* This paper has been written in the spring of 1975 when the author was visiting the University of Georgia, Athens, Georgia 30602.

Arrange the set of all normal pairs into a sequence and let $\{\mu_n\}$ denote the corresponding maps from $C(X)$ into R. One checks easily that the formula

$$\mu(A) = \sum_{n=1}^{\infty} \frac{\mu_n(A)}{2^n}$$

defines a Whitney map on $C(X)$.

In the first section we are concerned with saturated subsets of hyperspaces (the definition below). It was Kelley [11] who first observed that these subsets had nice homotopy properties. Our main result in the first section states that closed upper-saturated subsets of $C(X)$ have trivial shape. Since compacta with trivial shape are acyclic in all known senses ([3], [18]) the latter result improves a recent result of Rogers [20]. We also introduce the notion of horizontal subsets of $C(X)$. We observe that horizontal subsets of $C(X)$ have dimension less than that of $C(X)$ provided $C(X)$ is finite-dimensional, which follows from some considerations in [20] (comp. also [12]).

In the next section we prove a theorem on inverse limits of disks and derive a corollary to it. That corollary is a stronger form of a theorem proved by Bennett and Transue [2].

The results from the preceding section are applied in Section 3 to the hyperspaces of proper circle-like continua. By a proper circle-like continuum we mean a circle-like continuum with non-trivial shape. In other words: that continuum which can be expressed as the inverse limit of circles with essential bonding maps. We show that those hyperspaces behave similarly to the cones over those spaces. The corollary from Section 2 applied to those hyperspaces gives a generalization of a result of Ball and Sher [1].

The author is indebted to Dr. S. Spież for his interesting suggestions.

1. Saturated subsets of hyperspaces and auxiliary results. In this section we study upper saturated and lower saturated subsets of hyperspaces. A subset M of $C(X)$ is said to be *upper saturated*, *lower saturated* respectively, if the following conditions are satisfied.

(u) $A \in M$ and $A \subset B \Rightarrow B \in M$,

(l) $A \in M$ and $B \subset A \Rightarrow B \in M$.

The collection of all upper (lower) saturated subsets of $C(X)$ we denote by $\mathrm{US}(X)$ ($\mathrm{LS}(X)$, respectively). Let us note that these families are complementary to each other, that is, if M is an element of one of these families then $C(X) \setminus M$ is an element of the other. These families are closed with respect to arbitrary unions and intersections:

(a) $M_t \in \mathrm{US}(X)$ for $t \in T \Rightarrow \bigcup_{t\in T} M_t, \bigcap_{t \in T} M_t \in \mathrm{US}(X)$,

(b) $M_t \in \mathrm{LS}(X)$ for $t \in T \Rightarrow \bigcup_{t\in T} M_t, \bigcap_{t \in T} M_t \in \mathrm{LS}(X)$.

Remark also that every upper saturated subset of $C(X)$ is arcwise connected because each of its points can be joined with X by a segment, and each such segment is a subset of the given set.

Let $C^2(X)$ denote the hyperspace $C(C(X))$ and let $\sigma\colon C^2(X) \to C(X)$ be given by

$$\sigma(\mathscr{A}) = \bigcup \{A\colon A \subset \mathscr{A}\}.$$

It is known that σ is continuous and nonexpansive [11]. We say that $M \subset C(X)$ is an invariant subset with respect to σ provided $\sigma(C(M)) = M$, where $C(M) = \{\mathscr{A} \in C^2(X)\colon \mathscr{A} \subset M\}$. In such a case σ is a retraction of $C(M)$ onto M, regarding M as the set of singletons in $C(M)$ (the base of $C(M)$). Observe that

(c) every set $M \in \mathrm{US}(X)$ is invariant with respect to σ.

(d) the intersection of a collection of sets invariant with respect to σ is a set invariant with respect to σ.

Now we give some examples.

1.1. Example. Let μ be a Whitney map on $C(X)$ and let $t \in \operatorname{im}\mu$. Then $\mu^{-1}([t, \infty)) \in \mathrm{US}(X)$, $\mu^{-1}([0, t]) \in \mathrm{LS}(X)$ and each is closed in $C(X)$.

1.2. Example. Let f be a map from X into Y and let $M \in \mathrm{US}(Y)$ ($M \in \mathrm{LS}(Y)$). Then $\hat{f}^{-1}(M) \in \mathrm{US}(X)$ ($\hat{f}^{-1}(M) \in \mathrm{LS}(X)$ respectively). In particular, if f is a surjection, then $\hat{f}^{-1}(\{Y\}) \in \mathrm{US}(X)$ and it is closed in $C(X)$.

