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Ordering probabilities on an ordered measurable space

by

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Abstract. Objects considered in the paper are ordered measurable spaces. The space of all probability measures on such a space X is also an ordered measurable space, denoted by X^*, with naturally defined σ-field and order in it. Since X can be considered as a subspace of X^*, the order in X^* is an extension of the initial order in X. The paper is devoted to the investigation of connections between spaces X and X^*, in particular we look for classes which are closed under the operation $X \to X^*$. E. g. such is the class of absolute measurable sets in the ordered Hilbert cube.

0. Introduction.

The objects considered in the paper are ordered spaces; an *ordered space* is a set X with a σ-field $\mathfrak{M}$ of its subsets and with an ordering relation $\leqslant$. The whole system $(X, \mathfrak{M}, \leqslant)$ will be, for simplicity, denoted by X. In cases where a confusion could arise as to what is the σ-field and what is the relation in X we also use subscripts, i.e. $\mathfrak{M}_X$ denotes the σ-field in X and $\leqslant_X$ denotes the order in X.

An ordered space X is *proper* iff it has a *base*, i.e. a family $\mathscr{A}$ of subsets of X which generates the σ-field $\mathfrak{M}_X$ and defines the order in the following sense: for every $x, y \in X$, $x \leqslant y$ iff, for every $A \in \mathscr{A}$, $x \in A$ implies $y \in A$.

For every proper ordered space X we shall define another ordered space X^* (which is also proper) called a *probabilistic extension of* X. Its elements are all probabilities on $\mathfrak{M}_X$; for the σ-field $\mathfrak{M}_{X^*}$ the only reasonable definition is accepted: it is the smallest σ-field in X^* such that for every $A \in \mathfrak{M}_X$ the function $P \to P(A)$ is measurable. However, it is not quite obvious what would be a "natural" definition of the relation $\leqslant$ in X^*.

There is a natural embedding ϑ of the set X into the set X^* (which associates with every x the probability ϑ_x concentrated at x). Thus the order in X^* is expected to be an extension of the order in X in the sense that $x \leqslant y$ iff $\vartheta_x \leqslant \vartheta_y$, for every x, y.

Consider as an example the real line $\mathscr{R}$ with the Borel σ-field and the usual order $\leqslant$. If "$x \leqslant y$" has the meaning "y is (in some sense) not worse than x", then obviously every probability concentrated on an interval $[a, b]$ should be "better" (in the sense of the extended relation $\leqslant$) than any probability concentrated on $[c, d]$ whenever $d \leqslant a$. More generally, a probability Q on Borel subsets of the real line should be "preferred" to any other P whenever the graph of the distribution function F_Q lies below the graph of F_P; however, there is no objective reason why $\frac{1}{2}\vartheta_0+\frac{1}{2}\vartheta_2$ should be "preferred" to ϑ_1, or conversely. This suggests the following definition of $\leqslant$ in X^*: $P\leqslant Q$ iff $P(A)\leqslant Q(A)$ for every set A of the form $(a, +\infty)$ or $[a, +\infty)$. A similar definition will be proposed in the general case: $P\leqslant Q$ iff $P(A)\leqslant Q(A)$ for every measurable increasing set A (A is increasing iff $A = \{x\in X|\ \exists a\in A\ a\leqslant x\}$). An additional justification of this definition will be given below and in Section 4 (cf. also the Open Problem at the end of the paper).

For every isotone (1) measurable function u: $X\to\mathscr{R}$, there is a natural way to compare probabilities on $\mathfrak{M}_X$, namely $P\underset{u^*}{\lesssim} Q$ iff $\int u\,dP\leqslant\int u\,dQ$ (for the time being we neglect integrability questions); in economic applications, when u is considered to be a utility function, this rule of comparison is called the Expected Utility Hypothesis — cf. Borch [4], p. 30. In Section 4 we shall find, for a proper ordered space X, some classes $\mathcal{O}$ of real-valued measurable isotone functions on X (e.g. all such functions, bounded functions, etc.) for which the following proposition holds:

for every $P, Q\in X^*$, $P\leqslant Q$ iff $[P\underset{u^*}{\lesssim} Q$ for every function $u\in\mathcal{O}]$.

Our attention will mainly be restricted to the examination of properties of ordered spaces which are related to the notions of a base and a superbase of an ordered space. A *superbase* of a space X is a family $\mathscr{A}$ such that the family of all sets $\{P\in X^*|\ a<P(A)\}$ with $a\in[0, 1]$, $A\in\mathscr{A}$ is a base of X^*.

In Section 2 we study elementary properties of ordered spaces. We also find some classes of ordered spaces which are closed under the operation $X\to X^*$. Sections 3 and 5 are devoted to the investigation of the extended relation $\leqslant$ in X^* and of properties of superbases of ordered spaces.

In Section 6 the notion of a *semi-regular* ordered space is introduced (Definition 6.1). A very important property of semi-regular spaces is that for every absolute measurable subspace X_0 of a semi-regular space X and for every superbase $\mathscr{A}$ of X the restricted family $\mathscr{A}|_{X_0}$ is a superbase of X_0 (Corollary 6.5). It is also proved that the *ordered Hilbert cube* $\boldsymbol{H}$ (with the σ-field of Borel subsets and coordinatewise order) is semi-regular and that an absolute measurable subspace of a semi-regular space is semi-regular (Theorems 6.6 and 6.9).

In Section 7, the last, we consider *regular* ordered spaces, i.e., spaces which have a countable base and which, considered as measurable spaces, are almost Borel (the definition in Subsection 1.C). A space is regular iff it is (up to an isomorphism) an absolute measurable subspace of $\boldsymbol{H}$ (Theorem 7.3). It is also proved (Theorem 7.7) that for every base $\mathscr{A}$ of a regular space X the lattice $\mathscr{A}_{sd}$ generated by $\mathscr{A}$ is a superbase of X. Finally, it follows (Theorem 7.8) that the class of all regular spaces is closed under the operation $X\to X^*$.

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(1) I.e., for every $x, y\in X$, $x\leqslant y$ implies $u(x)\leqslant u(y)$.

1. Preliminaries.

1.A. Notions and notation. All the notions used in the paper are standard except for the notions concerning ordered spaces and the following: "exhaustive family of sets", "almost Borel measurable spaces", "increasing part of a set" and "defining family". These notions will be explained in the present section.

Throughout we shall use the following notation:

$\mathscr{N}$ = the set of all nonnegative integers;

$\mathscr{I}$ = the interval $[0, 1]$;

$\mathscr{R}$ = the set of all reals;

$\overline{\mathscr{R}} = \mathscr{R}\cup\{-\infty, +\infty\}$;

$\mathscr{R}^n$ = the n-dimensional Euclidean space;

$\mathscr{Q}$ = the set of all rationals in $\mathscr{I}$;

$\mathscr{H}$ = the Hilbert cube;

$\mathscr{C}$ = the Cantor set (understood as the product of $\aleph_0$ copies of the two-element set $\{0, 1\}$ and with the product topology).

For a family $\mathscr{A}$ of subsets of a fixed set X, the symbols by $\sigma(\mathscr{A})$, $\mathscr{A}_s$, $\mathscr{A}_\sigma$, $\mathscr{A}_d$, $\mathscr{A}_\delta$, $\mathscr{A}_c$ will denote the σ-field generated by $\mathscr{A}$, the family of all finite unions, countable unions, finite intersections, countable intersections and the family of all complements of sets in $\mathscr{A}$, respectively. If $X_0\subset X$, then $\mathscr{A}|_{X_0}$ will denote the restriction of $\mathscr{A}$ to X_0: $\{A\cap X_0|\ A\in\mathscr{A}\}$.

Given any family of sets $\mathscr{A}$, we define $\mathscr{A}_0 = \mathscr{A}$ and inductively for all countable ordinals α: $\mathscr{A}_\alpha = (\mathscr{A}_{\alpha-1})_\sigma$ or $\mathscr{A}_\alpha = (\bigcup_{\beta<\alpha}\mathscr{A}_\beta)_\delta$, according as α is odd or even.

We say that a family $\mathscr{A}$ of subsets of a set X is *exhaustive* iff $\mathscr{A}_\alpha\subset(\mathscr{A}_\beta)_c$ for some countable ordinals α, β.

We say that a family $\mathscr{A}$ of subsets of a set X *separates points* iff for every pair (x, y) of distinct points in X there exists an $A\in\mathscr{A}$ such that the set $A\cap\{x, y\}$ has exactly one element.

By $\mathscr{B}_M$ we denote the σ-field of Borel subsets of a topological space M. If a topological space has to be understood as a measurable space, the σ-field under consideration is that of Borel subsets.

The symbol $(x)_i$ stands for the ith coordinate of the vector (sequence) x.

For measurable spaces we shall use the following notation: a measurable space $(X, \mathfrak{M})$ will usually be denoted by the single letter X; the σ-field in X is then assumed to be denoted by $\mathfrak{M}$, sometimes with an appropriate subscript (here: $\mathfrak{M}_X$). When speaking about a σ-field $\mathfrak{M}_X$, we understand that it is a σ-field in the set X.

A *subspace* of a measurable space X is any set $X_0\subset X$ with the σ-field $\mathfrak{M}_{X_0} = \mathfrak{M}_X|_{X_0} = \{A\cap X_0|\ A\in\mathfrak{M}_X\}$.

Given two measurable spaces X and Y, we call a function f: $X\to Y$ an *embedding* iff f is an isomorphism (1) from X onto $f(X)$ which is a subspace of Y. We say that

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(1) Here and elsewhere in the paper isomorphisms are understood as one-to-one functions.

a measurable space X *can be embedded* into a space Y iff there exists an embedding $f: X \to Y$.

A measurable space X is said to be *separable* iff $\mathfrak{M}_X$ is countably generated and separates points in X. Obviously, for every separable space X, the σ-field $\mathfrak{M}_X$ contains all one-point sets.

It is known (see Marczewski [9], §§ 2, 3 and Halmos [6], Thms 1.5.D and 1.5.E) that:

1.A.1. *Every separable measurable space can be embedded into $(\mathcal{I}, \mathcal{B}_{\mathcal{I}})$. Every measurable space X with an $\mathfrak{m}$-generated σ-field $\mathfrak{M}_X$ which separates points in X can be embedded into the measurable space $2^{\mathfrak{m}}$, which is the product of $\mathfrak{m}$ copies of the two-element space $2 = \{0, 1\}$ with the σ-field $\{\emptyset, \{0\}, \{1\}, 2\}$.* ■

1.A.2. *If a family $\mathcal{A}$ generates an $\mathfrak{m}$-generated σ-field $\mathfrak{M}$ (for $\mathfrak{m} \geqslant \aleph_0$), then there exists a family $\mathcal{A}_0 \subset \mathcal{A}$ with $\operatorname{Card}(\mathcal{A}_0) \leqslant \mathfrak{m}$ generating $\mathfrak{M}$.* ■

1.A.3. *Let X be a measurable space and let X_0 be its subspace. If a family of sets $\mathcal{A}$ generates $\mathfrak{M}_X$, then $\mathcal{A}|_{X_0}$ generates $\mathfrak{M}_{X_0}$.* ■

Probabilities and measures are always understood as countably additive functions defined on σ-fields.

For a measurable space X and a measure μ on $\mathfrak{M}_X$, $\underline{\mu}$ (resp. $\bar{\mu}$) will denote the *inner* (resp.: *outer*) measure of μ (i.e. for $A \subset X$, $\underline{\mu}(A) = \sup_{\substack{B \in \mathfrak{M}_X \\ B \subset A}} \mu(B)$ and $\bar{\mu}(A) = \inf_{\substack{B \in \mathfrak{M}_X \\ A \subset B}} \mu(B)$).

For a measurable space X, a set $A \subset X$ is said to be *absolute measurable* (in X) iff $\underline{\mu}(A) = \bar{\mu}(A)$ for every finite measure μ (equivalently: for every probability μ) on $\mathfrak{M}_X$.

For a given measurable space X and for $a \in \mathcal{I}$, $A \in \mathfrak{M}_X$, we shall denote by $[a|A]$ and $[a||A]$ the set of all probabilities P on $\mathfrak{M}_X$ such that $a < P(A)$ and such that $a \leqslant P(A)$, respectively. For $\mathcal{I}_0 \subset \mathcal{I}$ and $\mathcal{A} \subset \mathfrak{M}_X$ we shall write $[\mathcal{I}_0|\mathcal{A}] = \{[a|A] \,|\, a \in \mathcal{I}_0, A \in \mathcal{A}\}$ and $[\mathcal{I}_0||\mathcal{A}] = \{[a||A] \,|\, a \in \mathcal{I}_0, A \in \mathcal{A}\}$.

