THEOREM 3. *If* $J_\alpha \models V = HC$, *then the following conditions are equivalent*

(a) J_α *is recursively inaccessible*,

(b) $J_\alpha \cap \wp(\omega) \models \Delta^1_2$-CA.

Again (a) implies that $J_\alpha \cap \wp(\omega)$ is then a β-model. Theorem 2 generalizes to n bigger than 2 as follows:

THEOREM 4. *If* $J_\alpha \models V = HC$, *then the following conditions are equivalent.*

(a) $J_\alpha \cap \wp(\omega) = \Sigma^1_{n+1}$-CA,

(b) J_α *is nonprojectible by means of a* Σ_n-*function*,

(c) $J_\alpha \models \Sigma_n$-*separation scheme*,

(d) J_α *possesses a cofinal tower of transitive* Σ_n-*elementary subsystems.*

Clearly if (d) then $J_\alpha \cap \wp(\omega)$ is a β-model.

"In limit" this is nothing else but a version of the "gap theorem"

THEOREM 5 ([2]): *If* $J_\alpha \models V = HC$, *then the following conditions are equivalent*:

(a) J_α *is a gap* (*i.e.* $(J_{\alpha+1} - J_\alpha) \cap \wp(\omega) = \emptyset$,

(b) $J_\alpha \cap \wp(\omega)$ *is a model of* CA,

(c) J_α *is a model of full replacement.*

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INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY

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Barely Baire spaces

by

W. G. Fleissner * (Pittsburgh, Pa.) and **K. Kunen** ** (Madison, Wis.)

Abstract. We give new examples of Baire spaces whose products are not Baire. In particular, we construct a Baire X with X^2 nowhere Baire, and, for each $\varkappa$, a family of spaces $\{X_\beta\colon \beta \leqslant \varkappa\}$ such that $\prod\{X_\beta\colon \beta < \varkappa\}$ is nowhere Baire but for all $\gamma < \varkappa$, $\prod\{X_\beta\colon \beta < \varkappa, \beta \neq \gamma\}$ is Baire. We indicate the relation of our technique to the forcing technique of P. E. Cohen.

Introduction. If the reader will bear with us, we will bare the facts about barely Baire spaces (1).

The Baire Category Theorem, that in complete metric spaces the intersection of countably many dense open sets is dense, is of fundamental importance in analysis ([5] and [21]). Following Bourbaki, we call a space in which the intersection of countably many dense open sets is dense a Baire space. That a compact Hausdorff space is Baire plays a key role in the Rasiowa–Sikorski proof of Gödel's Completeness Theorem [20]. It also motivates the form of Martin's Axiom most accessible to the nonlogician. In general topology that various other types of spaces are Baire is important ([7] and [24]).

Because of the usefulness of Baire spaces, it is natural to ask about the closure properties of the class of Baire spaces. For example, locally Baire spaces, images of Baire spaces under open maps, and dense G_δ's of Baire spaces are Baire. Images of Baire spaces under closed maps, or arbitrary G_δ's of Baire spaces need not be Baire. The more stubborn question of whether the product of Baire spaces or metric Baire spaces is Baire has been raised in [5], [16] and [22].

We call a Baire space X *barely Baire* if there is a Baire space Y such that $X \times Y$ is not Baire. We call a space X *nowhere Baire* if there is a family $\mathscr{D} = \{D_i\colon i \in \omega\}$ of dense open sets so that $\bigcap \mathscr{D}$ is empty.

Oxtoby [16] showed that the continuum hypothesis implies that there is a barely Baire space. More recently, P. E. Cohen improved this to an absolute result. Cohen's

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(1) The title was suggested by Judy Roitman; Eric van Douwen informs us that he had already used the phrase in a different sense.

proof involved techniques of forcing. In this paper (§ 4), we present a direct combinatorial proof. In § 5 we shall indicate the relationship between our methods and Cohen's.

§ 4 also includes a number of other examples obtained by our techniques. In particular, we find a Baire X such that X^2 is nowhere Baire, and, for every cardinal $\varkappa$, a family of spaces $\{X_\beta\colon \beta<\varkappa\}$ such that $\prod\{X_\beta\colon \beta<\varkappa\}$ is nowhere Baire, but for all $\gamma<\varkappa$, $\prod\{X_\beta\colon \beta<\varkappa, \beta\neq\gamma\}$ is Baire.

1. History. In this section we review the history of the problem. Let us begin with the proof that complete metric space X is Baire. Let U be a nonempty open set of X and $\mathscr{D}=\{D_i\colon i\in\omega\}$ a family of dense open sets of X.

We inductively choose a sequence $\{U_i\colon i\in\omega\}$ of non empty open sets satisfying: (i) $U_0=U$, (ii) $\overline{U}_{i+1}\subset U_i\cap D_i$, (iii) diameter $U_{i+1}<2^{-i}$.