1.3. Example. Denote by $2^{C(X)}$ the collection of all subsets of $C(X)$ and define functions u and l from $2^{C(X)}$ into $2^{C(X)}$ by the formula

$$u(M) = \{A \in C(X)\colon A \supset B \text{ for some } B \in M\},$$

$$l(M) = \{A \in C(M)\colon A \subset B \text{ for some } B \in M\}.$$

Observe that $\operatorname{im} u \subset \mathrm{US}(X)$ and $\operatorname{im} l \subset \mathrm{LS}(X)$. Moreover, if M is closed in $C(X)$, then both $u(M)$ and $l(M)$ are closed.

If $U_1, U_2, \ldots, U_n$ are open subsets of X, then we denote

$$\langle U_1, \ldots, U_n\rangle = \{A \in C(X)\colon A \subset \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for } i = 1, 2, \ldots, n\}.$$

The sets of this form constitute a base Σ for a topology called the *Vietoris topology on* $C(X)$. In our case (that is for metric continua) this topology coincides with that induced by the Hausdorff metric [16, p. 47].

Observe that

(e) every element G of Σ is invariant with respect to σ; if X is locally connected, then G is locally arcwise connected.

The second assertion follows from the following observation. If $G = \langle U_1, \ldots, U_n\rangle$ and V is a component of $\bigcup_{i=1}^{n} U_i$, then $\langle V, U_1, \ldots, U_n\rangle$ is an arc-component of G.

Moreover, if $A, B \in \langle V, U_1, \ldots, U_n\rangle$ and $C = A \cup B \cup L$, where L is an arc in V joining A and B, then $AC \cup BC$ is an arcwise-connected subset of

$$u(\{A, B\}) \cap \langle V, U_1, \ldots, U_n\rangle$$

containing A and B. Since $u(\{A, B\}) \subset M$ for every $M \in \mathrm{US}(X)$ which contains A and B, we have

(f) if X is locally connected and $M \in \mathrm{US}(X)$, then M is locally arcwise-connected.

Since the hyperspace $C(X)$ is a metrizable space, every open covering of a subset of $C(X)$ has a star-refinement. The characterization of absolute neighborhood retracts given in [8, p. 122], and the methods developed by J. L. Kelley [11] together with (e) and the above observation give the following result.

1.4. THEOREM. *Every locally arcwise-connected and invariant with respect to σ subset M of $C(X)$ is an absolute neighborhood retract for metrizable spaces. Moreover, M is an absolute retract provided it is arcwise-connected.*

1.5. COROLLARY. *If X is a locally connected continuum, then every set $M \in \mathrm{US}(X)$ and every connected element of the base Σ is an absolute retract for metrizable spaces* (comp. [11]).

Using 1.5 we now prove that closed upper saturated subsets of the hyperspaces have trivial shape. The reader is referred to [3] for the notions and fundamental results from shape theory. The following proposition is proved in [15] (see also [9]).

1.6. PROPOSITION. *For a compactum X the following conditions are equivalent:*

(i) $\mathrm{Sh}\,X = 0$, *i.e., the shape of X is trivial,*

(ii) *X can be represented as the intersection of a decreasing sequence of compact absolute retracts,*

(iii) *each map from X into a neighborhood retract is homotopic to a constant map,*

(iv) *if $X = \mathrm{invlim}\{X_n, \alpha_{nm}\}$, where X_n is a compact ANR-set, then for each n there exists an $m \geqslant n$ such that the projection $\alpha_m\colon X \to X_m$ is homotopic to a constant map.*

1.7. THEOREM. *If M is a closed subset of $C(X)$, then* $\mathrm{Sh}(u(M)) = 0$.

Proof. Consider X as a subset of the Hilbert cube Q and let $\{X_n\}$ be a decreasing sequence of locally connected subcontinua of Q converging to X, i.e. $\bigcap_n X_n = X$. Hence M is closed in each $C(X_n)$. Let u_n denote the operation u in $C(X_n)$. By Corollary 1.5 $u_n(M)$ is a compact absolute retract. Moreover the sequence $\{u_n(M)\}$ converge to $u(M)$. By Proposition 1.6 we obtain the conclusion of the theorem.