For a measurable space X we denote by X^* the measurable space of all probabilities on $\mathfrak{M}_X$ with the σ-field $\mathfrak{M}_{X^*}$ generated by the family $[\mathcal{I}|\mathfrak{M}_X]$. Clearly, the same obtains if we let $\mathfrak{M}_{X^*}$ be generated by one of the families $[\mathcal{Q}|\mathfrak{M}_X]$, $[\mathcal{I}||\mathfrak{M}_X]$ and $[\mathcal{Q}||\mathfrak{M}_X]$. Observe that all sets of the form $\{P \in X^* |\ P(A) \in B\}$, where $A \in \mathfrak{M}_X$, $B \in \mathcal{B}_{\mathcal{I}}$, are measurable in X^* (1).

Let X be a measurable space and let $S'' \subset S \subset S'$ be its subspaces. For a measure μ on $\mathfrak{M}_S$ we denote by $\mu||^{S'}$ the measure on $\mathfrak{M}_{S'}$, such that $\mu||^{S'}(A) = \mu(A \cap S)$ for every $A \in \mathfrak{M}_{S'}$; and by $\mu||_{S''}$ the measure on $\mathfrak{M}_{S''}$ such that $\mu||_{S''}(A) = \bar{\mu}(A)$ for every $A \in \mathfrak{M}_{S''}$.

If μ is a probability, then also $\mu||^{S'}$ is a probability; if $\bar{\mu}(S'') = 1$, then $\mu||_{S''}$ is a probability.

(1) Suppose that X is a compact metric space and $\mathfrak{M}_X = \mathcal{Q}_X$. Then $\mathfrak{M}_{X^*}$ is equal to the σ-field of Borel subsets with respect to the weak * topology in X^* (cf. Dubins and Freedman [5]).

Obviously, for every $S'' \subset S'$ and every measure μ on $\mathfrak{M}_{S''}$ and ν on $\mathfrak{M}_{S'}$ such that $\bar{\nu}(S'') = \nu(S')$ we have: $\mu||^{S'}||_{S''} = \mu$ and $\nu||_{S''}||^{S'} = \nu$. It is easy to check that.

1.A.4. *For every measurable space X and its subspace X_0 the function $\mu \to \mu||^X$ is an embedding of X_0^* into X^* and moreover the image of X_0^* by this embedding is equal to $\{P \in X^* |\ \bar{P}(X_0) = 1\}$.* ■

1.B. Elementary properties of probabilities. The space of probabilities on a given σ-field. The following lemmas are needed for proving the theorems which will be considered in this subsection:

1.B.1. Lemma. *The σ-field generated by an exhaustive and multiplicative family $\mathcal{A}$ coincides with the smallest class of sets $\mathcal{E}$ satisfying the following conditions:* (i) $\mathcal{A}_c \subset \mathcal{E}$; (ii) $\mathcal{E} = \mathcal{E}_\delta$; (iii) *if (A_i) is an increasing sequence of elements of $\mathcal{E}$, then $\bigcup A_i \in \mathcal{E}$.*

1.B.2. Lemma. *The σ-field generated by a family $\mathcal{A}$ coincides with the smallest class of sets $\mathcal{D}$ satisfying the following conditions:* (i) $\mathcal{A}_d \subset \mathcal{D}$; (ii) $\mathcal{D} = \mathcal{D}_c$; (iii) *if (A_i) is a sequence of mutually disjoint elements of $\mathcal{D}$, then $\bigcup A_i \in \mathcal{D}$.*

The proof of Lemma 1.B.1 is easy and will be omitted; the Lemma is analogous to a well-known theorem on Borel sets in metric spaces, cf. Kuratowski [7], § 30II. The proof of Lemma 1.B.2 can be found e.g. in Bauer [1] (Ch. I, Theorem 2.3). ■

We shall now prove a theorem of which many versions may be found in the literature (e.g. in Neveu [11], Ch. II § 7 or Meyer [10], Ch. III T 32). Since, however, we need it in a special form, we shall formulate another version:

1.B.3. Theorem. *Let $\mathfrak{M}$ be a σ-field generated by an exhaustive and multiplicative family $\mathcal{A}$. For every probability P on $\mathfrak{M}$, every set $A \in \mathfrak{M}$ and every $\varepsilon > 0$ there exists a set $B \in \mathcal{A}_{c\delta}$ contained in A and such that $P(A \setminus B) \leqslant \varepsilon$.*

Proof. Denote by $\mathcal{E}$ the class of all sets $A \in \mathfrak{M}$ which satisfy the following condition: "for every probability P on $\mathfrak{M}$ and every $\varepsilon > 0$ there exists a set $B \in \mathcal{A}_{c\delta}$ contained in A and such that $P(A \setminus B) \leqslant \varepsilon$". We are going to show that $\mathcal{E}$ satisfies conditions (i), (ii) and (iii) of Lemma 1.B.1. Since (i) is satisfied trivially, we begin by proving (ii). Let $A_i \in \mathcal{E}$ for $i \in \mathcal{N}$; denote $A = \bigcap A_i$. Choose a probability P and $\varepsilon > 0$. By assumption, for every i there exists a set $B_i \in \mathcal{A}_{c\delta}$ contained in A_i with $P(A_i \setminus B_i) \leqslant \varepsilon/2^{i+1}$. Clearly $B = \bigcap B_i$ belongs to $\mathcal{A}_{c\delta}$, is contained in A, and we have $P(A \setminus B) \leqslant \varepsilon$; hence $A \in \mathcal{E}$. Finally, in order to prove (iii), let $A_i \in \mathcal{E}$ for $i \in \mathcal{N}$ and let $A_i \nearrow A$. Choose a probability P and $\varepsilon > 0$. There exists an $i_0 \in \mathcal{N}$ such that $P(A \setminus A_{i_0}) \leqslant \frac{1}{2}\varepsilon$. By assumption there exists also a $B \in \mathcal{A}_{c\delta}$ contained in A_{i_0} and such that $P(A_{i_0} \setminus B) \leqslant \frac{1}{2}\varepsilon$. Now $B \subset A$ and $P(A \setminus B) \leqslant \varepsilon$ and hence $A \in \mathcal{E}$.

It follows from Lemma 1.B.1 that every set in $\mathfrak{M}$ belongs to $\mathcal{E}$ and this completes the proof of the theorem. ■

1.B.4. Corollary. *If a σ-field $\mathfrak{M}$ is generated by an exhaustive and multiplicative family $\mathcal{A}$, then for every probability P on $\mathfrak{M}$ and every set $A \in \mathfrak{M}$ there exists a set $C \in \mathcal{A}_{c\delta\sigma}$ contained in A and such that $P(A) = P(C)$.* ■

We are now going to study some relations between families generating a σ-field $\mathfrak{M}_X$ and families generating $\mathfrak{M}_{X^*}$.

1.B.5. THEOREM. *If a family of sets $\mathcal{A}$ generates a σ-field $\mathfrak{M}_X$, then the family $[\mathcal{Q}|\mathcal{A}_d]$ generates the σ-field $\mathfrak{M}_{X^*}$.*

Proof. By definition $\mathfrak{M}_{X^*} = \sigma([\mathcal{I}|\mathfrak{M}_X])$. Let us denote $\mathcal{F} = \sigma([\mathcal{Q}|\mathcal{A}_d])$. Since $\mathcal{F}\subset\mathfrak{M}_{X^*}$, in order to prove that $\mathcal{F} = \mathfrak{M}_{X^*}$, it suffices to show that $[a|X_0]\in\mathcal{F}$ for every $a\in\mathcal{F}$ and every $X_0\in\mathfrak{M}_X$. Let $\mathcal{D} = \{D\in\mathfrak{M}_X|\ [a|D]\in\mathcal{F}$ for all $a\in\mathcal{I}\}$. Thus we have to prove that $\mathcal{D}\supset\mathfrak{M}_X$. In view of Lemma 1.B.2 it suffices to check that $\mathcal{D}$ satisfies conditions (i), (ii) and (iii) of this lemma.

The Condition (i) follows from the obvious equality $[a|A] = \bigcup_{\substack{q\in\mathcal{Q}\\ q\geqslant a}} [q|A]$, where $a\in\mathcal{I}$, $A\in\mathcal{A}_d$. In order to prove (ii) choose $D\in\mathcal{D}$. Since for every $a\in\mathcal{I}$

$$[a|D^c] = X^*\setminus\bigcup_{\substack{q\in\mathcal{Q}\\ q<1-a}}[q|D]\in\mathcal{F}\ (^1),$$

we have $D^c\in\mathcal{D}$. Finally, we shall check (iii). Let (D_i) be a sequence of mutually disjoint elements of $\mathcal{D}$. Since for every $a\in\mathcal{I}$,

$$[a|\bigcup D_i] = \bigcup_{\substack{q\in\mathcal{Q}\\ q>a}}\ \bigcup_{n\in\mathcal{N}}\ \bigcap_{\substack{(q_0,\dots,q_n)\in\mathcal{Q}^{n+1}\\ \sum_0^n q_i<q}}\ \bigcup_{i=0}^{n}[q_i|D_i],$$

we have also $\bigcup D_i\in\mathcal{D}$. Thus the family $\mathcal{D}$ satisfies conditions (i), (ii) and (iii) of Lemma 1.B.2 and hence $\mathcal{D} = \mathfrak{M}_X$. ■

1.B.6. COROLLARY. *If a family $\mathcal{A}$ generates a σ-field $\mathfrak{M}_X$, then the family $[\mathcal{Q}|\mathcal{A}_s]$ generates $\mathfrak{M}_{X^*}$.*

Proof. Since $\mathcal{A}$ generates $\mathfrak{M}_X$, also $\mathcal{A}_c$ generates $\mathfrak{M}_X$. Apply Theorem 1.B.5 to the family $\mathcal{A}_c$. Thus $[\mathcal{Q}|\mathcal{A}_{cd}]$ generates $\mathfrak{M}_{X^*}$ and so does the family $[\mathcal{Q}|\mathcal{A}_{cd}]_c$, which is equal to

$$\{[q|A]_c|\ q\in\mathcal{Q}, A\in\mathcal{A}_{cd}\} = \{[1-q||A^c]|\ q\in\mathcal{Q}, A\in\mathcal{A}_{cd}\} = \{[q||A^c]|\ q\in\mathcal{Q}, A\in\mathcal{A}_{cd}\}$$
$$= \{[q||B]|\ q\in\mathcal{Q}, B\in\mathcal{A}_{cdc}\} = [\mathcal{Q}||\mathcal{A}_{cdc}] = [\mathcal{Q}||\mathcal{A}_s]$$

(because $\mathcal{A}_{cdc} = \mathcal{A}_s$). But obviously the families $[\mathcal{Q}||\mathcal{A}_s]$ and $[\mathcal{Q}|\mathcal{A}_s]$ generate the same σ-field. ■

Observe that in Theorem 1.B.5 the family $[\mathcal{Q}|\mathcal{A}_d]$ can be replaced by any of the following: $[\mathcal{Q}||\mathcal{A}_d]$, $[\mathcal{I}||\mathcal{A}_d]$ and $[\mathcal{I}|\mathcal{A}_d]$, and similarly in Corollary 1.B.6.

(1) D^c denotes the complement of the set D.