If $x_i\in\overline{U}_i$, then $\{x_i\colon i\in\omega\}$ is a Cauchy sequence, and by completeness converges to a point x. Now $x\in U\cap\bigcap\mathscr{D}$ so X is Baire.

Let us note several things about this proof. First, the proof is to define a nested sequence of open sets, satisfying certain conditions (ii), (iii) so that the intersection is a point. Second, we could have required the open sets to be in a given basis. Third, the choice of U_{i+1} depended on U_i and $\{D_n\colon 0\leqslant n\leqslant i\}$; the choice was made without knowledge of $\{D_n\colon i<n<\omega\}$.

It is often useful to consider a game associated with a space X with a basis B. For convenience let us exclude the empty set from B. Two players α and β alternately choose elements of B to form a nested sequence $\mathscr{U}=\{U_i\colon i\in\omega\}$ Player α wins if $\bigcap\mathscr{U}\neq\emptyset$; player β wins if $\bigcap\mathscr{U}=\emptyset$. Formalization of the game and a precise definition of winning strategy can be found in [5], [17] and [25].

For the game in which α moves first, "β has a winning strategy that depends on knowledge of all of the previous moves" implies "X is nowhere Baire" implies "β has a winning strategy $\sigma\colon B\to B$" [17]. If α has a winning strategy (depending on the previous moves) in the game in which β moves first, X is called (weakly) α-favorable [5]. The nice properties of weakly α-favorable spaces are listed in [25]. A weakly α-favorable space is complete in the sense of the next paragraph.

There is a plethora of completeness properties in the literature (see [1] for a survey.) Fortunately, we need not be concerned with them. We make the informal definition that a space is complete if it can be proved Baire by the argument above. The relation of complete spaces to products of Baire spaces is described in the theorems below.

THEOREM. *Any product of complete spaces is Baire* ([4], [16] *and* [25]).

Proof. We require the U_i's to have finite support. On the factors in the support, we use the completeness argument. We ignore the other factors. The same proof show that box products of π_m products [9] of complete spaces are Baire.

THEOREM. *If X is complete and Y is Baire, then $X\times Y$ is Baire* ([2] *and* [25]).

Idea of the proof. The dense open sets are refined so that we can find a fiber $\{y\}\times X$ on which the completeness property of X can be used.

Let us hasten to give an example of a metric Baire space which does not contain a dense complete subspace. One can readily verify that if Z is a subset of $\boldsymbol{R}$, the reals, such that neither Z or $\boldsymbol{R}-Z$ contains a perfect subset, then Z and $\boldsymbol{R}-Z$ are metric not complete Baire spaces. Bernstein constructed such a Z by transfinite induction ([12], § 40, I). But it is not barely Baire because

THEOREM. *If X is a second countable Baire space and Y is a Baire space, then $X\times Y$ is Baire* ([16] and [22]).

Idea of the proof. A category analogue of Fubini's theorem.

Oxtoby's fundamental paper [16] contains two other important results. The following theorem is generalized in Theorem 1, § 3.

THEOREM. *Any product of second countable Baire spaces is Baire.*

And, as mentioned in the introduction, assuming the continuum hypothesis, Oxtoby gave the first example of a barely Baire space, inductively defining a subspace of the Stone space of the measure algebra. Actually, the assumption used is that the union of $<c$ sets of measure zero has measure zero. White's analogous construction of a subspace of the real line with the density topology should be noted [26].

Two recent papers complete our survey of the history of the problem Krom [10] constructed a map $^{\#}$ from the class of topological spaces to the class of metric spaces such that $X^{\#}\times Y$ is Baire iff $X\times Y$ is Baire. This result had two important effects. First it reemphasized the importance of games, and second it encouraged people to look at "bad" topological spaces. The problem of the existence of barely Baire spaces was finally completely solved when P. E. Cohen [6] suggested forcing spaces and gave an absolute example of a barely Baire space.

2. Preliminaries. An ordinal is the set of its predecessors. A cardinal is an initial ordinal. The first infinite ordinal (cardinal) is ω. A function is a set of ordered pairs. The image of a set V under a function f is denoted $f''V$, the preimage is denoted $f^{-1}V$. The restriction of f to j is $f\restriction j$.

If $\{X_\alpha\colon \alpha\in I\}$ is a family of topological spaces, $\prod\{X_\alpha\colon \alpha\in I\}$ is the usual (Tychonoff, or finite) product. We use the standard base of open cylinders with finite support. We denote the support of B by $\operatorname{supp}B$.

S^T is the set of functions from T to S. FS_α, the set of finite sequences in α, is $\bigcup\{\alpha^n\colon n\in\omega\}$. If $\sigma\in FS_\alpha$, $\sigma^\frown\gamma$ is $\sigma\cup\{\langle\operatorname{dom}\sigma,\gamma\rangle\}$; that is, σ with γ stuck on the end.

