construction of Φ, a ↘ chain from k' to an element k'' of S_h^M (note that $T_i \subset \Phi(k') = \Phi(k'')$, since $f_i(k') = f_i(h) = 1$). By the argument of the previous paragraph k'' is ↘ chained to an element of K with degenerate Φ-image. This completes the proof that condition (iv), d) of the C-monotone Definition 7.1 is satisfied, and with it the proof of Theorem 7.4

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A degree theory for almost continuous functions

by

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Abstract. A degree theory is developed for almost continuous functions. This theory is used to prove certain fixed point theorems as well as a generalization of the Borsuk–Ulam theorem.

I. Introduction. In recent years non-continuous functions have been studied and applied to fixed point theory.

Let $f\colon X \to Y$ be a function from a topological space X to a topological space Y. For $C \subseteq X$, the graph over C is defined to be $\{(x, f(x))\colon x \in C\}$, a subspace of the topological space $X \times Y$. The graph of f, denoted by Γf, is defined to be the graph over X. A function $f\colon X \to Y$ is called a *connectivity function* if the graph over each connected set is connected. O. H. Hamilton [7] initiated the study of connectivity functions when he proved the following theorem:

THEOREM 1. *Every connectivity function from the n-cell I^n to the n-cell has a fixed point.*

Let $\mathrm{bd}(N)$ denote the boundary of N. In order to prove Theorem 1, Hamilton defined an additional class of functions:

DEFINITION 1. If $f\colon X \to Y$ is a function, then f is *peripherally continuous* if for each $x \in X$, each open $V \subseteq X$ for which $x \in V$, and each open $U \subseteq Y$ for which $f(x) \in U$, there exists a neighborhood N of x such that $\overline{N} \subseteq V$ and $f(\mathrm{bd}(N)) \subseteq U$.

He then proceeded to show, for $n \geqslant 2$, that every connectivity function is peripherally continuous and every peripherally continuous function has a fixed point. John Stallings [11] discovered a gap in Hamilton's argument, corrected it, and generalized the result to polyhedra. In doing so he defined a third class of functions:

DEFINITION 2. A function $f\colon X \to Y$ is *almost continuous* if for every open subset U of $X \times Y$ with $\Gamma f \subseteq U$ there exists a continuous function $g\colon X \to Y$ with $\Gamma g \subseteq U$.

As a consequence of a key theorem in Stallings paper we have:

THEOREM 2. *If f is a peripherally continuous function from either I^n or S^n, $n \geqslant 2$, into R^n then f is almost continuous.*

* This paper is dedicated to the memory of William Carroll Chewning, a friend and a bright young mathematician who was a source of inspiration to the second author.

Generalizations and related results, to mention a few, are contained in papers by Chewing [1, 2], Cornette [3, 4], Girolo [6], Kellum [10], Hildebrand and Sanderson [8], and Whyburn [12].

The purpose of this paper is to study almost continuous functions. We develop a degree theory for almost continuous functions in a natural way such that the properties of degree theory for continuous functions carry over, at least in part, to almost continuous functions. We use this theory to obtain certain fixed point theorems concerning almost continuous functions, as well as a generalization of the Borsuk–Ulam theorem. We also extend Tietze's extension theorem to almost continuous functions.

II. Local degree theory. Unless otherwise stated, D or D_n will always represent a bounded open subset of R^n and q will always denote a point of R^n.

DEFINITION 3. Let $f\colon \bar{D}\to R^n$ be almost continuous. Suppose $q\notin f(\mathrm{bd}(D))$. We define the *local degree*, $d(f,\bar{D},q)$, as follows. Let $\mathscr{U}$ be the collection of all open sets in $\bar{D}\times R^n$ that contain Γf. If $U\in\mathscr{U}$ then $\deg(U)$ consists of those integers n for which there exists a continuous function $g\colon \bar{D}\to R^n$ such that $\Gamma g\subseteq U$ and $d(g,\bar{D},q)=n$ together with $+\infty$ $(-\infty)$ if there exists an unbounded set of positive (negative) integers in $\deg(U)$. (See Cronin [5] for the definition of local degree for continuous functions.) Then the local degree of f with respect to $\bar{D}$ and q is

$$d(f,\bar{D},q)=\bigcap_{U\in\mathscr{U}}\deg(U).$$

We proceed to show that $d(f,\bar{D},q)\neq\emptyset$. First let $U\in\mathscr{U}$. The set $\mathrm{bd}(D)\times\{q\}$ is closed and $q\notin f(\mathrm{bd}(D))$ which implies $\Gamma f\subseteq U-(\mathrm{bd}(D)\times\{q\})$ an open set. There exists a continuous function $g\colon \bar{D}\to R^n$ with $\Gamma g\subseteq U-(\mathrm{bd}(D)\times\{q\})$. It follows that $d(g,D,q)$ is defined (see Cronin [5]), that $d(g,\bar{D},q)\in\deg(U)$ and hence that $\deg(U)\neq\emptyset$.

Next we show $\bigcap_{U\in\mathscr{U}}\deg(U)\neq\emptyset$. Toward that end we consider two cases.

Case 1. Suppose that for each $U\in\mathscr{U}$ it is true that $+\infty\in\deg(U)$ or $-\infty\in\deg(U)$. Let $U,V\in\mathscr{U}$. Then $U\cap V\in\mathscr{U}$ and either $+\infty\in\deg(U\cap V)$ or $-\infty\in\deg(U\cap V)$ which implies that $+\infty\in\deg(U)\cap\deg(V)$ or $-\infty\in\deg(U)\cap\cap\deg(V)$. So $+\infty\in\bigcap_{U\in\mathscr{U}}\deg(U)$ or $-\infty\in\bigcap_{U\in\mathscr{U}}\deg(U)$.