Now we shall investigate some other connections between a measurable space X and the space X^*. It follows from 1.A.5 that for every $P\in X^*$

$$\{P\} = \bigcap_{A\in\mathcal{A}_d}\{Q\in X^*|\ Q(A) = P(A)\},$$

where $\mathcal{A}$ is any family generating $\mathfrak{M}_X$. This statement and Theorem 1.B.5 give us

1.B.7. *If $\mathfrak{M}_X$ is countably generated, then X^* is separable.* ■

For a fixed measurable space X, we shall denote by ϑ the function from X into X^* which maps each point x into the probability ϑ_x concentrated at x (i.e. $\vartheta_x(A) = 1$ if $x\in A$ and 0 otherwise). We have

1.B.8. *If $\mathfrak{M}_X$ separates points in X, then ϑ is an embedding of X in X^*.*

1.B.9. *If $\mathfrak{M}_X$ is countably generated, then $\vartheta(X)\in\mathfrak{M}_{X^*}$.*

The proof of 1.B.8 follows from the definition of X^*. For the proof of 1.B.9 choose a family $\{A_i\}$ generating $\mathfrak{M}_X$. It suffices to show that

$$\vartheta(X) = \bigcap_i\{Q\in X^*|\ Q(A_i)\cdot Q(X\setminus A_i) = 0\}.$$

The inclusion $\subset$ is obvious. Let

$$P\in\bigcap_i\{Q\in X^*|\ Q(A_i)\cdot Q(X/A_i) = 0\}.$$

Let $B_i = A_i$ if $P(A_i) = 1$ and $B_i = X\setminus A_i$ if $P(A_i) = 0$. Clearly $\bigcap B_i$ is nonempty and $P = \vartheta_x$ for any $x\in\bigcap B_i$; this completes the proof of the inclusion $\supset$. ■

Finally, the following proposition completes Theorem 1.B.5 and Corollary 1.B.6 and follows from 1.B.8 and 1.A.3:

1.B.10. *Let a family $\mathcal{A}$ be contained in a σ-field $\mathfrak{M}_X$ which separates points in X. If the family $[\mathcal{I}|\mathcal{A}]$ generates $\mathfrak{M}_{X^*}$, then $\mathcal{A}$ generates $\mathfrak{M}_X$.* ■

1.C. Almost Borel measurable spaces. A measurable space X is said to be *almost Borel* (resp.: *standard analytic, standard Borel*) iff it is separable and for every metric separable complete space M and every embedding f of X into $(M, \mathcal{B}_M)$, $f(X)$ is an absolute measurable (resp.: analytic, Borel) subset of M.

Clearly, every standard Borel space is standard analytic, every standard analytic space is almost Borel. We have, moreover, the following:

1.C.1. THEOREM. *A measurable space X is almost Borel (resp.: standard analytic, standard Borel) if and only if there exists an absolute measurable (resp.: analytic, Borel) subset A of some metric separable complete space M such that the measurable spaces X and $(A, \mathcal{B}_A)$ are isomorphic.*

Proof. The implication $\Leftarrow$ follows in all cases from the definition and 1.A.1. The implication $\Leftarrow$ follows in the standard Borel and standard analytic cases from known theorems (see Kuratowski [7], § 39V, Th. 1 and Blackwell [2], cf. also Meyer [10], Ch. III § 1 T 16).

Let us now prove the implication $\Leftarrow$ in the almost Borel case. Let A be an absolute measurable subset of a metric separable complete space M, let f be an embedding of $(A, \mathscr{B}_A)$ into $(N, \mathscr{B}_N)$, where N is a metric separable complete space and let P be a probability on $\mathscr{B}_N$. Define a probability P_0 on $\mathscr{B}_M$ by the formula

$$P_0(X_0) = \overline{P}(f(X_0 \cap A)) \quad \text{for} \quad X_0 \in \mathscr{B}_M .$$

It is easy to check that P_0 is a well-defined probability (in fact, if for a probability μ on $\mathscr{B}_{f(A)}$ we denote by μ' the probability on $\mathscr{B}_A$ such that $\mu'(A_0) = \mu(f(A_0))$, then we shall have $P_0 = [P||_{f(A)}]'||^M)$ and $\overline{P_0}(A) = \overline{P}(f(A))$. Since $\underline{P}_0(A) = \overline{P_0}(A)$, we find a Borel subset M_0 of M contained in A with $P_0(M_0) = \overline{P_0}(A)$. The set $f(M_0)$, as a one-to-one measurable image of M_0, is Borel in N; thus $P(f(M_0))$ is defined and equal to $\overline{P}(f(M_0)) = P_0(M_0) = \overline{P_0}(A) = \overline{P}(f(A))$; hence $\underline{P}(f(A)) = \overline{P}(f(A))$. Since P was arbitrary, $f(A)$ is absolute measurable in N. ■

1.C.2. *For every standard Borel subspace X_0 of a separable measurable space X, X_0 belongs to $\mathfrak{M}_X$.*

Proof. As follows from 1.A.1, the space X can be embedded into the measurable space $(\mathscr{I}, \mathscr{B}_{\mathscr{I}})$; let us call such an embedding φ. Since $\varphi|_{X_0}$ is also an emdedding (namely, of the measurable space X_0 into $(\mathscr{I}, \mathscr{B}_{\mathscr{I}})$), we infer by the definition of standard Borel spaces that $\varphi(X_0) \in \mathscr{B}_{\mathscr{I}}$. Thus $\varphi(X_0) \in \mathscr{B}_{\varphi(X)}$ and consequently, by the definition of an embedding, $X_0 \in \mathfrak{M}_X$. ■

The proofs of the next statement and Theorem 1.C.4 will be omitted, because they need only standard methods, very similar to those already used in the proofs of Theorem 1.C.1 and 1.C.2.

1.C.3. *A subspace X_0 of an almost Borel measurable space X is almost Borel iff X_0 is an absolute measurable subset of X.* ■

1.C.4. THEOREM. *Let a measurable space X be separable. Then X is almost Borel if and only if for every $P \in X^*$ there exists a standard Borel subspace X_0 of X such that $P(X_0) = 1$.* ■

1.C.5. THEOREM. *If a measurable space X is almost Borel (resp.: standard analytic, standard Borel), then also X^* has this property.*

Proof. The standard Borel case is well known (see e.g. Parthasarathy [12], Ch. I). The standard analytic can be found in Blackwell, Freedman, Orkin [3], Lemma (25) and Proposition (4). Let us present here a short proof of the almost Borel case, given by C. Ryll-Nardzewski [13].

It follows from 1.B.7 that X^* is separable. Thus it suffices to find, for an arbitrary $\pi \in X^{**}$, a standard Borel subspace Z of X^* such that $\pi(Z) = 1$ (cf. Theorem 1.C.4). Let us consider the probability P_0 on $\mathfrak{M}_X$ given by the formula

$$P_0(A) = \int_{X^*} P(A)\pi(dP) \quad \text{for} \quad A \in \mathfrak{M}_X .$$

From the hypothesis we obtain a standard Borel subspace E of X with $P_0(E) = 1$. Consequently, $\pi(\{P \in X^* |\ P(E) = 1\}) = 1$. Write $Z = \{P \in X^* |\ P(E) = 1\}$. Since the spaces Z (considered as a subspace of X^*) and E^* are isomorphic (cf. 1.A.4), it follows from the standard Borel case of our theorem that Z is standard Borel. ■

A measurable space X is said to be a *Blackwell space* iff it is separable and such that, for every σ-field $\mathscr{X} \subset \mathfrak{M}_X$ which makes the space $(X, \mathscr{X})$ separable, there is an $\mathscr{X} = \mathfrak{M}_X$. It is known (Blackwell [2], Corollary 1) that every standard analytic space is a Blackwell space. However, an almost Borel space need not be a Blackwell space (see Maitra [8] and Example 2.7).

1.D. Ordered sets. The order in an ordered set X will be denoted by $\leqslant$ (or by $\leqslant_X$ if a confusion could arise). Subsets X_0 of an ordered set X are considered as ordered sets with the ordering relation $\leqslant_{X_0}$ defined as $x \leqslant_{X_0} y$ iff $x \leqslant_X y$ for every $x, y \in X_0$.

For a set $A \subset X$ we define

$$\overrightarrow{A} = \{x \in X |\ \exists a \in A\ a \leqslant x\}, \qquad \overleftarrow{A} = \{x \in X |\ \exists a \in A\ x \leqslant a\} .$$

In cases where a confusion could arise we shall write $\overrightarrow{A}^X$, $\overleftarrow{A}^X$ instead of $\overrightarrow{A}$, $\overleftarrow{A}$. A set A will be called *increasing* (*decreasing*) iff $A = \overrightarrow{A}$ $(A = \overleftarrow{A})$. For a family $\mathscr{A}$ of subsets of X, we write $\overrightarrow{\mathscr{A}} = \{\overrightarrow{A} |\ A \in \mathscr{A}\}$ and $\overleftarrow{\mathscr{A}} = \{\overleftarrow{A} | A \in \mathscr{A}\}$.

Clearly, $\overrightarrow{\overrightarrow{A}} = \overrightarrow{A} \supset A$ and $\overleftarrow{\overleftarrow{A}} = \overleftarrow{A} \supset A$. Moreover we have,

1.D.1. *A set $A \subset X$ is increasing iff A^c is decreasing.* ■

1.D.2. *The union and the intersection of any family of increasing sets is an increasing set.* ■

1.D.3. *If A is an increasing subset of X, then for every $X_0 \subset X$ $A \cap X_0$ is increasing in X_0.* ■

By the *increasing part* of a set $A \subset X$ we shall mean the set $A^{\odot} = (\overleftarrow{A^c})^c$. The sense of this definition explains 1.D.4:

1.D.4. *For every $A \subset X$, $A^{\odot}$ is the greatest increasing set contained in A (i.e.* (i) *$A^{\odot}$ is increasing,* (ii) *$A^{\odot} \subset A$,* (iii) *if B is increasing and $B \subset A$, then $B \subset A^{\odot}$).*

Proof. (i) follows from 1.D.1. Since $A^c \subset \overleftarrow{A^c}$, we have $A^{\odot} = (\overleftarrow{A^c})^c \subset A^{cc} = A$ and this gives (ii). Finally, let B be increasing and contained in A. We have $A^c \subset B^c$ and hence $\overleftarrow{A^c} \subset \overleftarrow{B^c} = B^c$. Thus $B = B^{cc} \subset (\overleftarrow{A^c})^c = A^{\odot}$ and this completes the proof. ■

1.D.5. LEMMA. *Let an increasing set C be of the form: $C = \bigcap_{t \in T} A_t$, where A_t are any subsets of X. Then $C = \bigcap A_t^{\odot}$.*

Proof. $A_t^{\odot} \subset A_t$ for all $t \in T$; hence $\bigcap A_t^{\odot} \subset \bigcap A_t = C$. On the other hand, $C \subset A_t$ for every $t \in T$; thus $C = C^{\odot} \subset A_t^{\odot}$ for every t and $C \subset \bigcap A_t^{\odot}$. ■

A family $\mathscr{A}$ of subsets of an arbitrary set X *defines* the following pre-order $\precsim$ in X (1):

$x \precsim y$ iff for every $A \in \mathscr{A}$, $x \in A$ implies $y \in A$.

(1) By a pre-order we mean a relation which is reflexive and transitive.

The pre-order defined by a family $\mathscr{A}$ is an order if and only if $\mathscr{A}$ separates points.

A *defining family* in an ordered set X is any family of subsets of X such that the pre-order defined by it is just $\leqslant_X$. It is very convenient to think of defining families in an ordered set X in the following way: a family $\mathscr{A}$ is a defining family iff it consists of increasing sets and has the following "separation property": for every $x, y \in X$ with $x \not\leqslant y$ there exists an $A \in \mathscr{A}$ such that $x \in A$ and $y \notin A$.

We call an ordering relation $\mathfrak{m}$-*definable* iff there exists a defining family of cardinality $\leqslant \mathfrak{m}$.

The following facts are easy to check:

1.D.6. *For every ordered set, the family of all increasing sets is a defining family; the union of a defining family and a family whose every member is an increasing set is also a defining family; every defining family separates points.* ■

1.D.7. *If $\mathscr{A}$ is a defining family in an ordered set X, then $\mathscr{A}|_{X_0}$ is a defining family in the ordered set X_0, for any $X_0 \subset X$.* ■

Some trivial examples show that in general a family consisting of increasing sets which separates points need not be a defining family.

2. Ordered spaces. In the Introduction we have defined the notion of ordered spaces. Here we shall introduce some further notions concerning these spaces and give some examples.

By an *isomorphism* between ordered spaces X and Y we mean any one-to-one function mapping X onto Y which is simultaneously an isomorphism in the sense of measurable spaces and an isomorphism in the sense or ordered sets. Two spaces are *isomorphic* iff there exists an isomorphism between them.

A *subspace* of an ordered space X is any subset $X_0 \subset X$ with the σ-field $\mathfrak{M}_{X_0}$ and ordering relation $\leqslant_{X_0}$.

An *embedding* of an ordered space X into an ordered space Y is any function $f\colon X \to Y$ which is an isomorphism between ordered spaces X and $f(X)$.

The product of a family $\{X_\xi \mid \xi \in \Xi\}$ of ordered spaces is the ordered space $B = \bigtimes_{\xi \in \Xi} X_\xi$ with the product σ-field $\mathfrak{M}_B = \bigotimes_\xi \mathfrak{M}_{X_\xi}$ and the coordinatewise order $\leqslant_B$ (i.e. $x \leqslant_B y$ iff $(x)_\xi \leqslant_{X_\xi} (y)_\xi$ for every $\xi \in \Xi$). If $X_\xi = X$ for every $\xi \in \Xi$ we shall also write $X^{\mathfrak{m}}$, where $\mathfrak{m} = \mathrm{Card}(\Xi)$, instead of $\bigtimes_{\xi \in \Xi} X$.