We call $\varkappa^\omega$ $J_\varkappa$ and give it a metric d; $d(f,g)=2^{-n}$ where n is least such that $f(n)\neq g(n)$. Thus $2^\omega=J_2$ is the Cantor set. The cardinality of a set S is denoted $\operatorname{card}S$. The cardinality of 2^ω is c. The next cardinal after c is denoted c^+.

If $\sigma\in FS_\varkappa$, and we are discussing $J_\varkappa$, then $B_\sigma=\{f\in J_\varkappa\colon \sigma\subset f\}$. A basis for $J_\varkappa$ is $\{B_\sigma\colon \sigma\in FS_\varkappa\}$. If D is a dense open set of $J_\varkappa$, then there is a function $\tilde{D}\colon FS_\varkappa\to FS_\varkappa$ such that

$$B_{\tilde{D}(\sigma)}\subset B_\sigma\cap D\,.$$

When $\operatorname{cf}\varkappa>\omega$, we define a map $^*\colon J_\varkappa\to\varkappa$; f^* is the least α greater than $f(n)$ for all $n\in\omega$. If $A\subset\varkappa$, $A^*=\{f\in J_\varkappa\colon f^*\in A\}$.

An infinite cardinal $\varkappa$ is *regular* if it is not the union of less than $\varkappa$ sets of cardinality less than $\varkappa$. A subset C of $\varkappa$ is called *cub* if it is closed unbounded. A subset A of $\varkappa$ is called *stationary* in $\varkappa$ if A intersects every C cub in $\varkappa$.

If $\varkappa$ is regular, the intersection of less $\varkappa$ sets cub in $\varkappa$ is cub in $\varkappa$; the contrapositive is that the union of less than $\varkappa$ set not stationary in $\varkappa$ is not stationary in $\varkappa$. If $\varkappa > \omega$ is a regular cardinal, then any stationary subset of $\varkappa$ can be split into $\varkappa$ disjoint stationary subsets of $\varkappa$ [23].

If $\tilde{D}: FS_\varkappa \to FS_\varkappa$ call γ a fixed point of $\tilde{D}$ if $\tilde{D}''FS_\gamma \subset FS_\gamma$. If $\varkappa$ is uncountable and regular, the fixed points of $\tilde{D}$ are cub in $\varkappa$ (see [11] for proofs).

$C_\omega \varkappa$ is the subset of $\varkappa$ of ordinals of cofinality ω; if $\varkappa$ is uncountable and regular, then $C_\omega \varkappa$ is stationary.

The following lemma is Lemma 3.4 of [8].

LEMMA 1. *Let $\varkappa > \omega$ be regular. If $K \subset J_\varkappa$ is closed, and $W = \{f^*: f \in K\}$ is stationary, then there is C cub in $\varkappa$ such that $C \cap C_\omega \varkappa \subset W$.*

Proof. Let $W_\sigma = \{f^*: \sigma \subset f \in K\}$. Let $\Sigma = \{\sigma: W_\sigma \text{ is stationary}\}$. By hypothesis the empty sequence is in Σ. Using the Pressing Down Lemma, we can define a function $\theta: \Sigma \times \varkappa \to \Sigma$ such that

(i) $\theta(\sigma, \alpha) \supset \sigma$;

(ii) $\theta(\sigma, \alpha) \notin FS\alpha$.

Let C be the set of γ such that

$$\theta''(\Sigma \cap FS_\gamma) \times \gamma \subset FS_\gamma .$$

3. New theorems. Although the main thrust of this paper is the new examples, we also have several new theorems.

THEOREM 1. *Suppose for all $\beta \in I$, X_β has a (pseudo, or π) base of cardinality $\leqslant \varkappa$. Then if $X = \prod \{X_\beta: \beta \in I\}$ is nowhere Baire, there is $I' \subset I$, $\operatorname{card} I' \leqslant \varkappa$, such that $\prod \{X_\beta: \beta \in I'\}$ is nowhere Baire* [16].

Proof. Direct from Lemmas 2 and 3.

LEMMA 2. *If each X_β has a (pseudo, or π) base of cardinality $\leqslant \varkappa$, then X has cellularity $\varkappa$. That is, every family of disjoint open sets of X has cardinality $\leqslant \varkappa$* [14].

Proof. It is clear if I is finite. By the Δ-system lemma, it then follows for all I.

LEMMA 3. *Suppose $X = \prod \{X_\beta: \beta \in I\}$ has cellularity $\varkappa$ and is nowhere Baire. Then there is $I' \subset I$, $\operatorname{card} I' \leqslant \varkappa$, $\prod \{X_\beta: \beta \in I'\}$ is nowhere Baire.*

Proof. Let $\mathscr{D} = \{D_n: n \in \omega\}$ be a family of dense open sets of X, $\bigcap \mathscr{D} = \emptyset$. Let $\{G_\beta^n: \beta \in K_n\}$ be a maximal collection of disjoint basic open subsets of D_n. It is easy to check that $\bigcup \{G_\beta^n: \beta \in K_n\}$ is dense open. Let

$$I' = \bigcup \{\operatorname{supp} G_\beta^n: \beta \in K_n, n \in \omega\} .$$

Remark 1. We may replace "nowhere Baire" with "not Baire" because a space is not Baire iff some non-empty (basic) open set is nowhere Baire.