Case 2. Suppose there exists a $U\in\mathscr{U}$ such that $\deg(U)\subseteq\{0,\pm1,\ldots,\pm m\}=M$. Further suppose for each $n\in M$ there exists a $U_n\in\mathscr{U}$ such that $n\notin\deg(U_n)$. Contrary to what we have established, the set $V\in\mathscr{U}$ defined by

$$V=\bigcap_{n\in M}(U_n\cap U)$$

would have $\deg(V)=\emptyset$. Thus in both cases we conclude

$$d(f,\bar{D},q)=\bigcap_{U\in\mathscr{U}}\deg(U)\neq\emptyset.$$

Suppose $f\colon \bar{D}\to R^n$ is continuous and $q\notin f(\mathrm{bd}(D))$. Then there exists an open set U with $\Gamma f\subseteq U$ such that any continuous function whose graph lies in U has the same local degree at q. (See Cronin [5].) Thus we conclude,

PROPOSITION 1. *If f is a continuous function the degree, as just defined, is the set consisting of the degree as defined by Cronin* [5].

THEOREM 3. *Let $f\colon \bar{D}\to R^n$ be almost continuous with $q\notin f(\mathrm{bd}(D))$. If $d(f,\bar{D},q)\neq\{0\}$ then there exists a $p\in D$ such that $f(p)=q$.*

Proof. Suppose no such p exists. Let U be an open subset of $\bar{D}\times R^n$ containing Γf. Then $V=U\cap(\bar{D}\times R^n-\bar{D}\times\{q\})$ is an open subset of $\bar{D}\times R^n$ containing Γf and hence containing the graph of a continuous function g such that $d(g,\bar{D},0)\neq\{0\}$. By Theorem 6.6 [5] there exists a p such that $g(p)=q$ contrary to $\Gamma g\subseteq V$.

Let D be the open unit disk in the x, y plane centered at $(0,0)$ and S the boundary of D. Let

$$z=r\exp[i\theta]=x+iy,\quad z_1=\exp[\pi i/4],$$

$$z_{n+1}=\exp[i(\pi/4)\sum_{k=0}^{n}(1/2)^k]=\exp[i\theta_n]$$

and

$$w_n=\exp(i\pi)/z_n,\quad n=1,2,\ldots$$

EXAMPLE 1. Select a closed disk E_n such that the boundary circle passes through z_n and z_{n+1} and if $z\in E_n\cap\bar{D}$ then $\theta_n\leqslant\theta\leqslant\theta_{n+1}$. Set $P_n=E_n\cap\bar{D}$. We define $f_n\colon \bar{D}\to R^2$ as follows. If $r=1$ and $\theta_n\leqslant\theta\leqslant\theta_{n+1}$ set

$$f_n(z)=\exp[i(\theta_n+(\theta_{n+1}-\theta_n+2\pi)(\theta-\theta_n)/(\theta_{n+1}-\theta_n))].$$

If $z\in\overline{\bar{D}-P_n}$ set $f_n(z)=z$. By Tietze's extension theorem f_n can be extended to all of $\bar{D}$. We set

$$f(z)=\begin{cases}f_n(z), & z\in P_n,\\ z & \text{otherwise.}\end{cases}$$

EXAMPLE 2. Define $t\colon R^2\to R^2$ by

$$t(x,y)=\begin{cases}(-x,y) & \text{if } x\leqslant 0.\\ (x,y) & \text{otherwise}\end{cases}$$

and $g(z)=f\circ t(z)$.

EXAMPLE 3. Let C_z denote the circle with center i and passing through z. Let $h\colon \bar{D}\to R^2$ be defined by

$$h(z)=\begin{cases}g(z) & \text{if } |z|=1,\\ g(S\cap C_z) & \text{otherwise.}\end{cases}$$

In each of the preceeding examples the functions are almost continuous and $d(f,\bar{D},0)=\{+\infty\}$, $d(g,\bar{D},0)=Z\cup\{+\infty\}$ and $d(h,\bar{D},0)=\{0\}$. We establish the last identity leaving the others to the reader. Let ϱ denote the metric in $\bar{D}\times R^2$,

S the boundary of D, B_n the continuum in $\bar{D}$ bounded by $C_{z_n} \cup C_{z_{n+1}}$, ε the distance from $(1, 0)$ to C_{z_1}, $V = \{(z, w)\colon z \neq i$ and $\varrho(h(z), w) < \varepsilon\}$, Q the component of $\bar{D} - C_{z_1}$ containing i, $N = Q \times Q$ and $U = V \cup N - S \times \{0\}$.

Suppose $\Gamma h \subseteq U$ an open subset of $\bar{D} \times R^2$. Consequently there exists a continuous function $a\colon \bar{D} \to R^2$ with $\Gamma a \subseteq U$. We show $d(a, D, 0) = 0$. We do this by showing that a is homotopic to a map $b\colon \bar{D} \to R^2$ and $d(b, D, 0) = 0$.

If $z \in \overline{S - Q}$ then $\varrho(a(z), b(z)) < \varepsilon < 1$. Thus,

(i) for each $z \in \overline{S - Q}$, $0 \notin \overline{h(z)a(z)}$, the line segment between $h(z)$ and $a(z)$.

From the continuity of a at $z = i$ we have

(ii) there exists an integer M such that if $z \in \bigcup_{m \geqslant M} (B_m \cap S) \cup \{i\}$ then $0 \notin \overline{z_M a(z)}$.

Finally,

(iii) if for some $n < M$ there exists a $w \in B_n \cap S$ such that $0 \in \overline{a(w)h(w)}$ then $0 \notin \overline{t(z)a(z)}$ for all $z \in B_n \cap S$.

The validity of (iii) can be argued as follows. Clearly $(w, a(w)) \in N$, for if this were not the case $\varrho(h(w), a(w)) < 1$ would imply $0 \notin \overline{a(w)h(w)}$. Hence $0 \notin \overline{wa(w)}$. Now suppose there is some point $v \in B_n \cap S$ such that $0 \in \overline{t(v)a(v)}$. Then $(v, a(v)) \in V$ and $h(v) \in \{\exp[i\theta]\colon -\pi \leqslant \theta \leqslant 0\}$. Set

$$B = \{z\colon z \in C_z \text{ and } C_z \text{ separates } w \text{ and } v\}.$$

The definition of h implies $h(B) \subseteq \{\exp[i\theta]\colon -\pi \leqslant \theta \leqslant 0\}$. If $z \in B$ then $a(z) \notin C_{z_1}$. For if it was then $(z, a(z)) \in V$ contrary to the fact that if $(z, a(z)) \in V$ then $\varrho(a(z), h(z)) < \varepsilon$. Then set $a(B)$ has nonempty intersection with $\{z\colon y \leqslant 0\}$ and Q. This is not compatible with the property that the continuous image of a connected set is connected. Thus $0 \notin \overline{t(v)a(v)}$ and the conclusion of (iii) follows.