Observe that for every countable family $\{X_i\}$ of ordered spaces which are simultaneously topological spaces with $\mathfrak{M}_{X_i} = \mathscr{B}_{X_i}$ we have $\mathfrak{M}_B = \mathscr{B}_B$, where $B = \bigtimes_i X_i$.

For an ordered space X we shall denote by $\mathscr{S}_X$ (or simply by $\mathscr{S}$) the family of all measurable increasing subsets of X (i.e. $\mathscr{S} = \vec{\mathfrak{M}}_X \cap \mathfrak{M}_X$).

Sometimes we shall not distinguish between subsets and subspaces of an ordered space, e.g. we shall say that X_0 is an absolute measurable subspace of X, which means that X_0, considered as a set, is absolute measurable in X.

We shall also use for ordered spaces qualifications which have been defined for measurable spaces, e.g. we shall say that an ordered space X is almost Borel, which means that X, considered as a measurable space, is almost Borel.

The following ordered spaces are of special interest for our further purposes:

$\boldsymbol{R}$, which is the real line with the σ-field $\mathfrak{M}_R = \mathscr{B}_{\mathscr{R}}$ of Borel subsets of $\mathscr{R}$ ($\mathscr{R}$ is the real line considered as a topological space) and the usual order $\leqslant_R = \leqslant$;

$\boldsymbol{I}$, which is the unit interval $[0, 1]$ considered as a subspace of $\boldsymbol{R}$; thus $\mathfrak{M}_I$ is equal to $\mathscr{B}_{\mathscr{I}}$ and $\leqslant_I$ is also the usual order $\leqslant$;

$\boldsymbol{2}$, the two-element set $\{0, 1\}$, considered as a subspace of $\boldsymbol{R}$.

We shall also consider powers of already introduced spaces, e.g. $\boldsymbol{R}^n$, $\boldsymbol{I}^{\aleph_0}$, $\boldsymbol{I}^c$, $\boldsymbol{2}^{\aleph_0}$, etc. We also write $\boldsymbol{C} = \boldsymbol{2}^{\aleph_0}$ and $\boldsymbol{H} = \boldsymbol{I}^{\aleph_0}$. Obviously $\mathfrak{M}_C = \mathscr{B}_{\mathscr{C}}$, where $\mathscr{C}$ is the Cantor set and $\mathfrak{M}_H = \mathscr{B}_{\mathscr{H}}$, where $\mathscr{H}$ is the Hilbert cube.

An ordered space is in fact a combination of two abstract systems: a measurable space and an ordered set. Mathematical routine in such cases requires some connections between those systems. The natural condition:

$\vec{X}_0$ is measurable for every measurable X_0, would be too strong here, because it rejects the spaces $\boldsymbol{R}^n$ for $n \geqslant 3$, as is shown by the following:

2.1. Theorem. *The relation $\leqslant$ in $\boldsymbol{R}^n$ for $n \geqslant 3$ does not satisfy the following measurability condition: for every $C \in \mathscr{B}_{\mathscr{R}^n}$ also $\vec{C} \in \mathscr{B}_{\mathscr{R}^n}$.*

The proof will be given for $n = 3$; the constructed example can easily be transferred to higher dimensions; its idea is similar to that of the construction (due to Dubins and Freedman [5]) of a Borel set whose convex hull is not Borel.

Let A be a Borel subset of $\mathscr{I}^2$ whose projection A' on the axis $y = 1$ is not Borel (A can even be a G_δ). Define a mapping $\varphi\colon \mathscr{I}^2 \to \mathscr{R}^3$ by: $\varphi((x, y)) = (x, y, 1-x)$. Since φ is a one-to-one function, $C = \varphi(A) \in \mathscr{B}_{\mathscr{R}^3}$.

Suppose that $\vec{C}$ is Borel. Then also $B = \varphi(\mathscr{I}^2) \cap \vec{C} \in \mathscr{B}_{\mathscr{R}^3}$, and $\varphi^{-1}(B) \in \mathscr{B}_{\mathscr{I}^2}$. Any section of $\varphi^{-1}(B)$, for instance $\{(x, y) \in \varphi^{-1}(B) \mid y = 1\}$, would also be Borel, but just this one is equal to A'. The contradiction shows that $\vec{C}$ is not Borel. ■

However, some weaker conditions of this kind will be considered. They are the generability and definability properties, which are defined below.

2.2. Definition. A *d-base* of an ordered space is a defining family composed of measurable sets; a *g-base* of an ordered space X is a family composed of increasing sets which generates the σ-field $\mathfrak{M}_X$; a *base* of a space is its d-base which is also its g-base.

We say that an ordered space has the *definability property* (*$\mathfrak{m}$-definability property*, *generability property*, *$\mathfrak{m}$-generability property*) iff it has a d-base (a d-base of cardinality $\leqslant \mathfrak{m}$, a g-base, a g-base of cardinality $\leqslant \mathfrak{m}$, respectively).

We call an ordered space *$\mathfrak{m}$-proper* iff it has the $\mathfrak{m}$-definability and $\mathfrak{m}$-generability properties, for $\mathfrak{m}$ being an infinite cardinal. A space is *proper* iff it is $\mathfrak{m}$-proper for some $\mathfrak{m}$.

We see that a d-base, as well as a g-base is a family, whose every element is a measurable increasing set; however, a d-base defines the order and a g-base generates the σ-field; a base must have both these properties.

It is obvious that if we take a union of a d-base, g-base or a base with any family consisting of measurable increasing sets, we obtain again a d-base, g-base or a base. Thus we have (cf. 1.D.6):

2.3. *If $\mathscr{D}$ is a d-base and $\mathscr{G}$ is a g-base of the same ordered space, then $\mathscr{D} \cup \mathscr{G}$ is its base.* ■

Thus, for $\mathfrak{m} \geqslant \aleph_0$, an ordered space is $\mathfrak{m}$-proper if and only if it has a base of cardinality $\leqslant \mathfrak{m}$. A space is proper iff it has a base.

Obviously, for $\mathfrak{m} \leqslant \mathfrak{m}'$, the $\mathfrak{m}$-definability property implies the $\mathfrak{m}'$-definability property and the $\mathfrak{m}$-generability property implies $\mathfrak{m}'$-generability property; if a space is $\mathfrak{m}$-proper then it is $\mathfrak{m}'$-proper.

An ordered space X has the definability property iff the family $\mathscr{S}_X$ is a defining family; X has the generability property iff $\mathscr{S}_X$ generates the σ-field $\mathfrak{M}_X$; X is proper iff $\mathscr{S}_X$ is its base.

For every ordered space X and $\mathfrak{m} \geqslant \aleph_0$ the $\mathfrak{m}$-definability property is equivalent to the following condition:

(*) there exists a family $\{f_\xi |\ \xi \in \Xi\}$ with $\mathrm{Card}(\Xi) = \mathfrak{m}$ of isotone measurable functions $f_\xi\colon X \to \overline{\mathscr{R}}$ such that for every $x, y \in X$ with $x \not\leqslant y$ there exists a ξ such that $f_\xi(x) > f_\xi(y)$.

In fact, if $\{f_\xi |\ \xi \in \Xi\}$ is such a family of functions, then the family $\{Z_{\xi,q} |\ \xi \in \Xi, q \in \mathscr{Q}\}$, where $Z_{\xi,q} = \{x \in X |\ f_\xi(x) \geqslant q\}$, is a d-base of X. If $\{S_\lambda |\ \lambda \in \Lambda\}$ is a d-base of X, then the family of indicators $\{\chi_{S_\lambda} |\ \lambda \in \Lambda\}$ clearly satisfies (*).

In general, it may happen that an ordered space X has the definability property and the relation $\leqslant_X$ is $\mathfrak{m}$-definable, but X does not have the $\mathfrak{m}$-definability property. However, on the other hand, we have (cf. 1.A.2):

2.4. *If a space X has the generability property and the σ-field $\mathfrak{M}_X$ is $\mathfrak{m}$-generated, where $\mathfrak{m} \geqslant \aleph_0$, then X has the $\mathfrak{m}$-generability property.* ■

Immediately from the definition of Blackwell spaces we obtain also:

2.5. *For every ordered Blackwell space, the $\aleph_0$-definability property implies the $\aleph_0$-generability property. In this case every countable d-base is a base.* ■

Obviously, in general the generability property does not imply the definability property.

There are spaces with $\aleph_0$-generability property and definability property which do not have the $\aleph_0$-definability property. Such a space can even be standard Borel. We present below an example given by C. Ryll-Nardzewski [13]:

2.6. Example. Consider the ordered space $(\mathscr{R}, \mathscr{B}_{\mathscr{R}}, \zeta)$, where $x\zeta y$ iff $x \leqslant y$ and $y-x$ is rational. One can prove that every increasing set which belongs to $\mathscr{B}_{\mathscr{R}}$ is of the form

$$[(a, +\infty) \cup D] \setminus C\,,$$

where $a \in \overline{\mathscr{R}}$ and $D, C \in \mathscr{B}_{\mathscr{R}}$ are sets of Lebesgue measure zero. Suppose there exists a countable d-base; let us call it $\{A_i\}$ and let $A_i = [(a_i, +\infty) \cup D_i] \setminus C_i$, where $a_i \in \overline{\mathscr{R}}$ and D_i, C_i are sets of Lebesgue measure zero. Write

$$T = \{(x, y) \in \mathscr{R}^2 |\ x \leqslant y,\ y-x \text{ is irrational and } \exists_i\ x \in A_i,\ y \notin A_i\}\,.$$

Since $\{A_i\}$ is a d-base,

$$T = \{(x, y) \in \mathscr{R}^2 |\ x \leqslant y,\ y-x \text{ is irrational}\}\,;$$

thus T is equal to a halfplane without a countable number of lines and $\lambda(T) = +\infty$ (λ denotes the 2-dimensional Lebesgue measure). On the other hand, $T \subset \bigcup_i T_i$ where

$$T_i = \{(x, y) \in \mathscr{R}^2 |\ x \leqslant y,\ x \in A_i,\ y \notin A_i\}\,;$$

observe that, for every i, $T_i \subset (D_i \times \mathscr{R}) \cup (\mathscr{R} \times C_i)$; hence

$$\lambda(T_i) \leqslant \lambda(D_i \times \mathscr{R}) + \lambda(\mathscr{R} \times C_i) = 0$$

and consequently $\lambda(T) = 0$, a contradiction.

There are also ordered spaces X with the $\aleph_0$-definability property (being almost Borel), which do not have the generability property:

2.7. Example. Consider the ordered space $(\mathscr{I}, \boldsymbol{K}, \leqslant)$, where $\boldsymbol{K} = \sigma(\mathscr{B}_{\mathscr{I}} \cup \{K\})$, and K does not belong to $\mathscr{B}_{\mathscr{I}}$. The set K can be chosen so that the measurable space $(\mathscr{I}, \boldsymbol{K})$ becomes almost Borel, e.g. if K is an absolute measurable subset of $\mathscr{I}$, then $(\mathscr{I}, \boldsymbol{K})$, as a direct sum of two almost Borel spaces, $(K, \mathscr{B}_K)$ and $(\mathscr{I} \setminus K, \mathscr{B}_{\mathscr{I} \setminus K})$, is almost Borel itself. Clearly, the family $\{\{\vec{q}\} |\ q \in \mathscr{Q}\}$ is a countable d-base but no family of increasing sets generates $\boldsymbol{K}$.

From Examples 2.6 and 2.7 if follows that in general neither a d-base nor a g-base need be a base.