Remark 2. This shows that the spaces of Example 2 cannot be smaller.

THEOREM 2. *Suppose $X^\varkappa$ is nowhere Baire. Then X^ω is nowhere Baire.*

Proof. Let σ be a winning strategy for β in the game associated with $X^\varkappa$ and B, the basis of open sets with finite support. Then β (with perfect memory) has a winning strategy in the game associated with X^ω and B', the basis of open sets with finite support. The idea is this: β relabels the index set as he goes along so that he is in effect playing according to σ in $X^\varkappa$.

Remark 3. Again nowhere Baire can be replaced by not Baire because the support of a basic open set is finite.

THEOREM 3. *Let X be a metric space without isolated points. Then there is a pair X', f such that*

(a) *$X - X'$ is a countable union of nowhere dense sets;*

(b) *f is a map from X' to 2^ω, the Cantor set;*

(c) *the preimage of a nowhere dense set of 2^ω is nowhere dense in X.*

COROLLARY 3.1. *Every metric space without isolated points is the union of c nowhere dense sets.*

COROLLARY 3.2. *Assuming Martin's every Baire metric space without isolated points has cardinality $\geqslant c$.*

COROLLARY 3.3. *It is consistent with the usual axioms of set theory that $\omega_1 < c$ and every metric space without isolated points is the union of ω_1 nowhere dense sets.*

Proof. It is consistent that $\omega_1 < c$ and 2^ω is the union of ω_1 nowhere dense sets. (Consider the random real model.)

Proof of Theorem 3. By Bing's theorem [3], X has a σ-discrete base B, $B = \bigcup \{B_i: i \in \omega\}$. Then F_i, the frontier of $\bigcup B_i$, is nowhere dense. $(F_i = X - (\bigcup B_i \cup \operatorname{int}(X - \bigcup B_i)))$. Let

$$X' = X - \bigcup \{F_i: i \in \omega\} .$$

From B on X, we can define a base $B' = \bigcup \{B'_i: i \in \omega\}$ on X' satisfying

(i) $\bigcup B'_i = X'$, B'_i disjoint;

(ii) B'_{i+1} everywhere properly refines B'_i.

Making (ii) more explicit, there are $\mathscr{W}_i^0$, $\mathscr{W}_i^1$ such that

(iii) $\mathscr{W}_i^0 \cap \mathscr{W}_i^1 = \emptyset$, $\mathscr{W}_i^0 \cup \mathscr{W}_i^1 = B'_i$;

(iv) for every $B \in B'_i$ there are B^0, B^1 such that $B^0 \in \mathscr{W}_{i+1}^0$, $B^1 \in \mathscr{W}_{i+1}^1$, $B^0 \subset B$, $B^1 \subset B$.

Now for $x \in X'$, define $f(x)$ to be the unique element of 2^ω satisfying $x \in \mathscr{W}_i^{f(x)(i)}$. To check (c), given N nowhere dense in 2^ω and U open in X', we need to find a non-empty open V, $V \subset U$, $V \cap f^{-1}(N) = \emptyset$. Without loss of generality, $U \in B'_i$. Then $f''U$ is basic open in 2^ω. Choose V' basic open in 2^ω, $V' \subset f''U - N$. Let $V = f^{-1}V'$.

4. New examples. Consistent examples of barely Baire spaces have been given in [6], [16] and [26]. In this section we present new and absolute examples.

Throughout this section, $\mathscr{D} = \{D_i: i \in \omega\}$ is a countable family of dense open sets and V a non-empty basic open set of the space in question.

EXAMPLE 1. An absolute barely Baire space. Let A be a stationary subset of ω_1 (or of $C_\omega\varkappa$). Then A^* is Baire.

Now each D_i induces a function $\tilde{D}_i: FS_i \to FS_i$ ($FS_i \to FS_i$). Let C_i be the cub of fixed points of D_i. Let $C = \bigcap \{C_i: i \in \omega\}$. Let $V = B_\sigma$. Choose $\gamma \in C \cap A$ so that $\sigma \in FS_\gamma$. Let $\sup\{\gamma_i: i \in \omega\} = \gamma$. Inductively define

$$\sigma_0 = \sigma,$$
$$\sigma_{i+1} = \tilde{D}_i(\sigma_i)^\frown\gamma_i .$$

Let $f = \bigcap \{B_{\sigma_i}: i \in \omega\} = \bigcup \{\sigma_i: i \in \omega\}$. Then $f^* = \gamma \in A$ and $f \in V \cap \bigcap \mathscr{D}$.