1.8. COROLLARY. *Every closed upper saturated subset of $C(X)$ has trivial shape.*

1.9. COROLLARY. [12] $\mathrm{Sh}\,C(X) = 0$.

1.10. COROLLARY. *If μ is a Whitney map and $t \in \mu(C(X))$, then*

$$\mathrm{Sh}\,\mu^{-1}([t, \infty)) = 0.$$

1.11. COROLLARY. *Let f be a map from X into Y and let M be a closed subset of $C(Y)$ belonging to $\mathrm{US}(Y)$. Then $\mathrm{Sh}\,\hat{f}^{-1}(M) = 0$ unless it is empty. In particular, if f is onto, then $\mathrm{Sh}\,\hat{f}^{-1}(Y) = 0$.*

A subset M of $C(X)$ is said to be *horizontal* provided that every segment in $C(X)$ has at most one point in common with M. For instance, if $f\colon X\to Y$ is a monotone mapping onto Y, then $f^{-1}(Y) = \{f^{-1}(y)\colon y \in Y\} \subset C(X)$ is a horizontal subset of $C(X)$. Furthermore, if f is open then $f^{-1}\colon Y \to f^{-1}(Y)$ is a homeomorphism, hence Y embeds into $C(X)$ as a horizontal subset.

Using an analogous reasoning as in [12] we obtain the following

1.12. COROLLARY. [20] *If M is a closed horizontal subset of $C(X)$, then* $\dim C(X) \geqslant \dim M + 1$.

1.13. COROLLARY. *If Y is a continuous monotone open image of a continuum X, then* $\dim Y \leqslant \dim C(X) - 1$.

From now on we agree to use notation from the theory of inverse limits according to the following convention:

If $\{X_n\}$ is a sequence of spaces and $\{f_{n,n+1}\colon X_{n+1} \to X_n\}$ a sequence of maps, then f_{nn} denotes the identity map 1_{X_n}, f_{nm} for $n<m$ denotes the composition $f_{n,n+1} \circ \ldots \circ f_{m-1,m}$ and f_n denotes the projection from $\mathrm{invlim}\{X_n, f_{nm}\}$ into X_n.

A map $f\colon X\to Y$ is said to be an *ε-mapping* provided $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in f(X)$. Two following facts are stated only for future references.

1.14. PROPOSITION. *Let $X = \mathrm{invlim}\{X_n, \alpha_{nm}\}$ be the inverse limit of compacta. Then we have*

(i) *if all bonding maps are surjective, the projection is also surjective,*

(ii) *if the projection $\alpha_n\colon X \to X_n$ is homotopic to a constant map, then there is an index $m > n$ such that $\alpha_{nm}\colon X_m \to X_n$ is homotopic to a constant map.*

(iii) *for each $\varepsilon > 0$, there exists an index $n(\varepsilon)$ such that the projection $\alpha_n\colon X \to X_n$ is an ε-map for each $n \geqslant n(\varepsilon)$,*

(iv) *the map $\alpha\colon C(X) \to \mathrm{invlim}\{C(X_n), \hat{\alpha}_{nm}\}$ given by*

$$\alpha(A) = (\hat{\alpha}_1(A), \hat{\alpha}_2(A), \ldots)$$

is a homeomorphism [21].

2. On inverse limits of disks.

For the definition of movability and fundamental results about it the reader is referred to [4]. The letters S and D are used in this paper to denote the unit circle and the unit disk in the complex plane respectively, i.e. $S = \{z \in C\colon |z| = 1\}$, $D = \{z \in C\colon |z| \leqslant 1\}$. By a solenoid we mean the limit of an inverse sequence such that each factor space is S and each bonding map is a finite product of the identity map. A map $f\colon X \to Y$ is called a *shape equivalence* provided the fundamental sequence f is a shape equivalence [3].

In this section we shall prove a result on inverse limits of disks, which will be applied in the next section to the hyperspaces of circle-like continua.

2.1. Theorem. *For each $n \geq 1$ let $S_n = S$, $D_n = D$ and let w_n be an interior point of D_n. Assume $\alpha_{n,n+1}\colon S_{n+1} \to S_n$ and $\beta_{n,n+1}\colon D_{n+1} \to D_n$ are mappings satisfying the conditions:*

(1) $\alpha_{n,n+1} \not\simeq 0$,

(2) $\beta_{n,n+1}(x) = \alpha_{n,n+1}(x)$ *for* $x \in S_{n+1}$,

(3) $w_{n+1} \in \beta_{n,n+1}^{-1}(w_n)$,

(4) $\beta_{n,n+1}^{-1}(w_n)$ *is connected.*

Let $X = \operatorname{inv\,lim}\{S_n, \alpha_{nm}\}$, $Y = \operatorname{inv\,lim}\{D_n, \beta_{nm}\}$, $w = (w_1, w_2, \ldots)$ *and regard* X *as the subset of* Y. *Then there is a space* X^* *and a mapping*

$$f\colon Y \setminus \{w\} \to X^*$$

having the following properties:

(i) *the restriction* $f|X\colon X \to X^*$ *is a shape equivalence; consequently* $\operatorname{Sh} X^* = \operatorname{Sh} X$,

(ii) *if X is movable, then X^* is a simple closed curve; if X is not movable, then X^* is a solenoid,*

(iii) *if A is a compact subset of Y separating Y between X and w, then there is a component C of A such that $f(C) = X^*$. Moreover, there is an index n_0 such that $\beta_n(C)$ separates D_n between S_n and w_n for each $n \geq n_0$,*

(iv) *if C is a subcontinuum of Y separating Y between X and w, and for each compact subset M of $Y \setminus C$ there exist two disjoint continua A and B in $Y \setminus C$ such that $M \subset A \cup B$, then $f|C\colon C \to X^*$ is a shape equivalence,*

(v) *if C is a subcontinuum of $Y \setminus \{w\}$ containing X such that for every compactum $M \subset Y \setminus C$ there is a continuum in $Y \setminus C$ containing M, then $f|C\colon C \to X^*$ is a shape equivalence.*

Proof. If P denotes a perforated disc in the plane, then by $\mathring{P}$ we denote its interior.

Using (4) one can easily construct a sequence of sets $U_1, U_2, \ldots$ and a sequence of perforated discs $P_2, P_3, \ldots$ such that

(5) U_n is an open subset of D_n and $w_n \in U_n \subset \mathring{D}_n$,

(6) $\beta_{n,n+1}^{-1}(w_n) \subset \mathring{P}_{n+1} \subset P_{n+1} \subset \beta_{n,n+1}^{-1}(U_n)$,

(7) $U_{n+1} \subset P_{n+1}$,

(8) $\bigcap_n \beta_n^{-1}(\overline{U}_n) = \{w\}$.

Let $\gamma_{n,n+1}\colon S_{n+1} \to S_n$ be a mapping such that

(9) $\gamma_{n,n+1} \simeq \alpha_{n,n+1}$ and $\gamma_{n,n+1}$ is a covering projection.

The existence of such a map follows from (1). With a little effort one can construct (using (2), (3), (6), (9) and the homotopy lifting theorem [22]) a sequence of mappings $\delta_1, \delta_2, \ldots, \delta_n\colon D_n \setminus \{w_n\} \to S_n$, satisfying the conditions:

(10) $\delta_n | S_n \simeq 1_{S_n}$,

(11) $\beta_{n,n+1}(z) \notin U_n \Rightarrow \delta_n \beta_{n,n+1}(z) = \gamma_{n,n+1} \delta_{n+1}(z)$, for each $n \geq 1$ and $z \in D_{n+1}$.

By (6), (7) and (11) we get

(12) $\beta_{nm}(z) \notin U_n \Rightarrow \delta_n \beta_{nm}(z) = \gamma_{nm} \delta_m(z)$ for each $n \leq m$ and $z \in D_m$.

Let $G_n = Y \setminus \beta_n^{-1}(\overline{U}_n)$ for each $n \geq 1$.

Let $X^* = \operatorname{inv\,lim}\{S_n, \gamma_{nm}\}$. It follows from (12) that the function $f_n\colon G_n \to X^*$ given by the formula:

$$f_n(y) = (\gamma_{1n} \circ \delta_n \circ \beta_n(y), \gamma_{2n} \circ \delta_n \circ \beta_n(y), \ldots, \delta_n \circ \beta_n(y), \delta_{n+1} \beta_{n+1}(y), \ldots)$$

is well-defined and continuous (because each coordinate function is continuous). Since for $k \geq \max(n, m)$ we have

$$\gamma_k f_n(y) = \delta_k \circ \beta_k(y) = \gamma_k f_m(y) \quad \text{for} \quad y \in G_n \cap G_m,$$

we infer that

(13) $y \in G_n \cap G_m \Rightarrow f_n(y) = f_m(y)$.

Observe that $\bigcup_n G_n = Y \setminus \bigcap_n \beta_n^{-1}(\overline{U}_n) = Y \setminus \{w\}$ (see (8)). Setting $f(y) = f_n(y)$ for $y \in G_n$, and noting that each G_n is an open subset of Y, we obtain by (13) a mapping

$$f\colon Y \setminus \{w\} \to X^*.$$

This is the map we were looking for. To complete the proof it remains to check conditions (i)–(v).