We shall formulate now a useful fact which follows from 1.A.3, 1.D.3 and 1.D.7:

2.8. Lemma. *If a family $\mathscr{A}$ is a d-base (resp.: g-base, base) of a space X and X_0 is a subspace of X, then $\mathscr{A}|_{X_0}$ is a d-base (resp.: g-base, base) of X_0. Thus, every subspace of a space with definability (resp.: $\mathfrak{m}$-definability, generability, $\mathfrak{m}$-generability) property has the same property and every subspace of an $\mathfrak{m}$-proper (resp. proper) space is also $\mathfrak{m}$-proper (resp. proper).* ■

The following theorem shows that ordered spaces $\boldsymbol{I}^{\mathfrak{m}}$ and $2^{\mathfrak{m}}$ are in some sense "universal" for all $\mathfrak{m}$-proper spaces:

2.9. Theorem. *For every ordered space X and every cardinal $\mathfrak{m} \geqslant \aleph_0$ the following conditions are equivalent:*

(i) *X is $\mathfrak{m}$-proper,*

(ii) *X is isomorphic with a subspace of $2^{\mathfrak{m}}$,*

(iii) *X is isomorphic with a subspace of $\boldsymbol{I}^{\mathfrak{m}}$.*

Proof. (i)⇒(ii) Let $\mathscr{A}$ be a base of X of cardinality $\leqslant \mathfrak{m}$. Arrange $\mathscr{A}$ into a sequence $(A_\xi)_{\xi\in\Xi}$ with $\mathrm{Card}(\Xi) = \mathfrak{m}$ (elements of $\mathscr{A}$ may be duplicated in this sequence if needed) and define a function $h\colon X\to 2^\Xi$ by the formula

$$(h(x))_\xi = \begin{cases} 1 \text{ if } x\in A_\xi, \\ 0 \text{ otherwise,}\end{cases} \quad \text{for } \xi\in\Xi,\ x\in X\,(^1).$$

Since $\mathscr{A}$ is a defining family, it separates points in X; thus h is a one-to-one function.

The function h is an isomorphism between X and $h(X)$, considered as measurable spaces. In fact: the family $\mathscr{A}$ generates $\mathfrak{M}_X$; $\mathfrak{M}_{h(X)}$ is generated by the family $\{T_\xi\cap h(X)|\ \xi\in\Xi\}$, where $T_\xi = \{x\in 2^\Xi|\ (x)_\xi = 1\}$ (see 1.A.3); finally, h maps the set A_ξ onto $T_\xi \cap h(X)$, for every ξ.

Let us now prove that h is an isomorphism between X and $h(X)$, considered as ordered sets. Let $x, y\in X$. Suppose first $x\leqslant y$. Since every set A_ξ is increasing, $x\in A_\xi$ implies $y\in A_\xi$, for every ξ. This means that $(h(x))_\xi\leqslant (h(y))_\xi$ for every ξ and consequently $h(x)\leqslant h(y)$. Suppose now that $x\nleqslant y$. Since $\mathscr{A}$ is a defining family, there exists a ξ_0 such that $x\in A_{\xi_0}$ and $y\notin A_{\xi_0}$. Thus $(h(x))_{\xi_0} > (h(y))_{\xi_0}$ and $h(x)\nleqslant h(y)$.

The implication (ii)⇒(iii) is entirely trivial. The implication (iii)⇒(i) is also easy: without loss of generality assume that X is simply a subspace of I^Ξ, where $\mathrm{Card}(\Xi) = \mathfrak{m}$. Since I^Ξ is $\mathfrak{m}$-proper (the family

$$\{U_q^\xi|\ \xi\in\Xi, q\in\mathscr{Q}\},\quad \text{where}\quad U_q^\xi = \{x\in I^\Xi|\ (x)_\xi > q\},$$

is a base of I^Ξ), it suffices to apply Lemma 2.8. ■

In fact, when proving Theorem 2.9, we have proved something more than was formulated in it, namely:

2.10. *For every proper space X and its base $\{A_\xi|\ \xi\in\Xi\}$ there exists an embedding h of X into 2^Ξ (and into I^Ξ) such that, for every ξ, $h(A_\xi) = T_\xi\cap h(X)$, where $T_\xi = \{x\in 2^\Xi|\ (x)_\xi = 1\}$.* ■

Let us note an interesting consequence of Theorem 2.9. Namely, there exists an embedding Ψ of the measurable space $(\mathscr{H}, \mathscr{B}_\mathscr{H})$ into $(\mathscr{I}, \mathscr{B}_\mathscr{H})$ which is isotone, i.e., for every $x, y\in\mathscr{H}$, $x\leqslant y$ implies $\Psi(x)\leqslant\Psi(y)$. To show this let Ψ_1 be any embedding of the ordered space $\boldsymbol{H}$ into $\boldsymbol{C}$. Define $\Psi_2\colon \boldsymbol{C}\to\mathscr{I}$ by the formula $\Psi_2(x) = \Sigma(x)_i/3^{i+1}$. The function $\Psi = \Psi_2\Psi_1$ is the desired embedding.

One could object that our definition of ordered spaces is too general, and propose the following "natural" approach: to begin with an ordered set X and consider it as an ordered space with the "interval σ-field", which is the σ-field generated by all sets $\{\vec{a}\}$ and $\{\overleftarrow{a}\}$ with $a\in X$. Let us call ordered spaces of this form *natural*.

(¹) The function h is actually the Marczewski's [9] characteristic function of a sequence of sets.

In the author's opinion, however, this approach would be too restrictive; usually subspaces of $\boldsymbol{H}$ are not natural in this sense (an example: the set

$$\{(x, y)\in \boldsymbol{R}^2|\ x, y\geqslant 0,\ x+y = 1\}$$

considered as a subspace of $\boldsymbol{R}^2$).

The theory presented in the paper covers the case of natural spaces. It is easy to check that every natural space X is proper (the family of all sets $\{\vec{a}\}$ and $\{\vec{a}\}^c$ is its base). Moreover, for every $\aleph_0$-proper ordered space X, the σ-field $\mathfrak{M}_X$ contains the interval σ-field.

3. Extension of an ordering relation.

Our objective now is to define for every proper ordered space X a natural extension of the relation $\leqslant$ to the set X^* of all probabilities on $\mathfrak{M}_X$. We use the word "extension", because X can be considered as a subspace of X^*, when $x\in X$ is identified with ϑ_x.

For a proper ordered space X we define below an ordering relation in X^*. The sense of this operation can best be seen on the example of the ordered real line $\boldsymbol{R}$. Probabilities P, Q can be represented here by their distribution functions F_P, F_Q. Then $P\leqslant Q$ if and only if the graph of F_Q lies entirely below the graph of F_P.

3.1. Definition. The *probabilistic extension* of a proper ordered space X is the ordered space X^* of all probabilities on $\mathfrak{M}_X$ with the σ-field generated by the family $[\mathscr{I}|\mathscr{S}_X]$ and with the order defined by the same family. This σ-field will be denoted by $\mathfrak{M}_{X^*}$ and the order in question will be denoted by $\leqslant_{X^*}$ (or sometimes simply by $\leqslant$).

Since the space X under consideration is assumed to be proper, this definition of $\mathfrak{M}_{X^*}$ coincides (because of Theorem 1.B.5 and 1.D.2) with that given in Section 1.A.

For any two probabilities $P, Q\in X^*$ we obviously have $P\leqslant Q$ if and only if $P(A)\leqslant Q(A)$ for every measurable increasing set A or, equivalently, if and only if $Q(B)\leqslant P(B)$ for every measurable decreasing set B.

The first thing we should check is that X^* is in fact an ordered space, i.e. that the defined relation in X^* is antisymmetric. This will be done if we show that the defining family $[\mathscr{I}|\mathscr{S}_X]$ separates points in X^*. It is known (see e.g. Bauer [1], Ch. I, Thm. 5.5) that, for every σ-field $\mathfrak{M}$ and every family $\mathscr{A}$ which generates it, two probabilities on $\mathfrak{M}$ are equal iff they are equal on all sets in $\mathscr{A}_d$. But, in fact, the family $\mathscr{S}_X$ is multiplicative and generates $\mathfrak{M}_X$ (X is proper!); thus whenever the probabilities $P, Q\in X^*$ are different, they are different on some set $A\in\mathscr{S}_X$ and hence for some $a\in\mathscr{I}$, $P\in[a|A]$ and $Q\notin[a|A]$, or conversely, $Q\in[a|A]$ and $P\notin[a|A]$.

Obviously, $[\mathscr{I}|\mathscr{S}_X]$ is a base of X^* and hence X^* is proper. Also the families $[\mathscr{Q}|\mathscr{S}_X]$, $[\mathscr{I}||\mathscr{S}_X]$ and $[\mathscr{Q}||\mathscr{S}_X]$ are bases of X^*.

We can prove even more:

3.2. THEOREM. *For every* $\mathfrak{m}$*-proper ordered space* X, *the space* X^* *is* $\mathfrak{m}$*-proper if* $\mathfrak{m} \geqslant \mathfrak{c}$ *and* $\mathfrak{c}$*-proper if* $\mathfrak{m} \leqslant \mathfrak{c}$.

Proof. Let $\mathscr{A}$ be a base of X with $\mathrm{Card}(\mathscr{A}) \leqslant \mathfrak{m}$. Since every $\mathfrak{m}$-generated σ-field has at most $\aleph_1 \cdot \mathfrak{m}^{\aleph_0} = \mathfrak{m} \cdot \mathfrak{c}$ elements, we have $\mathrm{Card}(\mathscr{S}_X) \leqslant \mathfrak{m} \cdot \mathfrak{c}$. But $[\mathscr{I} | \mathscr{S}_X]$ is a base of X^* and hence X^* is $\mathfrak{c} \cdot \mathfrak{m} \cdot \mathfrak{c} = \mathfrak{m} \cdot \mathfrak{c}$-proper. ■

We shall now show that the relation $\leqslant$ in X^* is in fact an extension of $\leqslant$ in X.

3.3. THEOREM. *For every proper ordered space* X, *the function* ϑ: $X \to X^*$ *is an embedding of* X *into* X^*.

Proof. It follows from 1.B.8 that ϑ is an embedding in the sense of measurable spaces. We shall now prove that it is an embedding in the sense of ordered sets.

Choose $x, y \in X$ let $x \leqslant y$. Since $\mathscr{S}_X$ is a defining family, for every $A \in \mathscr{S}_X$, $x \in A$ implies $y \in A$; thus $\vartheta_x(A) \leqslant \vartheta_y(A)$ for every $A \in \mathscr{S}_X$ and hence $\vartheta_x \leqslant \vartheta_y$.

Suppose now that $x \nleqslant y$. Since $\mathscr{S}_X$ is a defining family, there exists a set $A_0 \in \mathscr{S}_X$ such that $x \in A_0$ and $y \notin A_0$; hence $1 = \vartheta_x(A_0) > \vartheta_y(A_0) = 0$ and $\vartheta_x \nleqslant \vartheta_y$. ■

From this Theorem and 1.D.7 we obtain:

3.4. *Let* X *be a proper ordered space and let* $\mathscr{A} \subset \mathfrak{M}_X$. *If* $[\mathscr{I} | \mathscr{A}]$ *is a defining family in* X^*, *then* $\mathscr{A}$ *is a defining family in* X.

4. Connection of the order in X^* with some ways of comparing probabilities related to isotone functions.

For a given proper ordered space X, we denote:

$\mathscr{M}$ = the class of all measurable isotone functions f: $X \to \overline{\mathscr{R}}$;

$\mathscr{M}^b$ = the subclass of $\mathscr{M}$ consisting of all bounded functions;

$\mathscr{T}^b$ = the subclass of $\mathscr{M}^b$ consisting of all functions f which are strictly isotone (i.e. $x \leqslant y$ and $y \nleqslant x$ implies $f(x) < f(y)$).

For every function $u \in \mathscr{M}$ we define u^*: $X^* \to \overline{\mathscr{R}}$ by the formula

$$u^*(P) = \begin{cases} \int u\,dP & \text{if the integral is finite or diverges to } -\infty \text{ or } +\infty; \\ -\infty, & \text{otherwise.} \end{cases}$$

For every $u \in \mathscr{M}$ we define also a pre-ordering relation $\underset{u^*}{\lesssim}$ in X^*: for $P, Q \in X^*$, $P \underset{u^*}{\lesssim} Q$ iff $u^*(P) \leqslant u^*(Q)$ (1).

(1) In the economic applications the class $\mathscr{M}$ can be interpreted as the class of all utility functions on the ordered space X. Sometimes utility functions are assumed to be bounded or strictly isotone and this assumptions correspond, respectively, to classes $\mathscr{M}^b$ and $\mathscr{T}^b$.

The rule which associates with a utility function u the relation $\underset{u^*}{\lesssim}$ in the set of probability distributions (prospects) is in the economic literature referred to as The Expected Utility Hypothesis (cf. Borch [4], pp. 30–31. If $X = \boldsymbol{R}$ and $u = \mathrm{id}_{\boldsymbol{R}}$, we simply obtain here the classical rule of comparison of probabilities on $\mathscr{B}_{\mathscr{R}}$ by means of their expectations.

The following theorem shows connections between the relation $\leqslant_{X^*}$ and the relations $\underset{u^*}{\lesssim}$ in X^*:

4.1. THEOREM. *For every proper ordered space* X *and every* $P, Q \in X^*$ *the following conditions are equivalent*:

(i) $P \leqslant Q$;

(ii) $P \underset{u^*}{\lesssim} Q$ *for every function* $u \in \mathscr{M}^b$;

(iii) $P \underset{u^*}{\lesssim} Q$ *for every function* $u \in \mathscr{M}$.