Let

$$B^* = \{z\colon z \in B_n,\ n < M \text{ and there exists a } w \in B_n \text{ such that } 0 \in \overline{a(w)h(w)}\}$$

and

$$C^* = \overline{(B_1 \cup B_2 \cup \ldots \cup B_{m-1}) - B^*}.$$

Then we define $b\colon S \to R^n$ by

$$b(z) = \begin{cases} h(z) & \text{if } z \in C^* \cup (\overline{S - Q} \cap S), \\ t(z) & \text{if } z \in (B^* \cap S), \\ z_{M+1} & \text{otherwise.} \end{cases}$$

We extend b, by Tietze's extension theorem to a mapping $b\colon D \to R^2$. By (i), (ii), (iii) and the Poincaré–Bohl theorem [5] a is homotopic to b. Clearly $d(b, \bar{D}, 0) = \{0\}$.

We observe that $h/S = g/S$ which shows that the degree is not determined by the boundary.

THEOREM 4. *Let $f, g\colon \bar{D} \to R^n$ be almost continuous functions such that $f/S = g/S$, $S = \mathrm{bd}(D)$ and the restrictions are continuous functions. Then $d(f, \bar{D}, q) = d(g, \bar{D}, q)$ provided the degree is defined.*

Proof. Let $F\colon \bar{D} \to R^n$ be a continuous extension of f/S,

$$\varepsilon = \inf\{\varrho(f(x), q)\colon x \in S\},$$

where ϱ is the metric in R^n, and $C = \{(x, y) \in S \times R^n,\ \varrho(f(x), y) \geqslant \varepsilon\}$. Then C is a closed subset of $\bar{D} \times R^n$, $U = \bar{D} \times R^n - C$ is an open subset of $\bar{D} \times R^n$ and $\Gamma F \subseteq U$. By the Poincaré–Bohl theorem [5] any continuous function $h\colon \bar{D} \to R^n$ with $\Gamma h \subseteq U$ must have the same degree as F. Thus

$$d(f, \bar{D}, q) = d(F, \bar{D}, q) = d(g, \bar{D}, q).$$

DEFINITION 4. An *almost continuous homotopy* is an almost continuous mapping $F\colon \bar{D} \times I \to R^n$. We set $F_0 = F/\bar{D} \times \{0\}$ and $F_1 = F/\bar{D} \times \{1\}$.

The next example shows that the degrees of F_0 and F_1 can be different.

EXAMPLE 4. In the plane let D_1 be the closed unit disk center $(0, 0)$ with boundary S, D_2 the closed disk with center i and radius $\sqrt{2 - \sqrt{2}}$, and $D = D_1 \cup D_2$. We define $k\colon D \to R^2$, $K\colon D \times I \to R^2$ and $F\colon D \times I \to R^2$ as follows:

$$k(z) = \begin{cases} g(z) & \text{if } z \in D_1, \\ h(C_z \cap S) & \text{otherwise,} \end{cases}$$

$$K(z, t) = \begin{cases} z & \text{if } z \in D_1, \\ z(1-t) + zt/|z| & \text{otherwise} \end{cases}$$

and

$$F = k \circ K,$$

where g and h are defined in Examples 2 and 3 respectively.

It is easy to see that k, and consequently F by Proposition 4 [10], is almost continuous. Further, F is an almost continuous homotopy with $d(F_0, \bar{D}, 0) = \{0\}$ while $d(F_1, \bar{D}, 1) = Z \cup \{\pm\infty\}$.

THEOREM 5. *Suppose $F\colon \bar{D} \times I \to R^n$ is an almost continuous homotopy and $q\colon I \to R^n$ such that for each $t \in I$, $q(t) \notin F(\mathrm{bd}(D) \times \{t\})$. Further suppose there is an open set $V_0 \subseteq \bar{D} \times R^n$, and an integer m with the property that $\Gamma F_0 \subseteq V_0$ and if $g\colon \bar{D} \to R^n$ is a continuous function, $\Gamma g \subseteq V_0$, then $d(g, \bar{D}, q(0)) = m$. Then $m \in d(F_1, \bar{D}, q(1))$.*

Proof. Let $V_1 \subseteq \bar{D} \times R^n$ be open with $\Gamma F_1 \subseteq V_1$. Set $C_i = \bar{D} \times R^n - V_i$, $i = 0, 1$. We think of C_i and V_i as being subsets of $\bar{D} \times \{i\} \times R^n$. Set

$$C_2 = \bigcup_{t \in [0,1]} (\mathrm{bd}(D) \times \{t\} \times \{q(t)\}),$$

$C = C_0 \cup C_1 \cup C_2$ and $U = \bar{D} \times I \times R^n - C$. Then U is an open subset of $\bar{D} \times I \times R^n$ and $\Gamma F \subseteq U$. Hence there exists a continuous function $G\colon \bar{D} \times I \to R^n$ with $\Gamma G \subseteq U$. By construction $\Gamma G_i \subseteq V_i$, $i = 0, 1$. Since $d(G_0, \bar{D}, q(0)) = m$ it follows, by the continuous homotopy Theorem [5] that $d(G_1, \bar{D}, q(1)) = m$. Therefore $m \in d(F_1, \bar{D}, q(1))$.

COROLLARY 1. *Theorem 5 is applicable provided one end of the almost continuous homotopy, F_0 or F_1, is a continuous function.*

III. The degree of f: $S^n \to S^n$.

THEOREM 6 [1]. *Let f: $K \to B^n$ be an almost continuous function with K a closed subset of B^n, the n ball. Then there exists an almost continuous F: $B^n \to B^n$ such that $F/K = f$.*

Proof. The technique to construct an extension F is similar to that used by Cornette in Theorem 1 [3].