Observe that for each $n \geq 1$ the following diagram commutes:

$$\begin{array}{ccccc}
S_n & \xleftarrow{\alpha_{n,n+1}} & S_{n+1} & \xleftarrow{\alpha_n} & X \\
\downarrow{\scriptstyle \delta_n|S_n} & & \downarrow{\scriptstyle \delta_{n+1}|S_{n+1}} & & \downarrow{\scriptstyle f|X} \\
S_n & \xleftarrow{\gamma_{n,n+1}} & S_{n+1} & \xleftarrow{\gamma_n} & X^*
\end{array}$$

(see (2), (5) and (12)). Moreover, by (10) the map $\delta_n | S_n$ is a homotopy equivalence. It follows that $f|X$ is a shape equivalence (see [18]). This proves (i).

If X is movable, then there is an index n_0 such that $|\deg \alpha_{n_0 n}| = 1$ for each $n \geq n_0$ (see [14]). It follows from (9) that $\gamma_{n_0 n}$ is a homeomorphism; thus X^* is homeomorphic to S_{n_0}. In case X^* is not movable, X is a selenoid by (1), (9) and an argument used in [14]. This proves (ii).

To prove (iii) assume that A is a compact subset of Y separating Y between X and w. Then there exist open subsets M and N of Y such that

$$Y \setminus A = M \cup N, \quad X \subset M, \quad w \in N \quad \text{and} \quad M \cap N = \emptyset.$$

Since $M \cup A$ is compact and $G_1 \subset G_2 \subset \ldots$ (see (6) and (7)), there is an index k such that

$$M \cup A \subset G_k .$$

We shall show that

(14) $$\gamma_k \circ f | A \not\simeq 0 .$$

Suppose, to the contrary, that (14) does not hold. Then by the Borsuk homotopy extension theorem the map $\gamma_k \circ f|A$ can be extended to a map $g\colon N \cup A \to S_n$. Setting

$$h(y) = \begin{cases} \gamma_k \circ f(y) & \text{for} \quad y \in M \cup A , \\ g(y) & \text{for} \quad y \in N \cup A \end{cases}$$

we obtain a map $h\colon Y \to S_n$. Since Sh $Y = 0$, by Proposition 1.6 we have $h \simeq 0$. In particular $h|X \simeq 0$. But for $x \in X$ we have

$$h(x) = \gamma_k \circ f(x) = \gamma_k \circ f_k(x) = \delta_k \circ \beta_k(x) = \delta_k \circ \alpha_k(x)$$

(see (2)). Thus by (10) we infer that $\alpha_k \simeq 0$, contrary to Proposition 1.14 (ii) and (1). This proves (14).

It follows from (14) that there is a component C of A such that

$$\gamma_k \circ f | C \not\simeq 0 \quad \text{(see [16, p. 425])} .$$

This implies that for each $n \geq k$ we have $\gamma_n \circ f | C \not\simeq 0$. In particular we have $\gamma_n(f(C)) = S_n$, for each $n \geq k$. Thus $f(C) = X^*$. Let $n_0 \geq k$ be an index such that $\beta_n(C) \cap (S_{n_0} \cup \{w_{n_0}\}) = \emptyset$. It follows that for each $n \geq n_0$ we have $\beta_n(C) \cap S_n = \emptyset$ (see (2)). Note that $\beta_n(C) \subset D_n \setminus \{w_n\}$. We claim that the inclusion map $\beta_n(C) \subset D_n \setminus \{w_n\}$ is not homotopic to a constant map. For otherwise $\delta_n | \beta_n(C)$ would be homotopic to a constant, and so would be the map $\delta_n \circ \beta_n | C = \gamma_n \circ f | C$, contrary to our previous observation. Thus the Borsuk theorem [16, p. 470] implies that $\beta_n(C)$ separates D_n between S_n and w_n for each $n \geq n_0$, which proves (iii).

Now we prove (iv). Let C be a continuum in Y satisfying hypothesis of (iv) and let $M_1, M_2, \ldots$ be an increasing sequence of compacta such that $Y \setminus C = \bigcup_n M_n$. Let $C_n = \beta_n(C)$. First we construct a sequence of natural numbers $n_1 < n_2 < \ldots$ and a sequence of annuli $Q_{n_1}, Q_{n_2}, \ldots$ satisfying the following conditions,

(15) $$C_{n_j} \subset \mathring{Q}_{n_j} \subset Q_{n_j} \subset \mathring{D}_{n_j} \setminus \{w_{n_j}\} ,$$

(16) $$\beta_{n_j n_{j+1}}(Q_{n_{j+1}}) \subset Q_{n_j} ,$$

(17) $$M_{n_j} \subset \beta_{n_{j+1}}^{-1}(D_{n_{j+1}} \setminus \mathring{Q}_{n_{j+1}}) ,$$

for each $j \geq 1$.