If A and B are disjoint stationary subsets of ω_1 (or $C_\omega\varkappa$), then $A^* \times B^*$ is not Baire. Define

$$D_i = \{\langle f, g\rangle: \min(f^*, g^*) > \max(f(i), g(i))\},$$

a dense open set. If $\langle f, g\rangle \in A^* \times B^*$, then $f^* \neq g^*$. So assume $f^* > g^*$. Then for some i, $f(i) \geqslant g^*$ and $\langle f, g\rangle \notin D_i$.

Remark 4. A^ω is Baire by a similar argument, and by Theorem 2 every power of A is Baire.

Remark 5. It is instructive to compare the proof that A^* is Baire with the proof that a complete metric space is Baire. Both proofs define nested sequences of open sets which intersect in a point. The difference is that in Example 1 we used knowledge of the entire family $\mathscr{D}$ to define γ and $\{\gamma_i: i \in \omega\}$.

Remark 6. This example was discovered as a simplification of the metric spaces defined from a forcing argument. With hindsight, it seems incredible that it was not discovered by investigating non-separable analogues to Bernstein's example. R. Pol has emphasized this analogy in private correspondence and in [18]. Pol [19] also has shown that every nowhere separable Baire metric space can be split into two dense subspaces of second category whose product is not Baire.

EXAMPLE 2. For every cardinal $\varkappa$, there is a family $\{X_\alpha: \alpha < \varkappa\}$ of metric spaces such that

(i) $\prod\{X_\alpha: \alpha < \varkappa\}$ is nowhere Baire.

(ii) For every $\beta < \varkappa$, $\prod\{X_\alpha: \alpha < \varkappa\ \alpha \neq \beta\}$ is Baire.

First note that it is sufficient to prove the assertion for arbitrarily large $\varkappa$, so we consider only finite or regular $\varkappa$. Let $\{A_\alpha: \alpha < \varkappa\}$ be disjoint stationary subsets of ω_1, if $\varkappa \leqslant \omega_1$, or of $C_\omega\varkappa$ if $\varkappa > \omega_1$. Let $B_\alpha = \bigcup\{A_\beta: \beta < \varkappa, \beta \neq \alpha\}$. Then for all $\alpha < \varkappa$, the product $\prod\{B^*_\beta: \beta < \varkappa, \beta \neq \alpha\}$ is Baire because it contains as a dense subset a power of A^*_α, which is Baire by Remark 4 and Theorem 2.

If (some basic open set of) $\prod\{B^*_\alpha: \alpha < \varkappa\}$ is nowhere Baire, as we can show for $\varkappa < \omega_1$, then we are done. If $\prod\{B^*_\alpha: \alpha < \varkappa\}$ is Baire, then we add to our family the space Z.

$$Z = \{z \in J_\varkappa: z^* \in A_{z(n)} \text{ for some } n \in \omega\}.$$

For every α, a dense open set of A^*_α is dense in Z. So it remains to show that $Z \times \prod\{B^*_\alpha: \alpha < \varkappa\}$ is nowhere Baire. Define

$$D_{ij} = \{\langle z, f_0, \ldots\rangle: \min(z^*, f^*_{z(j)}) > \max(z(i), f_{z(j)}(i))\}.$$

Suppose $\langle z, f_0, \ldots\rangle \in \bigcap\{D_{ij}: i, j < \omega\}$. Then $z^* = f^*_{z(0)} = \ldots$, and $z \in Z$ iff for some i, $f_{z(i)} \notin B^*_{z(i)}$. So $\bigcap\{D_{ij}: i, j < \omega\} = \emptyset$.

EXAMPLE 3. A new consistent barely Baire space. Assume $\diamondsuit_\omega\varkappa$. That is, there is a sequence $\{S_\gamma: \gamma \in C_\omega\varkappa\}$ such that $S_\gamma: \gamma \to 2$ and for all $x: \varkappa \to 2$,

$$A_x = \{\gamma: S_\gamma = x \upharpoonright \gamma\}$$

is stationary.

Say that X is $\varkappa$-Baire if the intersection of $\varkappa$ dense open sets is dense. Let $Y_x = \{x\} \times Y$.

LEMMA 4. *Let X be $\varkappa$-Baire and let $\{V_\alpha: \alpha < \varkappa\}$ be a base for Y. Let $\{D_\beta: \beta < \varkappa\}$ be a family of dense open sets of $X \times Y$. Then there is a dense set G of X such that for $x \in G$ and $\beta < \varkappa$ $D_\beta \cap Y_x$ is dense open in Y_x.*

Proof. Let $G_{\alpha\beta}$ be the projection onto X of $D_\beta \cap X \times V_\alpha$. $G_{\alpha\beta}$ is a dense open set in X. Let $G = \bigcap\{G_{\alpha\beta}: \alpha, \beta < \varkappa\}$.

Remark 7. This proof is the same that Kuratowski and Ulam used for $\varkappa = \omega$ [13].