Let π_1: $B^n \times B^n \to B^n$ be the projection into the first coordinate map and

$$\mathscr{C} = \{H\colon H \text{ is a closed subset of } B^n \times B^n \text{ and } \pi_1(H) - K \text{ is the cardinality of the continuum } \mathfrak{C}\}.$$

The cardinality of $\mathscr{C}$ is $\mathfrak{C}$. We will order $\mathscr{C}$ into $H_1, H_2, \ldots, H_\alpha, \ldots$ so that no element has $\mathfrak{C}$ predecessors. Using transfinite induction we will select an element from each H_α. Select an element P_1 from H_1 such that $\pi_1(P_1) \notin K$, and for each ordinal α assume that for each ordinal $\beta < \alpha$ an element $P_\beta \in H_\beta$, such that $\pi_1(P_\beta) \notin K$, has been selected. We select an element $P_\alpha \in H_\alpha$ such that

$$\pi_1(P_\alpha) \notin \bigcup_{\beta<\alpha} \{\pi_1(P_\beta)\} \cup K.$$

Such a P_α exists since there are $\mathfrak{C}$ choices and by the well ordering card $\big(\bigcup_{\beta<\alpha} \pi_1(P_\beta)\big) < \mathfrak{C}$. In each H_α a point P_α may be chosen by transfinite induction. We set

$$F(x) = \begin{cases} f(x) & \text{if } x \in K, \\ y & \text{if } (x, y) = P_\alpha \text{ for some } \alpha, \\ \text{anything otherwise.} & \end{cases}$$

Let $\Gamma F \subseteq U$ an open subset of $B^n \times B^n$. Let $C = B^n \times B^n - U$. Then $C \notin \mathscr{C}$ and it follows that $\pi_1(C) - K$ is countable. The set $V = U \cap (K \times B^n)$ is an open subset of $K \times B^n$ and $\Gamma f \subseteq V$. Since f is almost continuous there exists a continuous function a: $K \to B^n$ such that $\Gamma a \subseteq V$. By Tietze's extension theorem a can be extended to a continuous function A: $B^n \to B^n$. Thus $\Gamma A/K \subseteq U$ and by the continuity of A there exists an open set Q with $K \subseteq Q$ and $\Gamma A/\bar{Q} \subseteq U$. Since $\pi_1(C) - K$ is countable, Q can be chosen so that $\mathrm{bd}(Q) \cap \pi_1(C) = \emptyset$. Then we set $C^* = \pi_1(C) - Q$ which is a compact countable set. Let $C^* = \{c_1, c_2, \ldots\}$. For each c_k there exists an open set $N_k \times U_k$ such that $(c_k, F(c_k)) \in N_k \times U_k$, $\bar{N}_k \times U_k \subseteq U$ and $\mathrm{bd}(U_k) \cap C^* = \emptyset$. By the compactness of C^* there is a finite number of the N_k's say $N_{k1}, \ldots, N_{km}$ which cover C^*.

[1] Kenneth Kellum has proven this as well as a more general result which will appear in this journal.

We define

$$g(x) = \begin{cases} F(c_{ki}) & \text{if } x \in \bar{N}_{ki}, \\ A(x) & \text{if } x \in \bar{Q}. \end{cases}$$

By Tietze's extension theorem g can be extended to a continuous function G: $B^n \to B^n$ and clearly $\Gamma G \subseteq U$. Thus F is almost continuous.

DEFINITION 5. Let S^n denote the n-sphere and f: $S^n \to S^n$ be almost continuous. Then we define $\deg(f)$ as follows: $m \in \deg(f)$, if and only if, there exists an almost continuous extension F: $B^n \to B^n$ of f such that $m \in d(F, B^n, 0)$.

By the previous theorem the definition above is well defined.

DEFINITION 6. Let f: $S^n \to S^n$ be almost continuous. Let $\mathscr{U}$ be the collection of all open subset of $S^n \times S^n$ which contain Γf. If $U \in \mathscr{U}$ then $\deg^*(U)$ consists of those integers n for which there exists a continuous function g: $S^n \to S^n$ such that $\Gamma g \subseteq U$ and g has degree n together with $+\infty$ $(-\infty)$ if there exists an unbounded set of positive (negative) integers in $\deg^*(U)$.

We define the degree of f, $\deg^*(f)$ as follows:

$$\deg^*(f) = \bigcup_{U \in \mathscr{U}} \deg^*(U).$$

PROPOSITION 2. *If f: $S^n \to S^n$ is continuous then $\deg(f)$ agrees with the usual definition of degree for continuous functions.*

THEOREM 7. *Let f: $S^n \to S^n$ be almost continuous. Then $\deg(f) = \deg^*(f)$.*

Proof. Let m be an integer in $\deg(f)$ and suppose U is an open subset of $S^n \times S^n$ with $\Gamma f \subseteq U$. Set $C = S^n \times S^n - U$, $K = \{(x, ty)\colon (x, y) \in C \text{ and } 0 \leqslant t \leqslant 1\}$ and $V = B^n \times B^n - K$ an open subset of $B^n \times B^n$. Since $m \in \deg(f)$ there exists an almost continuous extension F: $B^n \to B^n$ of f with $m \in \mathrm{def}(F, B^n, 0)$. Furthermore $\Gamma f \subseteq V$. So there exists a continuous function G: $B^n \to B^n$ with $\Gamma G \subseteq V$ and $d(G, B^n, 0) = m$. Let $H = G/S^n$. Then $\Gamma H/\|H\| \subseteq U$ and $\deg(H/\|H\|) = m$. Thus $m \in \deg^*(f)$.

