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Accepté par la Rédaction le 14. 6. 1976

Absolute suspensions and cones

by

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Abstract. De Groot conjectured that if a finite-dimensional compact metric space is a suspension about every pair of distinct points, then it is a sphere. Szymański proved this for dimensions strictly less than 4. Here it is shown that such a space is a regular generalized manifold homotopy equivalent to a sphere, and that any space about which it is a suspension is a generalized manifold homotopy equivalent to a sphere. An analogous result is established for spaces which are open cones about each point. These results are special cases of the Bing–Borsuk conjecture about locally homogeneous ANR's.

De Groot [5] has conjectured that if a finite-dimensional compact metric space is a suspension about every pair of distinct points then it is a sphere. Szymański [10] proved this for dimensions up to 3. Here it is shown that such a space is always a regular generalized manifold homotopy equivalent to a sphere, and that any space about which it is a suspension is a generalized manifold homotopy equivalent to a sphere. An analogous result is proved for spaces which are open cones about every point. In both cases the spaces about which the space is a suspension or cone are called links; it is shown that links need not be homeomorphic, but that their products with the real line are necessarily homeomorphic. Notice that our main result is a special case of the Bing–Borsuk conjecture, [1], that a separable finite-dimensional locally homogeneous ANR is a generalized manifold.

I should like to thank Dr. A. Szymański for a helpful letter concerning an earlier version of this work.

Definitions. The suspension sL of a space L is the quotient of $L\times[0,1]$ obtained by identifying $L\times 0$ and $L\times 1$ to distinct points, called the *conepoints*.

The *open* (*closed*) *cone* on a space L, written c^0L (cL), is the quotient of $L\times[0,1)$ ($L\times[0,1]$) obtained by identifying $L\times 0$ to a point.

In all cases the point corresponding to (x,t) is written $x\wedge t$. In sL, given s with $0\leqslant s\leqslant 1$, we write $c_s^0L=\{x\wedge t\mid x\in L \text{ and } 0\leqslant t<s\}$.

A compact finite-dimensional metric space X is called an *absolute suspension* (AS) if for each pair of distinct points x, y there is a space $L(x,y)$ and a homeomorphism from X to $sL(x,y)$ carrying x to the bottom conepoint $L\times 0$ and y to the top conepoint $L\times 1$.

A locally compact finite-dimensional metric space X is called an *absolute cone* (AC) if for each $x \in X$ there is a space $L(x)$ and a homeomorphism from X to $c^0L(x)$ carrying x to the conepoint.

The n-sphere S^n is an example of an AS; Euclidean n-space $\boldsymbol{R}^n$ is an example of an AC.

A compact finite-dimensional metric pair $(X, *)$ is called an *A-space* if for each $x \neq *$ there is a space $L(x)$ and a homeomorphism from X to $sL(x)$ carrying x, $*$ to the bottom, top conepoints respectively. Clearly an AS is an A-space for any choice of "top-point" $*$, while the one-point compactification of an AC is an A-space when the point at infinity is chosen as top-point. The notion of A-space allows simultaneous treatment of the AC and AS cases in the theorems below. In an A-space X, the space $L(x)$ will be called the *link* of x; the term is also applied in the obvious way to an AC or an AS.

The first lemma is essentially due to Szymański ([10] and letter).

LEMMA 1. *An A-space X is a homogeneous* ANR *and each of its links is an* ANR. *Moreover if $x \neq y \neq * \neq x$, there is a homeomorphism of X fixing $*$ and throwing x onto y.*

Proof. As each point has a neighbourhood base of the form $\{c_s^0 L\}$, where L is some link and $s>0$, X is locally contractible. As it is finite-dimensional it is an ANR by a theorem of Borsuk [2]. For any x, the open subset $X \setminus \{x, *\} \cong L(x) \times (0, 1)$ is an ANR($\mathfrak{M}$). Thus again by [2] the compact set $L(x)$ is an ANR.

If $x \neq *$, writing $L = L(x)$, we have h: $(sL, L\times 0, L\times 1) \cong (X, x, *)$. The map $x \wedge t \mapsto x \wedge (1-t)$ is a homeomorphism of sL interchanging the conepoints. Hence, conjugating by h, we get a homeomorphism of X swapping x and $*$. Now if $x \neq y \neq *$, by reparametrising the "vertical" coordinate of sL we may assume that $h(y) \in L \times \frac{1}{2}$. Thus the homeomorphism f of X fixes y. Similarly there is a homeomorphism g of X which fixed x and swaps y and $*$. The composite fgf fixes $*$ and swaps x and y. ■

LEMMA 2. *The homology sheaf of an A-space is the constant sheaf $\tilde{H}_*(X)$.*

Proof. We use the notation of [3]. The homology sheaf $\mathscr{H}_*(X)$ is generated by the presheaf $U \mapsto H_*(X, X-U)$. (Here $H_*(\ )$ denotes Borel–Moore homology and $\tilde{H}_*(\ )$ denotes reduced Borel–Moore homology.)