Let X be $2^\varkappa$ topologized so that the αth basic open neighbourhood of x is $\{x': x' \upharpoonright \alpha = x \upharpoonright \alpha\}$. Note that X is $\varkappa$-Baire. Let $Y = J_\varkappa$. Let

$$K = \{\langle x, f\rangle \in X \times Y: f^* \in A_x\}.$$

Every dense open set of K comes from a dense open set of $X \times Y$, so by the lemma there is a dense set of $x \in X$ such that for each i, D_i is dense open set in Y_x. Now $K \cap Y_x$ is homeomorphic to A^*_x, a Baire space. Thus $\bigcap \mathscr{D} \cap K \cap Y_x$ is dense in Y_x, and K is Baire.

To show K^2 is nowhere Baire, let

$$D_i = \{\langle\langle x, f\rangle, \langle x', f'\rangle\rangle: \min(f^*, f'^*) > \max(f(i), f'(i)) \text{ and } x \upharpoonright f^* \neq x' \upharpoonright f^*\}.$$

Remark 8. If $\lambda^\omega = \lambda$, and $2^\lambda = \varkappa = \lambda^+$, then $\diamondsuit_\omega\varkappa$ (Gregory, Laver).

EXAMPLE 4 ([1]). An absolute nowhere Baire square. Let $\{A_x: x \in 2^\omega\}$ be disjoint stationary subsets of $C_\omega c^+$. Let $M = 2^\omega \times J_{c^+}$. Our space is

$$Y = \{\langle x, f\rangle \in M: f^* \in A_x\}.$$

([1]) We also have a Baire notion of forcing whose square adds a real.

Let $\mathscr{D} = \{D_i: i \in \omega\}$ be a family of dense open sets of M and V a non-empty open set of M Let

$$W = \{f^*: \langle x, f\rangle \in V \cap \bigcap \mathscr{D}\}.$$

We first check that W is stationary. Let C be cub in c^+. We can inductively choose nested basic open sets B_i of M in such a way that $B_0 \subset V$, $B_{2i+2} \subset D_i$, and B_{2i+1} insures $f^* \in C$, where $\langle x, f\rangle = \bigcap \{B_i: i \in \omega\}$.

Now for $\langle x, f\rangle \in M$, $h: \omega \to \omega$, and $i \in \omega$, let $B(x, f, h, i)$ be the ball of radius $2^{-h(i)}$ around $\langle x, f\rangle$. Explicitly,

$$B(x,f,h,i) = \{\langle x', f'\rangle \in M: x \restriction h(i) = x' \restriction h(i), \quad f \restriction h(i) = f' \restriction h(i)\}.$$

Let

$$W_{xh} = \{f^*: B(x,f,h,i) \subset D_i \cap V \text{ for all } i \in \omega\}.$$

Now W, a stationary subset of c^+, is not the union of c non-stationary sets. So for some x, h, W_{xh} is stationary. By Lemma 1, there is a cub C such that $C \cap C_\omega c^+ \subset W_{xh}$. Then, $A_x \cap W_{xh} \neq \emptyset$. So there is $\langle x, f\rangle \in Y \cap V \cap \bigcap \mathscr{D}$, and Y is Baire.

To show that Y^2 is nowhere Baire, let

$$\dot{D_i} = \{\langle\langle x,f\rangle, \langle x', f'\rangle\rangle: x \neq x', \ \min(f^*, f') > \max(f(i), f'(i))\}.$$

Remark 9. Let $m \leqslant \omega$. The above method generalizes to construct a space X so that X^m is nowhere Baire and X^n is Baire for all $n < m$. Together with Theorem 2 this covers all possibilities for powers.

Let $\{A_y: y \in m^\omega\}$ be disjoint stationary subsets of $C_\omega c^+$. Let

$$B_y = \bigcup \{A_{y'}: y(i) \neq y'(i) \text{ for all } i \in \omega\}.$$

Let

$$X = \{\langle y, f\rangle: y \in m^\omega, f \in J_{c^+}, f^* \in B_y\}.$$

Remark 10. Let X be constructed as in Remark 9 with $m = \omega_1$. Then the usual product of ω_1 copies of X is Baire, but the box product of ω_1 copies of X is nowhere Baire.

5. Forcing. In this section we describe how Example 1 was derived from a notion of forcing. We also prove a theorem which enables us to show that some "cross-products" of barely Baire spaces are Baire.

A full explanation of forcing is of course beyond the scope of this article. But we hope that the material presented below will give a feeling of the relation between Baire spaces and extending models of set theory. The presentation is aimed at those with some experience with the partial order form of Martin's Axiom.

Martin's Axiom talks about a certain type of forcing; note the title "Internal Cohen Extensions" [15]. Forcing is the process of defining a partial order P and getting a filter G generic over P. (In this paper, it is relevant to note that the existence

of G can be shown by applying the Baire Category Theorem to the Stone space of the regular open algebra of P^t.)