By (iii) there is an index n_0 such that C_n separates D_n between S_n and w_n for each $n \geq n_0$. Let $n_1 = n_0$ and let Q_{n_1} be an annulus satisfying (15) for $j = 1$. Assume the numbers n_j and the annuli Q_{n_j} have been constructed for each $1 \leq j \leq k$. We shall show that each of these sequences can be extended by one term.

Observe that $E = M_{n_k} \cup \beta_{n_k}^{-1}(D_{n_k} \setminus \mathring{Q}_{n_k})$ is a compact subset of $Y \setminus C$ (see (15)). By our assumption there exist continua A and B in $Y \setminus C$ such that $E \subset A \cup B$ and $A \cap B = \emptyset$. Withount loss of generality we can assume that $w \in A$ and $X \subset B$ because $X \cup \{w\} \subset E$ (see (15)). Since A, B and C are mutually disjoint continua in Y and each β_n is an ε_n-map such that $\lim_n \varepsilon_n = 0$, there is an index $n_{k+1} > n_k$ such that $A' = \beta_{n_{k+1}}(A)$, $C_{n_{k+1}}$ and $B' = \beta_{n_{k+1}}(B)$ are mutually disjoint subcontinua of $D_{n_{k+1}}$. Since $n_{k+1} > n_0$, $w_{n_{k+1}} \in A'$ and $S_{n_{k+1}} \subset B'$, it follows that $C_{n_{k+1}}$ separates $D_{n_{k+1}}$ between A' and B'. Now it is easy to find an annulus $Q_{n_{k+1}}$ in $D_{n_{k+1}}$ containing $C_{n_{k+1}}$ in its interior and missing $A' \cup B'$. Any such annalus satisfies conditions (15), (16) and (17) for $j = k+1$, which completes the construction.

The limit of the inverse sequence

$$Q_{n_1} \leftarrow Q_{n_2} \leftarrow \ldots ,$$

where the bonding maps are determined by the maps $\beta_{n_j n_{j+1}}$ (see (16)), is (homeomorphic to) C, which follows from (15) and (17). The map $\delta_{n_j} | Q_{n_j}\colon Q_{n_j} \to S_{n_j}$ is a homotopy equivalence for each $j \geq 1$ (see (10)). This implies that $f|C\colon C \to X^*$ is a shape equivalence (comp. the proof of part (i)). This proves (iv).

An argument similar to the above one works for the point (v).

2.2. Corollary. *Keeping the notation of the theorem we have: if Y embeds in Euclidean 3-space, then X embeds in the plane. Moreover, Y is always embeddable into Euclidean 4-space.*

Proof. Since Y is a disk-like continuum, it can be embedded in E^4 [10].

Now, suppose Y embeds in E^3 and let $h\colon Y \to E^3$ be such an embedding. We shall prove that X is planar using an idea from [2]. Let S^2 be a 2-sphere with centre at $h(w)$ separating E^3 between $h(X)$ and $h(w)$. Let $A = h^{-1}(S^2 \cap h(Y))$. Hence A is a compact set separating Y between X and w. So by (iv) there exists a component C of A such that $f(C) = X^*$. By (ii) the space X^* is either a solenoid or a simple closed curve. It follows that X is a simple closed curve, because C embeds in S^2 and no subcontinuum of S^2 can be mapped onto a solenoid [7]. So X is movable and by (i) we see that X is movable because movability is an invariant of shape [4]. Hence X is a movable circle-like continuum. It follows that it can be embedded in the plane [14]. This completes the proof.

The above corollary implies the following result proved by Bennett and Transue.

2.3. Corollary [2]. *The cone over a nonplanar circle-like continuum can not be embedded in E^3.*

(Clearly, the cone can be represented as the limit of disks satisfying the hypothesis of the theorem.)

Problem. Suppose X is a continuum such that the cone over it can be embedded in the Euclidean 3-space. Does it follow that X can be embedded in the sphere S^2?

3. Applications to hyperspaces of circle-like continua. Recall that by S and D we denote the unit circle and the unit disk in the complex plane respectively. Let $\mathrm{ex}\colon R\to S$ be the mapping of reals given by

$$\mathrm{ex}(t) = e^{2\pi i t}.$$

Given a point $A\in C(S)$ there is an interval $[t_1 t_2]\subset R$ such that $\mathrm{ex}([t_1 t_2]) = A$. Setting

$$\gamma(A) = \max(0, 1-|t_1-t_2|)\,\mathrm{ex}\left(\frac{t_1+t_2}{2}\right)$$

we obtain a map γ from $C(S)$ into D (it is easy to check that γ is well-defined). Moreover, γ is a homeomorphism between $C(S)$ and D. We will refer to it as the standard homomorphism.