Proof. (i)$\Rightarrow$(ii) Let $u \in \mathscr{M}^b$. It suffices to consider the case where $u(X) \subset \mathscr{I}$. Let us denote $B_i^n = \{x \in X |\ u(X) \geqslant i/2^n\}$ for $n \in \mathscr{N}$, $i = 0, \ldots, 2^{n-1}$. Let $u_i^n = 2^{-n}\chi_{B_i^n}$ ($\chi_{B_i^n}$ are indicator functions) and $u^n = \sum_{i=0}^{2^n-1} u_i^n$. Every set B_i^n is increasing, and hence $(u_i^n)^*(P) \leqslant (u_i^n)^*(Q)$ for all n, i and $(u^n)^*(P) \leqslant (u^n)^*(Q)$ for all n. The inequality $u^*(P) \leqslant u^*(Q)$ follows from the Monotone Convergence Theorem.

(ii)$\Rightarrow$(iii) Let $u \in \mathscr{M}$. Suppose first that u is bounded from below. Each of the functions $u_n = \min\{n, u\}$ is bounded and isotone, and so $u_n^*(P) \leqslant u_n^*(Q)$. The sequence (u_n) is nondecreasing and converges to u. Using the Monotone Convergence Theorem, we obtain $u^*(P) \leqslant u^*(Q)$, i.e. $P \underset{u^*}{\lesssim} Q$.

If u is bounded from above, the proof is analogous. Let u be an arbitrary function in $\mathscr{M}$. Let $u^+ = \max\{0, u\}$, $u^- = \max\{0, -u\}$. Obviously $u = u^+ - u^-$. The functions u^+ and $-u^-$ are isotone, first of them being bounded from below and the second from above. Thus we have $(u^+)^*(P) \leqslant (u^+)^*(Q)$ and $(-u^-)^*(P) \leqslant (-u^-)^*(Q)$. Then $u^*(P) \leqslant u^*(Q)$ (observe that we do not simply add the terms of these inequalities, but also use the definition of u^*, setting $u^*(P) = -\infty$ if $(u^+)^*(P) = (-u^-)^*(P) = +\infty$).

(iii)$\Rightarrow$(i) Let $A \in \mathscr{S}_X$. The indicator function χ_A belongs to $\mathscr{M}$ and hence $P(A) = \chi_A^*(P) \leqslant \chi_A^*(Q) = Q(A)$. ■

4.2. *If for a proper ordered space* X *the corresponding class* $\mathscr{T}^b$ *is nonempty, then for every* P, $Q \in X^*$ *each of the conditions* (i)$\Rightarrow$(iii) *of the previous theorem is equivalent to*:

(iv) $P \underset{u^*}{\lesssim} Q$ *for every function* $u \in \mathscr{T}^b$.

Proof. The implication (iii)$\Rightarrow$(iv) is trivial, and so we shall prove (iv)$\Rightarrow$(i). It suffices to consider the case when there exist $x, y \in X$ with $x \leqslant y$ and $y \nleqslant x$ (the opposite case is trivial). Then there exists a function $p \in \mathscr{T}^b$ with $\inf_x p(x) = 0$, $\sup_x p(x) = 1$. Let $A \in \mathscr{S}_X$ and consider the sequence of functions

$$g_n(x) = \frac{1}{n+1} p(x) + \frac{n}{n+1} \chi_A(x) \quad \text{for} \quad n \in \mathscr{N}.$$

Since each g_n belongs to $\mathscr{T}^b$, we have

$$\int_X g_n(x) P(dx) \leqslant \int_X g_n(x) Q(dx).$$

Now from the Monotone Convergence Theorem we obtain

$$P(A) = \int_X \chi_A(x)P(dx) \leqslant \int_X \chi_A(x)\,Q(dx) = Q(A)\,.$$

Since A was arbitrary, we find that $P \leqslant Q$. ■

Every $\aleph_0$-proper space X satisfies the assumptions of 4.2. In fact, as follows from Theorem 2.9, X can be assumed to be a subspace of C. The function $p\colon X \to \mathscr{R}$, where $p(x) = \sum (x)_i/3^{i+1}$, clearly belongs to $\mathscr{T}^b$.

5. Superbases of proper ordered spaces.

We shall now be interested in studying bases (g-bases, d-bases) in spaces X^*. In particular, we shall look for bases of especially simple form $[\mathscr{Q}|\mathscr{A}]$ for some $\mathscr{A} \subset \mathscr{S}_X$.

5.1. Definition. A *d-superbase* (*g-superbase*, *superbase*, respectively) of a proper ordered space X is any family $\mathscr{A} \subset \mathscr{S}_X$ such that the family $[\mathscr{Q}|\mathscr{A}]$ is a d-base (g-base, base, respectively) of X^*.

In the definition above, the family $[\mathscr{Q}|\mathscr{A}]$ can be replaced by any of the following: $[\mathscr{Q}||\mathscr{A}]$, $[\mathscr{I}|\mathscr{A}]$, $[\mathscr{I}||\mathscr{A}]$.

Clearly, the family $\mathscr{S}_X$ is always a superbase of X.

Let us formulate an equivalent condition for a family of sets to be a d-superbase: For an ordered space X and $P, Q \in X^*$ we denote by $\gamma(P, Q)$ (or $\gamma_X(P, Q)$) the family of all sets $A \in \mathscr{S}_X$ such that $P(A) \leqslant Q(A)$. Now we see that

5.2. *A family $\mathscr{A} \subset \mathscr{S}_X$ is a d-superbase if and if for every $P, Q \in X^*$ the following implication holds*:

$$\mathscr{A} \subset \gamma(P, Q) \Rightarrow P \leqslant Q\,.$$

Thus a d-superbase is such a family $\mathscr{A}$ that, for all probabilities P, Q, their values on the elements of $\mathscr{A}$ determine whether P and Q are in the relation $\leqslant$ or not. Proposition 5.2 will very often be used in considering d-superbases.

Observe also that:

5.3. *In every ordered space X the class $\gamma(P, Q)$ is monotone i.e. if $A_i \in \gamma(P, Q)$ and $A_i \nearrow A$ or $A_i \searrow A$, then $A \in \gamma(P, Q)$, for every $P, Q \in X^*$.* ■

We are now in a position to prove the following theorems concerning superbases:

5.4. Theorem. *Every d-superbase (g-superbase, superbase) of a proper ordered space is its d-base (g-base, base, respectively).*

5.5. Theorem. *For every g-base $\mathscr{A}$ of a proper ordered space X, each of the families $\mathscr{A}_d$ and $\mathscr{A}_s$ is a g-superbase of X.*

The proof of Theorem 5.4 follows from 3.4 and 1.B.10, while the proof of Theorem 5.5 follows from Theorem 1.B.5 and Corollary 1.B.6. ■

We shall also prove

5.6. Theorem. *For every proper ordered space X and its subspace X_0, if a family $\mathscr{A}$ is a g-superbase of X, then $\mathscr{A}|_{X_0}$ is a g-superbase of X_0.*

Proof. The family $[\mathscr{Q}|\mathscr{A}]$ generates $\mathfrak{M}_X$; thus by virtue of 1.A.3, the family $[\mathscr{Q}|\mathscr{A}]|_Z$, where $Z = \{P \in X^*|\ \bar{P}(X_0) = 1\}$, generates the σ-field $\mathfrak{M}_Z$ in the subspace Z.

As follows from 1.A.4, the function $\varphi\colon X_0^* \to Z$, where $\varphi(\mu) = \mu||^X$, is an isomorphism between measurable spaces $(X_0^*, \mathfrak{M}_{X_0^*})$ and $(Z, \mathfrak{M}_Z)$. Since for every $q \in \mathscr{Q}$, $A \in \mathscr{A}$, $\varphi^{-1}([q|\mathscr{A}] \cap Z) = [q|A \cap X_0]$ and the family $\{[q|A] \cap Z | q \in \mathscr{Q},\ A \in \mathscr{A}\}$ generates $\mathfrak{M}_Z$, we find that the family $\{[q|A \cap X_0]|\ q \in \mathscr{Q}, A \in \mathscr{A}\} = [\mathscr{Q}|(\mathscr{A}|_{X_0})]$ generates $\mathfrak{M}_{X_0^*}$, which means that $\mathscr{A}|_{X_0}$ is a g-superbase of X_0. ■

However, we are still unable to tell much about d-superbases and superbases of an ordered space. We give below an example of a base $\mathscr{A}$ such that even the family $\mathscr{A}_d$ is not a superbase:

5.7. Example. Let us consider the ordered space $\boldsymbol{R}^2$. The family $\mathscr{A}$ consisting of all sets of the form

$$A_q = \{x \in \boldsymbol{R}^2|\ (x)_1 \geqslant q)\}, \qquad B_r = \{x \in \boldsymbol{R}^2|\ (x)_2 \geqslant r\}\,,$$

where q, r are rational, is a base but not a superbase of $\boldsymbol{R}^2$. Define probabilities P, Q on $\mathscr{B}_{\mathscr{R}^2}$ by the formulas

$$P(\{(0,1)\}) = P(\{(1,0)\}) = 0.2, \qquad P(\{(0,0)\}) = P(\{(1,1)\}) = 0.3\,,$$

$$Q(\{(0,1)\}) = Q(\{(1,0)\}) = 0, \qquad Q(\{(0,0)\}) = 0.4, \qquad Q(\{(1,1)\}) = 0.6\,.$$

It is easy to check that for all $A \in \mathscr{A}_d$, $P(A) \leqslant Q(A)$, and hence $\mathscr{A}_d \subset \gamma(P, Q)$, but $P \leqslant Q$ does not hold, because $P(C) > Q(C)$, where $C = \{x \in \boldsymbol{R}^2|\ (x)_1 + (x)_2 > \frac{1}{2}\}$. Thus $\mathscr{A}_d$ is not a d-superbase of R^2.

We now turn to considering methods of contruction of d-superbases of proper ordered spaces.

5.8. Lemma. *Let X be a proper ordered space. If an additive family $\mathscr{Z} \subset \mathfrak{M}_X$ satisfies the conditions*

(i) $\vec{\mathscr{Z}} \subset \mathscr{Z}$,

(ii) *for every $P \in X^*$, $A \in \mathfrak{M}_X$, $\varepsilon > 0$ there exists a set $B \in \mathscr{Z}$, $B \subset A$ such that* $P(A \setminus B) \leqslant \varepsilon$.

Then the family $\vec{\mathscr{Z}}$ is a d-superbase of X.

Proof. Let P, Q be any probabilities on $\mathfrak{M}_X$ and let $P(\vec{Z}) \leqslant Q(\vec{Z})$ for every $Z \in \mathscr{Z}$. We have to prove that $P(A) \leqslant Q(A)$ for every $A \in \mathscr{S}_X$.

First, let $A \in \mathscr{Z}_\sigma \cap \mathscr{S}_X$, i.e. $A = \bigcup A_i$, where $A_i \in \mathscr{Z}$; let us denote $W_i = \bigcup_{j=0}^{i} A_j$. Obviously $W_i \nearrow A$, and hence also $\vec{W}_i \nearrow A$. Since $\mathscr{Z}$ is additive, $P(\vec{W}_i) \leqslant Q(\vec{W}_i)$ for every i. The family $\gamma(P, Q)$ is monotone (cf. 5.3); thus $P(A) \leqslant Q(A)$.

Let A be any element of $\mathscr{S}_X$. It follows from (ii) that there exists a set $B \in \mathscr{Z}_\sigma$, $B \subset A$, with $P(B) = P(A)$. From (i) we obtain $\vec{B} \in \mathscr{Z}_\sigma \cap \mathscr{S}_X$, and obviously also $\vec{B} \subset A$. Hence $P(A) = P(\vec{B}) \leqslant Q(\vec{B}) = Q(A)$. ■

5.9. THEOREM. *For a proper ordered space X let a family $\mathscr{A}\subset\mathfrak{M}_X$ satisfy the conditions*

(i) *$\mathscr{A}$ is multiplicative and exhaustive and generates $\mathfrak{M}_X$,*

(ii) $\vec{\mathscr{A}}\subset\mathfrak{M}_X$,

(iii) $\overrightarrow{\mathscr{A}_{c\delta}}\subset\mathscr{A}_{c\delta}$.

Then the family $(\overleftarrow{\mathscr{A}})_{cd}$ is a d-superbase of X.