Some differences between external Cohen extensions and Martin's Axiom are:

1. G intersects all dense subsets of P (in the countable model M).
2. P need not have the countable chain condition.
3. G will be a new set, not in the model M (except in trivial cases).

We define a topological space P^t from a given partial order P. The basic open set of the point p is $\{q: p \geqslant q\}$, so any intersection of open sets is open, and, if P has two comparable elements, P is not T_1. The relation to Baire spaces is the following. Let G be generic over P, H generic over Q.

4. $P^t \times Q^t$ is Baire iff no new ω-sequences of ordinals can be defined from G and H.

We now describe the forcing argument from which Example 1 was derived. Let A and B be disjoint stationary subsets of $C_\omega\varkappa$, $\varkappa$ regular, $>\omega$. Let $P(Q)$ be the collection of order preserving and limit preserving maps from an countable ordinal with last element to $A(B)$. Say that $p \geqslant p'$ iff $p \subseteq p'$. $P^t(Q^t)$ can be shown to be Baire by an argument similar to (but more complicated than) the argument in Example 1.

Because G and H intersect every dense set, range $\bigcup G$ and range $\bigcup H$ are cub in $\varkappa$. These cubs are disjoint because A and B are, so cf $\varkappa$ has become ω. So by 4 $P^t \times Q^t$ is not Baire. We can explicitly define a new ω-sequence of ordinals cofinal in $\varkappa$. Let β_0 be the first element of range $\bigcup G$. Inductively define β_{2n+1} to be the least element of range $\bigcup H$ greater than β_{2n}; β_{2n+2} to be the least element of range $\bigcup G$ greater than β_{2n+1}. Bringing this idea to $P^t \times Q^t$, we see that D_n, the set of $\langle p, q\rangle$ that "intertwine" at least n times, is dense open, and $\bigcap \{D_n: n \in \omega\} = \emptyset$.

THEOREM 4. *Suppose P^t is a Baire forcing space. Then $P^t \times X$ is Baire iff $V^P \models \hat{X}$ is Baire.*

COROLLARY 4.1. *The product of a Baire forcing space with Oxtoby's example is Baire.*

COROLLARY 4.2. *The product of a Jensen–Johnbräten tree space with a stationary set forcing space is Baire.*

COROLLARY 4.3. *If P^t forces an ω-closed unbounded set through A, an ω-stationary set of $\varkappa$, and Q through an ω-stationary set of λ, and $2^\varkappa \leqslant \lambda$, then $P^t \times Q^t$ is Baire.*

Remark 11. Of course these corollaries extend to Baire spaces derived from forcing spaces.

Proof of Theorem 4. By $\hat{X}$, $(\hat{T})$ we mean the set of points (open sets) of X in V. By $V^P \models \hat{X}$ is Baire, we mean that in V^P there are no $G_i \subset \hat{T}$ such that, letting $G_i^\circ = \bigcup G_i$,

1. $\forall 0 \in \hat{T}$, $G_i^\circ \cap 0 \neq \emptyset$,
2. $\bigcap \{G_i^\circ: i \in \omega\} = \emptyset$.

Assume $P^t \times X$ is not Baire; that $\{D_i\}$ are dense open in $P^t \times X$ and $\bigcap \{D_i\} = \emptyset$. Define $G_i \subset \hat{T}$ by $\|U \in G_i\| > B_p$ iff $B_p \times U \subset D_i$. Now

$$\|x \in G_i^\circ\| = \sup_{x \in U} \|U \in G_i\| = \operatorname{intcl} \bigcup_{x \in U} \{B_p \colon B_p \times U \subset D_i\}$$

which differs from $\bigcup \{B_p \colon B_p \times U \subset D_i, x \in U\}$ by a nowhere dense set H_{xi}.

(*) $\qquad p \in \|x \in G_i^\circ\| - H_{xi} \quad$ implies $\quad (p, x) \in D_i$.

Suppose $\|0 \cup G_i^\circ = \emptyset\| \geqslant B_p$. Now $B_p \times 0 \cap D_i \neq \emptyset$, so there are B_q and U such that $B_q \times U \subset B_p \times 0 \cap D_i$. Then $\|U \subset G_i^\circ\| \geqslant B_q$ and $\|0 \cap G_i^\circ \neq \emptyset\| \geqslant B_q$; a contradiction establishing 1.

Now suppose $\|x \in \bigcap G_i^\circ\| > B_p$. Because P^t is Baire, there is $q \in B_p - \bigcup H_{xi}$. Then by (*) $(q, x) \in D_i$; a contradiction establishing 2.