Let m be an integer in $\deg^*(f)$ and suppose F: $B^n \to B^n$ is an almost continuous extension of f as defined in Theorem 6. Let $U \subseteq B^n \times B^n$ be an open subset with $\Gamma F \subseteq U$ and set $V = (S^n \times S^n) \cap U$. Then V is an open subset of $S^n \times S^n$ with $f' \subseteq V$. Since $m \in \deg^*(f)$ there exists a continuous function g: $S^n \to S^n$ with $\Gamma g \subseteq V$ and $\deg(g) = m$. Now one can use essentially the same argument as was used in Theorem 6 to show that g can be extended to a continuous function G: $B^n \to B^n$ with $\Gamma G \subseteq U$. Thus $m \in \deg(f)$.

Similar arguments show $\pm\infty \in \deg(f)$, if and only if, $\pm\infty \in \deg^*(f)$.

DEFINITION 7. A mapping F: $S^n \times I \to S^n$ that is almost continuous is called an *almost continuous homotopy*. We set $F_0 = F/S^n \times \{0\}$ and $F_1 = F/S^n \times \{1\}$.

Analogus to Theorem 5 and Corollary 1 we have:

THEOREM 8. *Suppose that F: $S^n \times I \to S^n$ is an almost continuous homotopy. Further suppose there is an open set $V_0 \subseteq S^n \times S^n$ and an integer m with the property that*

$\Gamma F_0 \subseteq V_0$ *and if* $g\colon S^n \to S^n$ *is a continuous function,* $\Gamma g \subseteq V_0$, *then* $\deg(g) = m$. *Then* $m \in \deg(F_1)$.

COROLLARY 2. *Theorem 8 is applicable provided one end of the almost continuous homotopy is a continuous function.*

IV. Applications.

PROPOSITION 3. *Let* $f\colon X \to R^n$ *be almost continuous. If* $g\colon X \to R^n$ *is continuous then* $f+g\colon X \to R^n$ *is almost continuous. If* $g\colon X \to R$ *is continuous, then* $g \cdot f\colon X \to R^n$ *is almost continuous.*

Proof. Let $*$ denote either addition or multiplication. Let U be an open subset of $X \times R^n$ with $\Gamma f * g \subseteq U$. Consider the continuous mapping $G\colon X \times R^n \to X \times R^n$ defined by $G(x, y) = (x, y * g(x))$. Then $G^{-1}(U) \subseteq X \times R^n$ is an open subset and $\Gamma f \subseteq G^{-1}(U)$. So there exists a continuous function $h\colon X \to R^n$ with $\Gamma h \subseteq G^{-1}(U)$. Then $G(x, h(x)) = (x, h(x) * g(x)) \in U$ for each $x \in X$. So $h * g$ is a continuous function and $\Gamma h * g \subseteq U$. Thus $f * g$ is almost continuous.

The next theorem was first proven by John Stallings [11].

THEOREM 9. *Let* $f\colon I^n \to I^n$ *be almost continuous. Then there exists an* $x \in I^n$ *such that* $f(x) = x$.

Proof. We assume that I^n is the closed unit ball in R^n and for each $x \in \mathrm{bd}(I^n)$ $f(x) \neq x$. Consider

$$F\colon I^n \times I \to R^n \quad \text{and} \quad q\colon I \to I^n$$

defined by

$$F(x, t) = x - tf(x) \quad \text{and} \quad q(t) = 0\,.$$

It follows from Proposition 3 that F is an almost continuous homotopy. Further if $x \in \mathrm{bd}(I^n)$ then $F(x, t) \neq 0$. The homotopy fulfills the hypothesis of Corollary 1 thus $1 \in d(x - f(x), I^n, 0)$. By Theorem 1 there exists an $x \in I^n$ such that $x - f(x) = 0$ or $f(x) = x$.

In that which follows $||x||$ will denote the Euclidean norm in R^n.

THEOREM 10. *If* $f\colon S^n \to S^n$ *is almost continuous,* $g\colon S^n \to S^n$ *is continuous and* $f(x)$ *and* $g(x)$ *are never antipodal* $(f(x) \neq -g(x))$, *then there exists an almost continuous homotopy between* f *and* g.

Proof. The homotopy is given by

$$F(x, t) = \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||}\,.$$

Application of Proposition 3 and [Proposition 1, 11] show that F is almost continuous.

In the corollaries below we assume $f\colon S^n \to S^n$ is almost continuous.

COROLLARY 3. *If* f *sends no point to its antipode, then there is a homotopy between* f *and the identity map. Therefore* $1 \in \deg(f)$ *and* $\deg(f) \neq \{0\}$.

COROLLARY 4. *If* f *has no fixed point then* f *is homotopic to the antipode map. Hence by Corollary 2,* $(-1)^{n+1} \in \deg(f)$.

If X is a metric space and $x, y \in X$ then the distance between x and y will be denoted by $\varrho(x, y)$. If $\varepsilon > 0$ then $N(x, \varepsilon) = \{y\colon \varrho(x, y) < \varepsilon\}$.

PROPOSITION 4. *Let* X *be a metric space and* $f, g\colon X \to R^n$ *be functions such that* f *is continuous at* x_0 *and* g *is peripherally continuous at* x_0. *Then* $f+g$ *is peripherally continuous at* x_0.

Proof. Let V and U be neighborhoods of x_0 and $f+g(x_0)$ respectively. Let $\varepsilon > 0$ be such that $N(f+g(x_0), \varepsilon) \subseteq U$. Since f is continuous at x_0 there exists a $\gamma > 0$ such that if $\varrho(x, x_0) < \gamma$ then $\varrho(f(x), f(x_0)) < \frac{1}{2}\varepsilon$. Since g is peripherally continuous at x_0 there exists a neighborhood M of x such that $M \subseteq N(x_0, \gamma)$ and $g(\mathrm{bd}(M)) \subseteq$ $\subseteq N(g(x_0), \frac{1}{2}\varepsilon)$. Thus it follows that $f+g(\mathrm{bd}(M)) \subseteq N(f+g(x_0), \varepsilon)$.