Consider now a fixed proper circle-like continuum X. By [17] we can represent it as the limit of an inverse sequence $\{S_n, \alpha_{nm}\}$ where $S_n = S$ for each $n\geqslant 1$. According to 1.14 and [13] ([1]) there exists an index n_0 such that for each $n\geqslant n_0$ the projection $\alpha_n\colon X\to S_n$ is not homotopic to a constant map. Without loss of generality we may assume that $\alpha_n \not\simeq 0$ for each $n\geqslant 1$. Since $\alpha_n = \alpha_{nm}\circ\alpha_m$, $m\geqslant n$, it follows that $\alpha_{n,n+1}\not\simeq 0$ for each $n\geqslant 1$. For each $n\geqslant 1$ let $D_n = D$ and let $\gamma_n\colon C(S_n)\to D_n$ be the copy of the standard homeomorphism. Let $i_n\colon S_n\to C(S_n)$ and $\beta_{n,n+1}\colon D_{n+1}\to D_n$ be given by

$$i_n(x) = \{x\} \quad\text{and}\quad \beta_{n,n+1}(x) = \gamma_n\circ\hat{\alpha}_{n,n+1}\circ\gamma_{n+1}^{-1}.$$

Observe that α's and β's satisfy the assumptions of the theorem from Section 3 see Corollary 1.11). Let $Y = \mathrm{invlim}\{D_n\beta_{nm}\}$ and let $w_\infty = (0, 0, \ldots)\in Y$. Regard X as the subset of Y (the inclusions $\{\gamma_n\circ i_n\}$ define such an embedding). For each $n\geqslant 1$ the following diagram commutes:

$$\begin{array}{ccccc} S_{n+1} & \xrightarrow{i_{n+1}} & C(S_{n+1}) & \xrightarrow{\gamma_{n+1}} & D_{n+1} \\ \downarrow{\scriptstyle \alpha_{n,n+1}} & & \downarrow{\scriptstyle \hat{\alpha}_{n,n+1}} & & \downarrow{\scriptstyle \beta_{n,n+1}} \\ S_n & \xrightarrow{i_n} & C(S_n) & \xrightarrow{\gamma_n} & D_n \end{array}$$

For each $A\in C(X)$ put

$$h(A) = (\gamma_1\alpha_1(A), \gamma_2\alpha_2(A), \ldots).$$

The above discussion and Proposition 1.14 imply that

$$h\colon (C(X), \hat{X}, X)\to(Y, X, w_\infty)$$

defines a homeomorphism between the triples. We agree to identify the first triple with the second one according to that homeomorphism. So applying the theorem from Section 3 to the triple $(C(X), \hat{X}, X)$ we obtain the following result:

([1]) Theorems 3.3 and 3.4 from this paper have first been obtained by C. E. Burgess [5]. This information was communicated to the author by Professor C. E. Burgess.

3.1. THEOREM. *Let X be a proper circle-like continuum. Then there exist a space X^* and a continuous map*

$$f\colon C(X)\setminus\{X\}\to X^*$$

having the following properties:

(i) *the restriction $f|\hat{X}\colon \hat{X}\to X^*$ is a shape equivalence,*

(ii) *if X is movable, then X^* is a simple closed curve; if X is not movable, X^* is a solenoid,*

(iii) *if $A\subset C(X)$ is a compact set separating $C(X)$ between $\hat{X}$ and X, then there exists a component C of A such that $f(C) = X^*$,*

(iv) *if C is a subcontinuum of $C(X)$ separating $C(X)$ between $\hat{X}$ and X and for each compact subset $M\subset C(X)\setminus C$ there exist two disjoint continua A and B in $C(X)\setminus C$ such that $M\subset A\cup B$, then $f|C\colon C\to X^*$ is a shape equivalence.*

(v) *if C is a subcontinuum of $C(X)\setminus\{X\}$ containing $\hat{X}$ such that for every compact set $M\subset C(X)\setminus C$ there exists a continuum in $C(X)\setminus C$ containing M, then $f|C\colon C\to X^*$ is a shape equivalence.*

3.2. COROLLARY. *Let μ be a Whitney map on $C(X)$ with $\mu(X) = 1$ and let $\mu^*\colon S(X^*)\to I$ be the map from the cone over X^* given by $\mu^*([x, t)] = t$. Then there exists a surjective map f^* such that the diagram*

$$\begin{array}{ccc} C(X) & \xrightarrow{f^*} & S(X^*) \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu_*} & \\ & I & \end{array}$$

commutes, and for each closed connected subset $A\subset I$ the map defined by f^* between $\mu^{-1}(A)$ and $\mu_*^{-1}(A)$ is a shape equivalence.