A base for the topology consists of sets homeomorphic to $c_s^0 L$, where $s>0$ and L is a link. Any member of this base, U say, has a compact and contractible complement, and so the exact sequence of a pair gives a natural isomorphism $\tilde{H}_*(X) \mapsto H_*(X, X-U)$. Thus the presheaf sections over the whole space generate each stalk by restriction, and the restriction maps are all monomorphisms. Thus the associated sheaf is constant with the claimed stalks. ■

THEOREM 1. *An A-space of dimension n is a regular generalized manifold homotopy equivalent to S^n. Each of its links is a generalized manifold homotopy equivalent to S^{n-1}.*

Proof. By Lemma 1, $H_*(X)$ is finitely generated. Using Lemma 2 and taking coefficients in a field, we see that the spectral sequence of [3], v. 8.1 reduces to

$$E_2^{p,q} \cong H_p(X) \otimes \tilde{H}_{-q}(X) \Rightarrow H_{-p-q}(X).$$

Clearly X is connected, so $\tilde{H}_0(X) = 0$. If $N>n = \dim X$, $\tilde{H}_N(X) = 0$. By a result of Kodama [8] X has cohomological dimension n over any field. By [7] for some x, $\mathscr{H}_n(X)_x \neq 0$. Thus $\tilde{H}_n(X) \neq 0$.

Suppose inductively that $0 = \tilde{H}_j(x)$ for $j<p$. Then $H_n(X) \otimes \tilde{H}_p(X) \cong E_2^{n,-p} \cong E_\infty^{n,-p}$ is a summand in $H_{p-n}(X)$. If $p<n$ this is zero, so $\tilde{H}_*(X) = 0$ for $* \neq n$. The spectral sequence collapses to

$$H_n(X) \otimes \tilde{H}_n(X) \cong E_2^{n,-n} \cong E_\infty^{n,-n} \cong H_0(X).$$

Thus $H_n(X) \cong \tilde{H}_n(X)$ is one-dimensional. By Lemma 1, X is locally contractible, and so clc for any coefficients. Thus X is an $n-gm_k$ for any field k. Thus by [9] it is an $n-gm_{\boldsymbol{Z}}$ and is clearly regular in the sense of [11].

If $n \leqslant 1$, X is a sphere by Chapter IX of [11]. If $n>1$, by [10] $L(x)$ is connected and so $\pi_1(X) = 1$. Thus by the Hurewicz and Whitehead theorems $X \simeq S^n$.

Finally $L(x) \times (0, 1) \cong X \setminus \{x, *\}$ is an open subset of an $n-gm$ and so an $n-gm$. Thus by the factorization theorem ([3], V. 15.8) $L(x)$ is an $(n-1)-gm$. Now $\tilde{H}_*(L(x)) \cong \tilde{\mathscr{H}}_{*+1}(sL(x)) = \tilde{H}_{*+1}(X)$, so by the argument above $L(x) \simeq S^{n-1}$. ■

COROLLARY 1. *An n-dimensional* AS *X is a regular generalized n-manifold homotopy equivalent to S^n; all its links are generalized $(n-1)$-manifolds homotopy equivalent to S^{n-1}.* ■

COROLLARY 2. *An n-dimensional* AC *X is a regular generalized n-manifold proper homotopy equivalent to $\boldsymbol{R}^n$; all its links are generalized $(n-1)$-manifolds homotopy equivalent to S^{n-1}.* ■

Chapter IX of [11] yields

COROLLARY 3. *In dimensions less than* 4, *each* AS *is a sphere and each* AC *is a Euclidean space.* ■

It is interesting to enquire just how far the links of an A-space are determined.

THEOREM 2. *If X is an A-space and $x, y \in X$,* $L(x) \times (0, 1) \cong L(y) \times (0, 1)$.

Proof. By Lemma 2 there is a homeomorphism h of X swapping x and y and fixing $*$. Thus

$$L(x) \times (0, 1) \cong X \setminus \{x, *\} \cong X \setminus \{y, *\} \cong L(y) \times (0, 1). \ ■$$

This has the obvious corollary in each case. The theorem is best possible in the following sense.

EXAMPLE. In an AS or an AC, links need not be homeomorphic.

For the latter case there is constructed in [4] a generalized 3-manifold V such that $V \not\cong S^3$ but $c^0 V \cong \mathbf{R}^4$. Thus V is a possible link of $\mathbf{R}^4$, but is not homeomorphic to the link S^3.

For the former case, a recent result of Edwards [6] provides a homeomorphism $S^5 \cong s(sM^3)$, where M^3 is Mazur's 3-manifold. Thus sM^3 and S^4 can both be links of S^5, but $sM^3 \not\cong S^4$. Indeed sM^3 is not an AS. For if it were an AS, M^3 would be the link of a conepoint, while the link of any other point would be homotopy equivalent to S^3. But $S^3 \not\simeq M^3$, in contradiction of Theorem 2. This last observation shows that there is no simple proof of de Groot's conjecture by downwards induction on dimension.

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Accepté par la Rédaction le 21. 6. 1976