LEMMA 1. *Let* $a\colon S^n \to S^n$ *be the antipodal map, i.e.* $a(x) = -x$ *and let* $\mathcal{U}$ *be a collection of open subsets of* S^n *satisfying*

(i) *if* $x \in S^n$ *and* ε *is any positive number then there exists a* $U \in \mathcal{U}$ *such that* $x \in U$ *and* $\mathrm{dia}(U) < \varepsilon$, *and*

(ii) *if* $U \in \mathcal{U}$ *then* $a(U) \in \mathcal{U}$.

Then there exists a minimal finite subcover of S^n *such that* $a(U)$ *is an element of the subcover if and only if* U *is also.*

Proof. Let $S^k = \{x = (x_1, \ldots, x_{n+1})\colon x \in R^{n+1}, ||x|| = 1$ and $x_i = 0$ for $i \geqslant k+2\}$ where $0 \leqslant k \leqslant n$. We proceed by induction. Surely S^0 has such a finite subcover. Suppose S^k has a minimal finite subcover $U_1, U_2, \ldots, U_m$ such that $a(U_r)$ is a member of the subcover $1 \leqslant r \leqslant m$. Let

$$H^{k+1} = \{x \in S^{k+1}\colon x_{k+2} \geqslant 0\} \quad \text{and} \quad C^{k+1} = H^{k+1} - \bigcup_{i=1}^{m} U_i\,.$$

For each $x \in C^{k+1}$ we may select, by (i), a $U_x \in \mathcal{U}$ so that $x \in U_x$ and

$$U_x \cap \{x \in S^{k+1}\colon x_{k+2} \leqslant 0\} = \emptyset\,.$$

Since C^{k+1} is compact there exists a minimal finite subcover U_{x1}, U_{x2}, U_{xp}. Then

$$\{U_1, U_2, \ldots, U_m, U_{x1}, U_{x2}, \ldots, U_{xp}, a(U_{x1}), a(U_{x2}), \ldots, a(U_{xp})\}$$

is the appropriate subcover of S^{k+1}. By induction the result follows.

The technique of proof of the next lemma is similar to that used by John Stallings to prove Theorem 5 [11].

LEMMA 2. *Let* $f\colon S^n \to S^n$, $n \geqslant 2$, *be an odd,* $f(-x) = -f(x)$, *peripherally continuous function and* $W \subseteq S^n \times S^n$ *an open subset with* $\Gamma f \subseteq W$. *Then there exists an odd continuous function* $g\colon S^n \to S^n$ *such that* $\Gamma g \subseteq W$.

Proof. First, note the fact that there exists a number $\varepsilon^0 > 0$ such that if Q_1 and Q_2 are two connected open subsets of S^n with connected boundaries satisfying (i) $\mathrm{dia}(Q_i) < \varepsilon^0$ for $i = 1, 2$, (ii) $Q_1 \cap Q_2 \neq \emptyset$ and (iii) $Q_2 - Q_1 \neq \emptyset \neq Q_1 - Q_2$, then $\mathrm{bd}(Q_1) \cap \mathrm{bd}(Q_2) \neq \emptyset$. Since f is peripherally continuous there exists for each point $x \in X$ for each number $\varepsilon > 0$ a positive number $\delta = \delta(\varepsilon) \leqslant \varepsilon$ and a neighborhood $U_x \subseteq S^n$ such that $U_x \subseteq N(x, \delta)$ and $f(\mathrm{bd}(U_x)) \subseteq N(f(x), \frac{1}{2}\varepsilon)$. Since f is odd the reflection $a(U_x) = \{-y\colon\ y \in U_x\} \subseteq N(-x, \delta)$ and $f(\mathrm{bd}(a(U_x))) \subseteq N(f(-x), \frac{1}{2}\varepsilon)$. Without loss of generality we may assume U_x and $\mathrm{bd}(U_x)$ are connected [p. 255, 11]. Also, for each $x \in S^n$ there exists a positive number $e^* = e^*(x)$ such that

$$N(x, e^*) \times N(f(x), e^*) \subseteq W$$

and

$$N(-x, e^*) \times N(f(-x, e^*) \subseteq W.$$

We now define

$$\mathscr{U} = \{U_x\colon\ x \in U_x \subseteq S^n,\ U_x \text{ is a connected open set, } \mathrm{bd}(U_x) \text{ is connected, } \mathrm{dia}(U_x) < \min(\varepsilon^0, e^*(x)) \text{ and } f(\mathrm{bd}(U_x)) \subseteq N(f(x), \tfrac{1}{2}e^*(x))\}.$$

Then $\mathscr{U}$ satisfies the hypothesis of Lemma 1. Thus there is a minimal finite subcover of S^n, such that U_x is an element of the subcover if and only if $a(U_x)$ is too. Denote such a subcover by $\{U_{x1}, U_{x2}, \ldots, U_{xm}\}$ where x_i is the point associated with U_{xi}. Let η be less than the Lebesgue number of the covering and less than $\varrho(x_i, \mathrm{bd}(U_{xi}))$, $i = 1, \ldots, m$. Next, let S^n be triangulated in such a way that the mesh $< \frac{1}{2}\eta$, each x_i is a vertex and if Δ is a simplex of the triangulation so is $a(\Delta)$. Call the vertices of the triangulation $v_1, v_2, \ldots, v_r$. To each vertex v_i assign one of the vertices x_j, denoted by $p(v_i)$ such that the closed star of v_i, in this triangulation, is in $U(p(v_i))$. Also, make the assignment in such a manner that $p(a(v_i)) = a(p(v_i))$ and $p(x_i) = x_i$. We defined $g(v_i) = f(p(v_i))$. It is clear that for the $g(v)$ thus defined $g(-v) = -g(v)$. Let $\{v_{i0}, \ldots, v_{in}\}$ be the vertices of an n-simplex Δ of the triangulation. We show that there exists a vertex of Δ, which we call V, such that

$$\{g(v_{i0}), \ldots, g(v_{in})\} \subseteq N(g(V), e^*(p(V))).$$

Toward that end we let

$$e_0^* = \max\{e^*(p(v))\colon\ v \text{ is a vertex of the } n\text{-simplex } \Delta\}$$

and V be the corresponding vertex. Letting v be any other vertex of $\{g(v_{i0}), \ldots, g(v_{in})\}$.