Conversely, assume $V^P = X$ is not Baire; that $G_i \in V^P$ satisfy 1 and 2. Define $D_i \subset P^i \times X$ by

$$D_i = \bigcup \{B_p \times U \colon \|U \subset G_i^\circ\| \geqslant B_p\}.$$

It is easy to check that D_i is dense. Suppose that $(p, x) \in \bigcap D_i$ then $B_p \times \{x\} \subset \bigcap \{D_i\}$. And $\|x \in G_i^\circ\| \geqslant B_p$, contradicting the assumption on G_i.

6. $\varkappa$-Baire spaces. Call a space X *$\varkappa$-Baire* if the intersection of $\varkappa$ dense open sets of X is dense. For $\varkappa$ a regular cardinal, much of the theory of Baire (= ω-Baire) lifts to $\varkappa$-Baire. For example, $C_\varkappa \varkappa^+$, the set of ordinals less than $\varkappa^+$ of cofinality $\varkappa$, can be split into two disjoint stationary sets and the analogue of Example 1 can be constructed. Note that by Corollary 3.1, a $\varkappa$-Baire space cannot be required to be metric.

To get $\varkappa$-Baire spaces whose product is not Baire by this method, one needs disjoint $\varkappa$-fat subsets of a regular cardinal $> \varkappa$. A set A is a $\varkappa$-fat subset of λ if for every $\alpha < \varkappa$ and every cub C of λ, $C \cap A$ contains a closed copy of α.

It is consistent that disjoint $\varkappa$-fat subsets of $\varkappa^+$ exist. For example, the subset of $\varkappa^+$ added by Cohen forcing and its complement are $\varkappa$-fat. Moreover $V = L$ implies "fat" versions of $\diamondsuit$.

7. Questions.

1. Can the metric Baire not barely Baire spaces be generated from the complete metric spaces are locally separable metric spaces by products, dense superspaces, and dense G_δ subspaces?

2. Let X be metric barely Baire. Must there be a metric Baire space Y with the same weight as X such that $X \times Y$ is not Baire?

3. Does "α has a winning strategy that depends on knowledge of all the previous moves" imply "α has a winning strategy $\sigma\colon B \to B$"?

4. Suppose λ is singular and X is regular and $\varkappa$-Baire for all $\varkappa < \lambda$. Is X λ-Baire?

5. If the box product of a family of spaces is Baire, must the usual product of that family be Baire?

6. Is there absolutely a regular T_2 Baire space without isolated points of cardinality ω_1?

7. Is Z needed in Example 2?

8. Can $\varkappa < \lambda$ replace $2^\varkappa \leqslant \lambda$ in Corollary 4.3?

9. If X is Baire and X^ω nowhere Baire, is X barely Baire?

Remark 12. Shelah has informed us that in a model of Magidor there are not two disjoint far subsets of ω_2. Galvin has informed us that in a model of Magidor and Laver, the box product of ω_2 copies of a separable not weakly α favorable metric space is not Baire.

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Absolute suspensions and cones

by

W. J. R. Mitchell (Cambridge)

Abstract. De Groot conjectured that if a finite-dimensional compact metric space is a suspension about every pair of distinct points, then it is a sphere. Szymański proved this for dimensions strictly less than 4. Here it is shown that such a space is a regular generalized manifold homotopy equivalent to a sphere, and that any space about which it is a suspension is a generalized manifold homotopy equivalent to a sphere. An analogous result is established for spaces which are open cones about each point. These results are special cases of the Bing–Borsuk conjecture about locally homogeneous ANR's.

De Groot [5] has conjectured that if a finite-dimensional compact metric space is a suspension about every pair of distinct points then it is a sphere. Szymański [10] proved this for dimensions up to 3. Here it is shown that such a space is always a regular generalized manifold homotopy equivalent to a sphere, and that any space about which it is a suspension is a generalized manifold homotopy equivalent to a sphere. An analogous result is proved for spaces which are open cones about every point. In both cases the spaces about which the space is a suspension or cone are called links; it is shown that links need not be homeomorphic, but that their products with the real line are necessarily homeomorphic. Notice that our main result is a special case of the Bing–Borsuk conjecture, [1], that a separable finite-dimensional locally homogeneous ANR is a generalized manifold.

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DEFINITIONS. The suspension sL of a space L is the quotient of $L\times[0,1]$ obtained by identifying $L\times 0$ and $L\times 1$ to distinct points, called the *conepoints*.

The *open* (*closed*) *cone* on a space L, written c^0L (cL), is the quotient of $L\times[0,1)$ ($L\times[0,1]$) obtained by identifying $L\times 0$ to a point.

In all cases the point corresponding to (x,t) is written $x\wedge t$. In sL, given s with $0\leqslant s\leqslant 1$, we write $c^0_sL=\{x\wedge t\mid x\in L \text{ and } 0\leqslant t<s\}$.

A compact finite-dimensional metric space X is called an *absolute suspension* (AS) if for each pair of distinct points x, y there is a space $L(x,y)$ and a homeomorphism from X to $sL(x,y)$ carrying x to the bottom conepoint $L\times 0$ and y to the top conepoint $L\times 1$.