Proof. As follows from Theorem 1.B.3, the family $\mathscr{A}_{c\delta}$ satisfies the assumptions of Lemma 5.8 ($\mathscr{A}_{c\delta}$ is additive, because $\mathscr{A}$ is multiplicative). Hence $\overrightarrow{\mathscr{A}_{c\delta}} = \mathscr{A}_{c\delta}\cap\mathscr{S}_X$ is a d-superbase of X. Let P, Q be any probabilities on $\mathfrak{M}_X$. Thus it suffices to prove the implication

$$(\overleftarrow{\mathscr{A}})_{cd}\subset\gamma(P, Q) \Rightarrow \mathscr{A}_{c\delta}\cap\mathscr{S}_X\subset\gamma(P, Q).$$

Let $A\in\mathscr{A}_{c\delta}\cap\mathscr{S}_X$; then $A = \bigcap\limits_i A_i$, where $A_i\in\mathscr{A}_c$. As follows from Lemma 1.D.5, $A = \bigcap A_i^{\odot}$ (recall that $A_i^{\odot} = (\overleftarrow{A_i^c})^c$). Let $B_i = \bigcap\limits_{j=0}^{i} A_j^{\odot}$. Then $B_i\searrow A$. Since every $A_i\in\mathscr{A}_c$, we have $A_i^c\in\mathscr{A}$ and $A_i^{\odot}\in(\overleftarrow{\mathscr{A}})_c$; hence every $B_i\in(\overleftarrow{\mathscr{A}})_{cd}$. Thus every $B_i\in\gamma(P, Q)$ and, by virtue of 5.3, $A\in\gamma(P, Q)$. ■

5.10. COROLLARY. *If a family $\mathscr{A}$ of subsets of a proper ordered space X satisfies the conditions*

(i) *$\mathscr{A}$ is multiplicative and exhaustive and generates $\mathfrak{M}_X$,*

(ii) $\overleftarrow{\mathscr{A}}\subset\mathfrak{M}_X$,

(iii) $\overleftarrow{\mathscr{A}_{c\delta}}\subset\mathscr{A}_{c\delta}$

then the family $(\vec{\mathscr{A}})_s$ is a d-superbase of X.

To show this, let us consider the same ordered space, but with the inverse order: $\tilde{X} = (X, \mathfrak{M}_X, \geqslant)$ (i.e. $x\geqslant y$ iff $y\leqslant x$). All the assumptions of Theorem 5.9 are satisfied. Thus the family $[\overleftarrow{\mathscr{A}}^{\tilde{X}}]_{cd} = [\vec{\mathscr{A}}^X]_{cd}$ is a d-superbase of $\tilde{X}$. It is easy to see that

$$[A\in\gamma_X(P, Q) \text{ iff } A^c\in\gamma_{\tilde{X}}(P, Q)] \quad \text{for} \quad A\in\mathscr{S}_X,$$

$$[P\leqslant_{X^*}Q \text{ iff } Q\geqslant_{\tilde{X}^*}P] \quad \text{for} \quad P, Q\in X^*.$$

Thus for every $P, Q\in X^*$ we have

$$(\vec{\mathscr{A}}^X)_s\subset\gamma_X(P, Q) \text{ iff } [\overleftarrow{\mathscr{A}}^{\tilde{X}}]_{cd}\subset\gamma_{\tilde{X}}(P, Q) \text{ iff } Q\geqslant_{\tilde{X}^*}P \text{ iff } P\leqslant_{X^*}Q. \ ■$$

6. Semi-regular ordered spaces.

Semi-regularity is an auxiliary notion which is particularly helpful in investigations of $\aleph_0$-proper spaces. There are two very important properties of semi-regular spaces. The first is that the restriction of a d-superbase of a semi-regular space X to an absolute measurable subset X_0 is a d-superbase of the space X_0.

The second important property of semi-regular spaces is that every absolute measurable subspace of a semi-regular space is semi-regular. This, together with the fact that $\boldsymbol{H}$ is semi-regular, will give us a broad class of semi-regular spaces, namely that of all absolute measurable subspaces of $\boldsymbol{H}$.

6.1. DEFINITION. An ordered space X is said to be *semi-regular* iff it is proper and, for any probabilities $P, Q\in X^*$, $P\leqslant Q$ implies that, for every $A\in\mathfrak{M}_X$, $P(A)\leqslant Q(\vec{A})$.

The next three theorems show some connections between a semi-regular space and its absolute measurable subspaces:

6.2. THEOREM. *Let X_0 be an absolute measurable subspace of a semi-regular ordered space X. Then for all the probabilities μ, ν on $\mathfrak{M}_{X_0}$.*

$$\mu\leqslant_{X_0^*}\nu \quad \textit{iff} \quad \mu||^X\leqslant_{X^*}\nu||^X.$$

Proof. (A) *Let us assume first that $X_0\in\mathfrak{M}_X$*. We have to prove that the conditions

(a) $$\mu(A_0)\leqslant\nu(A_0) \quad \text{for every } A_0\in\mathscr{S}_{X_0}$$

and

(b) $$\mu||^X(A)\leqslant\nu||^X(A) \quad \text{for every } A\in\mathscr{S}_X$$

are equivalent.

The implication (a)$\Rightarrow$(b) is obvious: for every $A\in\mathscr{S}_X$ we have $A_0 = A\cap X_0\in\mathscr{S}_{X_0}$ and $\mu||^X(A) = \mu(A_0)\leqslant\nu(A_0) = \nu||^X(A)$.

Now, (b) implies (a): Let $A_0\in\mathscr{S}_{X_0}$. Since $X_0\in\mathfrak{M}_X$, we have $A_0\in\mathfrak{M}_X$. From (b) and the definition of semi-regularity we obtain $\mu||^X(A_0)\leqslant\nu||^X(\vec{A}_0^X)$. But $\nu||^X(\vec{A}_0^X) = \nu(A_0)$ and $\mu||^X(A_0) = \mu(A_0)$, and so we find that $\mu(A_0)\leqslant\nu(A_0)$.

(B) *Let X_0 be any absolute measurable subspace of X*. For any probabilities μ, ν on $\mathfrak{M}_{X_0}$ there is a set $K\in\mathfrak{M}_X$, $K\subset X_0$ with $\mu(K) = \nu(K) = 1$. Applying our theorem twice in the already proved case (A) (first $\mu||_K||^X = \mu||^X$, $\nu||_K||^X = \nu||^X$, second $\mu||_K||^{X_0} = \mu$, $\nu||_K||^{X_0} = \nu$), we obtain (K will now be considered as a subspace)

$$\mu||_K\leqslant_{K^*}\nu||_K \quad \text{iff} \quad \mu||^X\leqslant_{X^*}\nu||^X$$

and

$$\mu||_K\leqslant_{K^*}\nu||_K \quad \text{iff} \quad \mu\leqslant_{X_0^*}\nu.$$

Thus $\mu\leqslant_{X_0^*}\nu$ iff $\mu||^X\leqslant_{X^*}\nu||^X$ and this completes the proof. ■

From this theorem and 1.A.4 it follows that

6.3. *For every semi-regular space X and its absolute measurable subspace X_0 the function φ: $X_0^*\to X^*$, where $\varphi(\mu) = \mu||^X$, is an embedding and, moreover, $\varphi(X_0^*) = \{P\in X^*|\ \bar{P}(X_0) = 1\}$.* ■

6.4. THEOREM. *Let X be a semi-regular ordered space and let X_0 be an absolute measurable subspace of X. If a family $\mathscr{A}$ is a d-superbase of X, then $\mathscr{A}|_{X_0}$ is a d-superbase of X_0.*

Proof. (A) *First let $X_0\in\mathfrak{M}_X$*. Suppose that $\mathscr{A}|_{X_0}$ is not a d-superbase of X_0. Then there exist probabilities μ, ν on $\mathfrak{M}_{X_0}$ such that $\mathscr{A}|_{X_0}\subset\gamma_{X_0}(\mu, \nu)$ but not $\mu\leqslant_{X_0^*}\nu$.

We shall now show that in this case $\mathscr{A}$ cannot be a d-superbase of X. Let us consider $\mu||^X$ and $\nu||^X$. Since $\mu(A\cap X_0)\leqslant\nu(A\cap X_0)$, we have $\mu||^X(A)\leqslant\nu||^X(A)$. Thus $\mathscr{A}\subset\gamma_X(\mu||^X,\nu||^X)$.

Since $\mu\leqslant_{X_0^*}\nu$ does not hold, there exists a $Z\in\mathscr{S}_{X_0}$ such that $\mu(Z)>\nu(Z)$. This means that $\mu||^X(Z)>\underline{\nu||}^X(\vec{Z}^X)$. Since $Z\in\mathfrak{M}_X$, the semi-regularity of X implies that $\mu||^X\not\leqslant_{X^*}\nu||^X$. This contradicts our assumption ($\mathscr{A}$ should be a d-superbase of X).

(B) *Let X_0 be any absolute measurable subspace of X.* Let μ,ν be any probabilities on $\mathfrak{M}_{X_0}$ such that $\mathscr{A}\subset\gamma_{X_0}(\mu,\nu)$. We have to prove $\mu\leqslant_{X_0^*}\nu$. There exists a set (which will be also considered as a subspace) $K\in\mathfrak{M}_X$, $K\subset X_0$ with $\mu(K)=\nu(K)=1$. It is easy to see that the condition $\mathscr{A}\subset\gamma_{X_0}(\mu,\nu)$ implies $\mathscr{A}|_K\subset\gamma_K(\mu||_K,\nu||_K)$ (cf. 1.D.3). As follows from the already proved case (A), $\mathscr{A}|_K$ is a d-superbase of K. Hence $\mu||_K\leqslant_{K^*}\nu||_K$. It follows from Theorem 6.2 that $\mu\leqslant_{X_0^*}\nu$. ■

From this theorem and Theorem 5.6 we obtain

6.5. Corollary. *For every semi-regular ordered space X and its absolute measurable subspace X_0, if a family $\mathscr{A}$ is a superbase of X, then $\mathscr{A}|_{X_0}$ is a superbase of X_0.* ■

Let us now prove the last theorem of this series:

6.6. Theorem. *An absolute measurable subspace X_0 of a semi-regular space X is also semi-regular.*

Proof. Let the probabilities μ,ν on $\mathfrak{M}_{X_0}$ satisfy the condition $\mu\leqslant_{X_0^*}\nu$ and let $Z\in\mathfrak{M}_{X_0}$. We should prove that $\mu(Z)\leqslant\underline{\nu}(\vec{Z}^{X_0})$. It follows from Theorem 6.2 that $\mu||^X\leqslant_{X^*}\nu||^X$. There exists a set $K\in\mathfrak{M}_X$, $K\subset Z$, with $\mu(K)=\mu(Z)$ and $\nu(K)=\nu(Z)$. By the semi-regularity of X we have $\mu||^X(K)\leqslant\nu||^X(\vec{K}^X)=\underline{\nu}(\vec{K}^{X_0})$. Hence $\mu(Z)\leqslant\underline{\nu}(\vec{Z}^{X_0})$ (because $\vec{K}^{X_0}\subset\vec{Z}^{X_0}$).

Now the theorem follows from Lemma 2.8. ■

We are now going to prove that the ordered space $\boldsymbol{H}$ is semi-regular. For this purpose the following lemmas are needed:

6.7. Lemma. *Let X be a proper ordered space. If there exists a family $\mathscr{Z}\subset\mathfrak{M}_X$ such that*

(i) $\vec{\mathscr{Z}}\subset\mathfrak{M}_X$,

(ii) *for every $P\in X^*$, every $A\in\mathfrak{M}_X$ and every $\varepsilon>0$ there exists a $B\in\mathscr{Z}$, $B\subset A$ such that $P(A\setminus B)\leqslant\varepsilon$.*

Then X is semi-regular.

Proof. Choose any probabilities P, Q on $\mathfrak{M}_X$ with $P\leqslant Q$. What we need to prove is that $P(A)\leqslant Q(\vec{A})$ for every $A\in\mathfrak{M}_X$. First let $A\in\mathscr{Z}_\sigma$, i.e. $A=\bigcup A_i$, where $A_i\in\mathscr{Z}$. Then obviously $\vec{A}=\bigcup\vec{A}_i\in\mathfrak{M}_X$ and we have $P(A)\leqslant P(\vec{A})\leqslant Q(\vec{A})=Q(A)$ (because $\vec{A}\in\mathscr{S}_X$).