It is clear that

$$\varrho(g(v), f(\mathrm{bd}(Up(V)))) < \tfrac{1}{2}e^*(p(v))$$

and

$$\varrho(g(V), f(\mathrm{bd}(Up(V)))) < \tfrac{1}{2}e^*(p(V)).$$

Since $\mathrm{bd}(Up(v)) \cap \mathrm{bd}(Up(V)) \neq \emptyset$,

$$\varrho(g(v), g(V)) < \tfrac{1}{2}e^*(p(v)) + \tfrac{1}{2}e^*(p(V)) \leqslant \varepsilon_0^*.$$

Thus $g(v) \in N(g(V), e_0^*)$.

From this and the fact that the mesh size is less than $\frac{1}{2}\eta$ is follows that for all vertices of the n-simplex Δ

$$(v, g(v)) \in U_{p(V)} \times N(g(V), e^*(p(V))) \subseteq W.$$

Extend g linearly on Δ. It is easy to see that g satisfies the requirements of the lemma.

DEFINITION 8. Let $f, g\colon X \to Y$ be functions mapping set X to set Y. An element x in X is called a *coincidence point* iff $f(x) = g(x)$.

LEMMA 3. *Let g and h denote real valued functions defined on the closed interval $[a, b]$. If*

(1) $(g(a) - h(a))(g(b) - h(b)) < 0$,

(2) *the graphs of g and h are connected, and*

(3) *the functions have the common property that at each point of the domain one of the functions is continuous then f and g have a coincidence point.*

Proof. Conditions (1), (2) and (3) hold if g and h are replaced by $\tan^{-1}(g)$ and $\tan^{-1}(h)$, respectively. Thus we may assume that f and g are bounded. Assume f and g have no coincidence point.

A point x_0 in the domain will be called a switching point if in every neighborhood of x_0 there exists two points x_1 and x_2 such that $(g(x_1) - h(x_1))(g(x_2) - h(x_2)) < 0$. Conditions (1), (2) and (3) imply that if x_1 and x_2 are two numbers such $(g(x_1) - h(x_1))(g(x_2) - h(x_2)) < 0$ then there is a switching point x satisfying $x_1 < x < x_2$. Thus the set of switching points is perfect. Clearly, the set of switching points is closed. The complement of this closed set is the union of a countable collection of open intervals, UI_i, such that on each interval $\bar{I}_i$, either $g(x) < h(x)$ or $g(x) > h(x)$.

We will show that within any neighborhood of a switching point x_0 there are switching points at which g is continuous and also switching points at which h is continuous. To accomplish this task we construct a new pair of functions.

Let $\bar{I}_i = [a_i, b_i]$. Define on the interval two coincidence free continuous functions $\bar{g}_i(x)$ and $\bar{h}_i(x)$ satisfying the conditions: $g(x) = \bar{g}_i(x)$ and $h(x) = \bar{h}_i(x)$ for $x = a_i$ and b_i and $|g - \bar{g}|, |G - \bar{G}|, |h - \bar{h}|, |H - \bar{H}| < 1/i$ where

$$g = \inf(g(x)), \qquad h = \inf(h(x)),$$
$$\bar{g} = \inf(\bar{g}_i(x)), \qquad \bar{h} = \inf(h_i(x)),$$
$$G = \sup(g(x)), \qquad H = \sup(h(x)),$$
$$\bar{G} = \sup(\bar{g}_i(x)), \qquad \bar{H} = \sup(\bar{h}_i(x)),$$

the supremum and infimum being taken over the interval I_i.

Define

$$\bar{g}(x) = \begin{cases} \bar{g}_i(x), & x \in I_i, \\ g(x) & \text{otherwise}, \end{cases}$$

$$\bar{h}(x) = \begin{cases} \bar{h}_i(x), & x \in I_i, \\ h(x) & \text{otherwise}. \end{cases}$$

The functions $\bar{g}$ and $\bar{h}$ have the same set of switching points as g and h and at a switching point $\bar{g}$ or $\bar{h}$ is continuous if g or h is continuous, respectively. Suppose the graph of $\bar{g}$ is not connected. Then there exists disjoint open sets U and V separating the graph of $\bar{g}$. Let

$$A = \{(x, g(x)):\ (x, \bar{g}(x)) \in U\},$$

$$B = \{(x, g(x)):\ (x, \bar{g}(x)) \in V\}.$$

The graph of g being connected implies that

$$\bar{A} \cap B \neq \emptyset \quad \text{or} \quad A \cap \bar{B} \neq \emptyset.$$

Assume $A \cap B \neq \emptyset$. Thus there exists a sequence $(x_n, g(x_n)) \in A$ and a point $(x, g(x)) \in B$ such that

$$(x_n, g(x_n)) \to (x, g(x)).$$

Clearly $(x, g(x))$ is a switching point from which it follows that $g(x) = \bar{g}(x)$. We may assume that $x_n \in I_{in}$ for some integer in. Since $\Gamma\bar{g}|I_{in}$ is connected it follows that $\Gamma\bar{g}|I_{in} \subseteq U$. By the construction of $\bar{g}$ it is possible to find a $y_n \in I_{in}$, for each n, such that

$$(y_n, \bar{g}(y_n)) \to (x, \bar{g}(x)),$$

a contradiction. Similarly $\bar{h}$ has a connected graph.

Now assume that in a neighborhood of some switching point x_0 the same function, say g, is continuous at all switching points. Then $\bar{g}$ is continuous on this neighborhood and provides a separation of the graph of $\bar{h}$, which is connected. Thus a contradiction.