Proof. Let $f^*(A) = [f(A), \mu(A)]$ for $A\neq X$ and $f^*(X) =$ the vertex of $S(X^*)$. The properties of f^* are easily verified.

Since movability is an invariant of shape and for circle-like continua movability is equivalent to embeddability in the plane we have:

3.3. COROLLARY. *Let μ be a Whitney map on $C(X)$ and let $0\leqslant t<\max\mu(C(X))$. Then*

(i) $\mathrm{Sh}\,\mu^{-1}(t) = \mathrm{Sh}\,X$,

(ii) *(X movable) $\Rightarrow$ ($\mu^{-1}(t)$ can be embedded in the plane).*

The second observation follows from the fact that $\mu^{-1}(t)$ is a proper circle-like continuum [13]. (That observation answers a question raised in [13].) By a pseudocircle we mean a hereditarily indecomposable proper circle-like continuum in the plane. It is known that any two pseudocircles are homeomorphic [6]. If X is hereditarily indecomposable, then there exists a monotone (open) map from X onto $\mu^{-1}(t)$, hence $\mu^{-1}(t)$ is hereditarily indecomposable. Hence we conclude that

3.4. COROLLARY. *If X is the pseudocircle, then $\mu^{-1}(t)$ is homeomorphic to X for each $0\leqslant t<\max\mu(C(X))$.*

(Since each proper subcontinuum of the pseudocircle is a pseudoarc, the above corollary follows also from the Moore theorem [16, p. 533].)

Applying Corollary 2.2 to our situation we obtain:

3.5. COROLLARY. *If X is a circle-like continuum, which can not be embedded in the plane then $C(X)$ can not be embedded into E^3.*

This result extends a theorem of Ball and Sher [1] who proved it in case where X is hereditarily indecomposable.

In [19] J. T. Rogers proved that $C(X)$ embeds into E^3 if X is a planar circle-like continuum. Combining this result with the corollary we have

3.6. COROLLARY. *If X is a circle-like continuum, then $C(X)$ embeds into E^3 iff X embeds in the plane.*

3.7. THEOREM. *If X is a circle-like continuum and $g\colon \hat{X}\to Y$ is a mapping into an ANR-set, then there exists a map $h\colon C(X)\setminus\{X\}\to Y$ which extends g.*

Proof. Consider $C(X)$ and $S(X^*)$ as subsets of the Hilbert cube and let $\bar{g}$ be an extension of g onto a neighborhood U of $\hat{X}$ in Q. Identify X^* with the base of $S(X^*)$. Let $\bar{f}\colon Q\to Q$ be an extension of $f|\hat{X}\colon \hat{X}\to X^*$ and let $f_n=\bar{f}$ for each $n\geqslant 1$. Since $f|\hat{X}$ is a shape equivalence the fundamental sequence $\underline{f}=\{f_n\}\colon \hat{X}\to X^*$ has the left homotopy inverse $\underline{f}'=\{f_n'\}\colon X^*\to\hat{X}$. Hence $\underline{f}'\circ\underline{f}\simeq\underline{1}_{\hat{X}}$, which implies the existence of an index n_0 such that

$$f_n'\circ\bar{f}|\hat{X}\simeq 1_{\hat{X}} \quad \text{in } U \quad \text{for each } n\geqslant n_0 .$$

Let v denote the vertex of $S(X^*)$ and let $p\colon S(X^*)\setminus\{v\}\to X^*$ be the projection onto the "first coordinate". Let $q\colon C(X)\setminus\{X\}\to Y$ be given by

$$q(A)=\bar{g}\circ f_{n_0}'\circ p\circ f^*(A),$$

where f^* is the map from 3.2. Let us observe that

$$q|\hat{X}=\bar{g}\circ f_{n_0}'\circ\bar{f}|\hat{X}\simeq\bar{g}|\hat{X}=g .$$

Since $\hat{X}$ is a compact subset of $C(X)\setminus\{X\}$, by the Borsuk homotopy extension theorem there exists a map h with desired properties.

Remark. Since each map from $C(X)$ into any ANR-set is homotopic to a constant map (§ 2), and there are essential maps from $\hat{X}$ (X is assumed to be proper circle-like) into ANR-sets (for instance into a circle), the above fact says that the vertex X of $C(X)$ fills, in some sense, the "hole" in $C(X)\setminus\{X\}$. If X is a solenoid then no other point from $C(X)\setminus\hat{X}$ has this property.

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Accepté par la Rédaction le 25. 5. 1976