Let A be any element of $\mathfrak{M}_X$. From (ii) follows the existence of a set $B'\in\mathscr{Z}_\sigma$, $B'\subset A$ with $P(A)=P(B')$. Thus we obtain $P(A)=P(B')\leqslant Q(\vec{B}')\leqslant Q(\vec{A})$. ■

6.8. Lemma. *If Z is a closed subset of the Hilbert cube $\mathscr{H}$, then $\vec{Z}$ (considered in the sense of $\boldsymbol{H}$) is also closed.*

Proof. Let (y^i) be any sequence such that $y^i\in\vec{Z}$ and $y^i\to y$. We have to prove that $y\in\vec{Z}$. For every i there exists an $x^i\in Z$ with $x^i\leqslant y^i$. Since Z is compact, there is a convergent subsequence $x^{k_i}\to x\in Z$. Since $x^{k_i}\leqslant y^{k_i}$, we have $x\leqslant y$; hence $y\in\vec{Z}$. ■

Now we can proceed to the following:

6.9. Theorem. *Every absolute measurable subspace of the ordered Hilbert cube $\boldsymbol{H}$ is semi-regular.*

Proof. As follows from Theorem 6.6, it suffices to prove that $\boldsymbol{H}$ is semi-regular.

The family of all closed subsets of $\mathscr{H}$ satisfies assumption (i) of Lemma 6.7. Let $\mathscr{A}$ be any countable base of open sets (in the topological sense) of $\mathscr{H}$. We can assume $\mathscr{A}$ to be multiplicative. Since $\mathscr{A}_\sigma\subset\mathscr{A}_{c\delta\sigma}$ (every open set is an F_σ), $\mathscr{A}$ is exhaustive. Thus, as follows from Theorem 1.B.3, the family $\mathscr{Z}=\mathscr{A}_{c\delta}$ satisfies assumption (ii) of Lemma 6.7 (in fact, it is not necessary to use Theorem 1.B.3 in its general form here; its usual, topological form would be sufficient). Thus from Lemma 6.7 it follows that $\boldsymbol{H}$ is semi-regular. ■

7. Regular ordered spaces.

We shall define here a class of ordered spaces which have some "nice" properties: these spaces have countable bases and superbases; for regular spaces there is a very simple rule of construction of a superbase from a given base; finally, the class of regular spaces is closed under the operation $X\to X^*$.

7.1. Definition. An ordered space X is *regular* iff it is $\aleph_0$-proper and almost Borel.

7.2. *Every absolute measurable subspace of a regular ordered space is regular.*

7.3. Theorem. *For every ordered space X the following conditions are equivalent:*

(i) *X is regular,*

(ii) *X is isomorphic with an absolute measurable subspace of $\boldsymbol{C}$,*

(iii) *X is isomorphic with an absolute measurable subspace of $\boldsymbol{H}$.*

Proof. 7.2 follows from Lemma 2.8 and 1.C.3, while Theorem 7.3 is a consequence of Theorem 2.9 and 1.C.1. ■

From Theorems 7.3 and 6.9 we also obtain:

7.4. Corollary. *Every regular ordered space is semi-regular.* ■

We shall now use the results of Section 5 for the construction of a countable d-superbase of $\boldsymbol{H}$. Consequently, we shall be able to find a countable superbase of an arbitrary regular space.

Let us denote by $\mathscr{G}$ the family of all cyllinders in $\mathscr{H}$ over finite products of open intervals with rational end-points. The family $\mathscr{G}$ is a base of open sets (in the topological sense) of the Hilbert cube; clearly $\mathscr{G}$ is countable. The family of all open subsets of $\mathscr{H}$ is then equal to $\mathscr{G}_\sigma$ and the family of all closed subsets of $\mathscr{H}$ is equal to $\mathcal{G}_{c\delta}$. Finally, let us define $\mathcal{U}$ as the family of all sets U_q^i with $i \in \mathcal{N}$, $q \in \mathcal{Q}$, where $U_q^i = \{x \in \mathcal{H} |\ (x)_i > q\}$.

It is easily seen that $(\vec{\mathcal{G}})_{cd} = (\vec{\mathcal{G}})_s = \mathcal{U}_{sd} = \mathcal{U}_{ds}$.

7.5. Lemma. *The countable family* $(\vec{\mathcal{G}})_{cd} = \mathcal{U}_{sd}$ *is a superbase of the ordered space* $\boldsymbol{H}$.

Proof. Since $\mathcal{G}$ is countable, also $(\vec{\mathcal{G}})_{cd}$ is countable. In order to show that $(\vec{\mathcal{G}})_{cd}$ is a d-superbase of $\boldsymbol{H}$, we shall check that $\mathcal{G}$ satisfies all the assumptions of Theorem 5.9. In fact:

(i) $\mathcal{G}$ is multiplicative and exhaustive (every open set is an F_σ, i.e. $\mathcal{G}_\sigma \subset \mathcal{G}_{c\delta_\sigma}$) and generates $\mathcal{B}_{\mathcal{H}}$;

(ii) is obviously satisfied;

(iii) follows from Lemma 6.8.

Thus the family $(\vec{\mathcal{G}})_{cd}$ is a d-superbase of $\boldsymbol{H}$.

From Theorem 5.5 it follows that $(\vec{\mathcal{G}})_{cd}$ is also a g-superbase of $\boldsymbol{H}$. ∎

7.6. Corollary. *Every regular ordered space X has a countable superbase. If g is an embedding of X into* $\boldsymbol{H}$, *then the family* $\{g^{-1}(U) |\ U \in \mathcal{U}\}$ *is a superbase of X.*

In fact, without loss of generality we can assume that X is an absolute measurable subspace of $\boldsymbol{H}$ and $g = \mathrm{id}_X$. From Corollaries 7.4 and 6.5 it follows that the family $\mathcal{U}_{sd|X} = (\mathcal{U}|_X)_{sd}$ is a superbase of X. ∎

We now obtain the following:

7.7. Theorem. *For every countable base* $\mathcal{A}$ *of a regular space X, the family* $\mathcal{A}_{sd}$ *is a superbase of X.*

Proof. Let h be an embedding of X into $\boldsymbol{H}$ such that for every $i \in \mathcal{N}$, $h(A_i) = h(X) \cap T_i$, where $T_i = \{x \in \boldsymbol{C} |\ (x)_i = 1\}$ and (A_i) is any arrangement of the family $\mathcal{A}$ (cf. 2.10). It follows from Corollary 7.6 that the family $\{h^{-1}(U) |\ U \in \mathcal{U}\}_{sd} = \{h^{-1}(T_i) |\ i \in \mathcal{N}\}_{sd} = \mathcal{A}_{sd}$ is a superbase of X. ∎

The next theorem is an analogue of Theorem 3.2 for the case of $\aleph_0$-proper spaces:

7.8. Theorem. *The regularity of an ordered space X implies the regularity of* X^*.

Proof. It follows from Theorem 1.C.5 that X^* is almost Borel and from Theorem 7.7 that X^* is $\aleph_0$-proper. ∎

It also follows from Theorem 3.3, 1.B.9 and 7.2 that Theorem 7.8 can be inverted in the following way:

7.9. *For every proper ordered space X with a countably generated σ-field* $\mathfrak{M}_X$, *the regularity of* X^* *implies the regularity of X.* ∎

Theorem 7.8 can also be strengthened as follows:

7.10. Theorem. *For a regular space X with a countable base* $\{A_i\}$, *the function* $h\colon X^* \to \boldsymbol{H}$ *defined by the formula*

$$(h(P))_i = P(A_i) \quad \text{for} \quad i \in \mathcal{N},\ P \in X^*,$$

is an embedding. ∎

From Theorem 1.C.5 it follows that also the class of all $\aleph_0$-proper standard analytic ordered spaces and the class of all $\aleph_0$-proper standard Borel spaces are closed under the operation $X \to X^*$.

An open problem.

In Section 4 we have shown that, if a proper ordered space X is given, the ordering $\leqslant_{X^*}$ in X^* can be defined in an equivalent way by means of isotone functions. This is a formalization of some intuitions arising from economics, in the case where the ordering $\leqslant_X$ is understood as reflecting human preferences. However, one could try to define the extension of $\leqslant_X$ formalizing some mechanical intuitions. Assume a space X to be given. Every probability on $\mathfrak{M}_X$ represents a distribution of the unit mass on X. Suppose that "$x \leqslant y$" is understood as "there is a possibility of a flow from x to y". One could introduce the relation $\leqslant^0$ in X^* saying: "$P \leqslant^0 Q$ iff the mass represented by P can be shifted, according to the directions of a possible flow, so that it will represent Q".

This informal "definition" of $\leqslant^0$ described above can be formalized in many ways. We now give a simple formal definition of $\leqslant^0$.

Given measurable spaces X and Y, a measurable function f on X into Y and a probability P on $\mathfrak{M}_X$, we define Pf^{-1} as a probability on $\mathfrak{M}_Y$ such that $Pf^{-1}(Y_0) = P(f^{-1}(Y_0))$ for $Y_0 \in \mathfrak{M}_Y$.

Let X be a proper ordered space. By $\leqslant^0$ we denote the smallest relation ϱ in X^* satisfying the contitions:

(i) If $P \in X^*$ and $f\colon X \to X$ is a measurable function such that $P(\{x |\ x \leqslant f(x)\}) = 1$, then $P \varrho Pf^{-1}$.

(ii) If $P \in X^*$ and $g\colon X \to X$ is a measurable function such that $P(\{x |\ g(x) \leqslant x\}) = 1$, then $Pg^{-1} \varrho P$.

(iii) If the probabilities $P, Q \in X^*$ can be represented in the form $P = \alpha P_1 + (1-\alpha) P_2$, $Q = \alpha Q_1 + (1-\alpha) Q_2$, where $0 \leqslant \alpha \leqslant 1$, $P_1, P_2, Q_1, Q_2 \in X^*$ and $P_1 \varrho Q_1$ and $P_2 \varrho Q_2$, then $P \varrho Q$.

The question arises whether the relations $\leqslant^0$ and $\leqslant_{X^*}$, which is given by Definition 3.1, are equal.

The author knows only some partial answers: clearly $\leqslant^0 \subset \leqslant_{X^*}$; if $X = \boldsymbol{R}$, the answer is obviously positive; generally, if $\mathfrak{M}_X$ contains all one-point sets, then the relations $\leqslant^0$ and $\leqslant_{X^*}$ coincide on the set of all probabilities on $\mathfrak{M}_X$ with countable supports (the last is a consequence of the Min-Cut Max-Flow Theorem).

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Shape properties of hyperspaces

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Abstract. Using some ideas from shape theory several results on the hyperspaces of subcontinua are obtained. The hyperspaces of circle-like continua are studied in great detail.

0. Introduction. By a continuum we mean a compact connected metric space. Given a continuum X by $C(X)$ we denote the hyperspace of nonvoid subcontinua of X with the Hausdorff metric $\mathrm{dist}(\cdot,\cdot)$ (see for instance [11] where several facts about $C(X)$ are proved). A map f, i.e., a continuous function, from X into Y defines a map $\hat{f}\colon C(X)\to C(Y)$ given by $\hat{f}(A)=f(A)$, which is called the *map induced by f*. Throughout this paper maps with hats above will always denote the induced maps. By $\hat{X}$ we denote the base of $C(X)$, that is the set of all singletons in $C(X)$. This space is isometric to X and occasionally is identified with X. Continuum X regarded as a point of $C(X)$ is called the *vertex of* $C(X)$. For every two continua $A, B\in C(X)$ such that $A\subset B$ there is a maximal monotone collection of continua between them which forms an arc in $C(X)$. Such a collection will be denoted by AB and called a *segment in* $C(X)$. If A is a singleton and $B=X$, then AB is called a *maximal segment*. A map μ from $C(X)$ into reals R is called a *Whitney map on* $C(X)$ provided the conditions are satisfied:

(*) $$A\subset B \text{ and } A\neq B \Rightarrow \mu(A)<\mu(B),$$

(**) $$\mu(\{x\})=0 \quad \text{for each } x\in X.$$

Whitney maps always exist [23]. We take the opportunity to show how we can construct many Whitney maps on $C(X)$.

Let $U_1, U_2, \ldots$ be an open base for X and call a pair $\alpha=(U_i, U_j)$ normal if $\overline{U}_i\subset U_j$. For such a pair let f_α denote the Uryshon map from X into the unit interval $I=[0,1]$ sending $\overline{U}_i$ into 0 and $X\setminus U_j$ into 1, and let $\mu_\alpha\colon C(X)\to R$ be given by

$$\mu_\alpha(A)=\mathrm{diam} f_\alpha(A).$$

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