Suppose x_0 is a switching point at which g is continuous. Select positive numbers δ_0 and ε_0 such that if $|x-x_0|<\delta_0$ then $|g(x)-g(x_0)|<\varepsilon_0<1$. There exists a switching point x_1, $|x_1-x_0|<\delta_0$, at which h is continuous. Select positive numbers δ_1 and ε_1, less than $\frac{1}{2}$ such that if $|x-x_1|<\delta_1$ then $|x-x_0|<\delta_0$ and $|h(x)-h(x_1)|<\varepsilon_1$. There exists a switching point x_2, $|x_1-x_2|<\delta_1$, at which g is continuous. Select positive numbers δ_2 and ε_2, less than $1/2^2$ such that if $|x-x_2|<\delta_2$ then $|x-x_1|<\delta_1$ and $|g(x)-g(x_2)|<\varepsilon_2$. Continuing in this manner we get a sequence of points x_n such that x_n converges to some switching point at which both g and h are continuous. At a switching point both functions cannot be continuous. Thus we have a contradiction.

THEOREM 11. *Let $f\colon S^n \to R^n$ be a connectivity function such that for each x f is continuous at x or $-x$. Then there exists an x such that $f(x) = f(-x)$.*

Proof. Consider the case $n = 1$. Define $C_1\colon I \to S^1$ by $C_1(t) = \exp(i\pi t)$ and $C_2\colon I \to S^1$ by $C_2(t) = \exp(i\pi(1+t))$. Set $g = f \circ C_1$ and $h = f \circ C_2$. Then $g, h\colon I \to R$ are connectivity functions satisfying the hypothesis of Lemma 3. Thus they have a coincidence point x. Therefore $f(C_1(x)) = g(x) = h(x)$, $f(C_2(x)) = f(-C_1(x))$.

Now assume $n \geqslant 2$. Suppose there is no x for which $f(-x) = f(x)$. We think of $R^n = \{(x_1, \ldots, x_{n+1}):\ x_{n+1} = 0\}$. Define $F\colon S^n \to S^n$ by

$$F(x) = (f(x)-f(-x))/\|f(x)-f(-x)\|.$$

The function F is peripherally continuous by Proposition 4, [Corollary 2.8, 8] and [Theorem 4, 11]. Clearly $F(-x) = -F(x)$ which implies that F is odd.

We show $\deg(F) \neq 0$. By Lemma 2 each open set $W \subseteq S^n \times S^n$ that contains the ΓF also contains the graph of a continuous function $G\colon S^n \to S^n$ which is odd. Then $\deg(G) \neq 0$ and hence $\deg(W) \neq \{0\}$.

Suppose there exists an open set U containing the ΓF such that

$$\deg(U) = \{x_1, \ldots, x_r\},$$

where each x_1 is an integer. If no such U exists then $+\infty$ or $-\infty$ is in every open set containing the ΓF and hence $\deg(F) \neq 0$. Suppose for each $x_i \in \deg(U)$, $x_i \neq 0$, there exists an open set V_i containing the ΓF such that $x_i \notin V_i$. Then $\deg(U \cap V_1 \cap \ldots \cap V_n) = \{0\}$, which is a contradiction. Thus we conclude $\deg(F) \neq \{0\}$.

On the other hand $F\colon S^n \to S^n$ is not onto and hence $\deg(F) = \{0\}$. Thus we conclude that there is an x such that $f(x) = f(-x)$. This completes the proof.

The second case of the above theorem also follows from [Theorem 3.2, 9].

EXAMPLE 5. Let $t = \tan^{-1}(\frac{1}{2}\pi z)+1$ and $f\colon S^2 \to R^2$ defined by

$$f(x, y, z) = \begin{cases} (x-\sin t,\ y-\sin t) & \text{if } z \neq \pm 1, \\ (1, 0) & \text{if } z = 1, \\ (-1, 0) & \text{if } z = -1. \end{cases}$$

Then f is connectivity function and no (x, y, z) exists for which $f(x, y, z) = f(-x, -y, -z)$.

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Ordering probabilities on an ordered measurable space

by

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Abstract. Objects considered in the paper are ordered measurable spaces. The space of all probability measures on such a space X is also an ordered measurable space, denoted by X^*, with naturally defined σ-field and order in it. Since X can be considered as a subspace of X^*, the order in X^* is an extension of the initial order in X. The paper is devoted to the investigation of connections between spaces X and X^*, in particular we look for classes which are closed under the operation $X \to X^*$. E. g. such is the class of absolute measurable sets in the ordered Hilbert cube.

0. Introduction.

The objects considered in the paper are ordered spaces; an *ordered space* is a set X with a σ-field $\mathfrak{M}$ of its subsets and with an ordering relation $\leqslant$. The whole system $(X, \mathfrak{M}, \leqslant)$ will be, for simplicity, denoted by X. In cases where a confusion could arise as to what is the σ-field and what is the relation in X we also use subscripts, i.e. $\mathfrak{M}_X$ denotes the σ-field in X and $\leqslant_X$ denotes the order in X.

An ordered space X is *proper* iff it has a *base*, i.e. a family $\mathscr{A}$ of subsets of X which generates the σ-field $\mathfrak{M}_X$ and defines the order in the following sense: for every $x, y \in X$, $x \leqslant y$ iff, for every $A \in \mathscr{A}$, $x \in A$ implies $y \in A$.

For every proper ordered space X we shall define another ordered space X^* (which is also proper) called a *probabilistic extension of* X. Its elements are all probabilities on $\mathfrak{M}_X$; for the σ-field $\mathfrak{M}_{X^*}$ the only reasonable definition is accepted: it is the smallest σ-field in X^* such that for every $A \in \mathfrak{M}_X$ the function $P \to P(A)$ is measurable. However, it is not quite obvious what would be a "natural" definition of the relation $\leqslant$ in X^*.

There is a natural embedding ϑ of the set X into the set X^* (which associates with every x the probability ϑ_x concentrated at x). Thus the order in X^* is expected to be an extension of the order in X in the sense that $x \leqslant y$ iff $\vartheta_x \leqslant \vartheta_y$, for every x, y.

Consider as an example the real line $\mathscr{R}$ with the Borel σ-field and the usual order $\leqslant$. If "$x \leqslant y$" has the meaning "y is (in some sense) not worse than x", then obviously every probability concentrated on an interval $[a, b]$ should be "better" (in the sense of the extended relation $\leqslant$) than any probability concentrated on $[c, d]$ whenever $d \leqslant a$. More generally, a probability Q on Borel subsets of the real line