If ZF is consistent, so is the theory: $\mathrm{ZF}+\mathrm{AC}(\beth_1, \beth_2)+\neg\,\mathrm{CR}$ where CR is the following weak axiom of choice:

$$\forall r \subset \wp\wp(\omega)[\forall x \in r \forall y \in r(x = y \vee x \cap y = \emptyset)$$
$$\to \exists a \subset \wp(\omega) \forall x \in r(x \neq \emptyset \to \exists t\, x \cap a = \{t\})].$$

If $\mathscr{M}$ is a model of that theory, $\mathscr{M}^+$ is a model of $T_4+\neg\,\mathrm{PC}$.

Post-script. Independently, W. Marek, using the interpretation of second order arithmetic in set theory given here and in [11], gets some results on ω-models of second order arithmetic corresponding to sets L_α, for some countable α. These results are in print in Fund. Math. [10].

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Some comments on the paper by Artigue, Isambert, Perrin, and Zalc

by

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DEFINITION. (a) $\{J_\alpha\}_{\alpha \in On}$ *is the Jensen splitting of constructible sets into a hierarchy.*

(b) J_α *is projectible iff there is a* 1-1 $\Sigma_1^{J_\alpha}$ *function on* J_α *into some* $x \in J_\alpha$.

The following is well known although no proof of it has appeared.

THEOREM 1 (Kripke). *The following conditions are equivalent for admissible* J_α $(\alpha > \omega)$:

(a) J_α *is non-projectible,*

(b) J_α *satisfies the* Σ_1*-separation scheme,*

(c) J_α *possesses a cofinal tower of* Σ_1*-elementary transitive subsystems.*

Using the following proposition of Artigue, Isambert, Perrin and Zalc.

PROPOSITION. *Theories* Σ_2^1-CA *and* KP$+\Sigma_1$*-separation* + V = HC *are bicommutable by means of well-founded trees and restrictions to* $\wp(\omega)$.

We find that

THEOREM 2. *If* $J_\alpha \models \mathrm{V} = \mathrm{HC}$, *then the following conditions are equivalent:*

(a) $J_\alpha \cap \wp(\omega) \models \Sigma_2^1$-CA,

(b) J_α *is non-projectible,*

(c) J_α *satisfies the* Σ_1*-separation scheme,*

(d) J_α *possesses a cofinal tower of* Σ_1*-elementary transitive subsystems.*

We notice that (d) implies that $J_\alpha \cap \wp(\omega)$ is a β-model. As a corollary we find that

If $J_\alpha \cap \wp(\omega) \models \Sigma_2^1$-CA, then $J_\alpha \cap \wp(\omega)$ possesses a cofinal tower of Σ_2^1-elementary subsystems (each satisfying thus Δ_2^1-CA).

Using another proposition of Artigue, Isambert, Perrin and Zalc, namely

PROPOSITION. *Theories* Δ_2^1-CA *and* KP+"*Mostowski Contraction lemma*" +V = HC *are bicommutable.*

We get

THEOREM 3. *If* $J_\alpha \models V = HC$, *then the following conditions are equivalent*

(a) J_α *is recursively inaccessible*,

(b) $J_\sigma \cap \wp(\omega) \models \Delta_2^1$-CA.

Again (a) implies that $J_\alpha \cap \wp(\omega)$ is then a β-model. Theorem 2 generalizes to n bigger than 2 as follows:

THEOREM 4. *If* $J_\alpha \models V = HC$, *then the following conditions are equivalent.*

(a) $J_\alpha \cap \wp(\omega) = \Sigma_{n+1}^1$-CA,

(b) J_α *is nonprojectible by means of a* Σ_n-*function*,

(c) $J_\alpha \models \Sigma_n$-*separation scheme*,

(d) J_α *possesses a cofinal tower of transitive* Σ_n-*elementary subsystems.*

Clearly if (d) then $J_\alpha \cap \wp(\omega)$ is a β-model.

"In limit" this is nothing else but a version of the "gap theorem"

THEOREM 5 ([2]): *If* $J_\alpha \models V = HC$, *then the following conditions are equivalent*:

(a) J_α *is a gap* (*i.e.* $(J_{\alpha+1} - J_\alpha) \cap \wp(\omega) = \emptyset$,

(b) $J_\alpha \cap \wp(\omega)$ *is a model of* CA,

(c) J_α *is a model of full replacement.*

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Barely Baire spaces

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Abstract. We give new examples of Baire spaces whose products are not Baire. In particular, we construct a Baire X with X^2 nowhere Baire, and, for each $\varkappa$, a family of spaces $\{X_\beta\colon \beta \leqslant \varkappa\}$ such that $\prod\{X_\beta\colon \beta < \varkappa\}$ is nowhere Baire but for all $\gamma < \varkappa$, $\prod\{X_\beta\colon \beta < \varkappa, \beta \neq \gamma\}$ is Baire. We indicate the relation of our technique to the forcing technique of P. E. Cohen.

Introduction. If the reader will bear with us, we will bare the facts about barely Baire spaces (1).

The Baire Category Theorem, that in complete metric spaces the intersection of countably many dense open sets is dense, is of fundamental importance in analysis ([5] and [21]). Following Bourbaki, we call a space in which the intersection of countably many dense open sets is dense a Baire space. That a compact Hausdorff space is Baire plays a key role in the Rasiowa–Sikorski proof of Gödel's Completeness Theorem [20]. It also motivates the form of Martin's Axiom most accessible to the nonlogician. In general topology that various other types of spaces are Baire is important ([7] and [24]).

Because of the usefulness of Baire spaces, it is natural to ask about the closure properties of the class of Baire spaces. For example, locally Baire spaces, images of Baire spaces under open maps, and dense G_δ's of Baire spaces are Baire. Images of Baire spaces under closed maps, or arbitrary G_δ's of Baire spaces need not be Baire. The more stubborn question of whether the product of Baire spaces or metric Baire spaces is Baire has been raised in [5], [16] and [22].

We call a Baire space X *barely Baire* if there is a Baire space Y such that $X \times Y$ is not Baire. We call a space X *nowhere Baire* if there is a family $\mathscr{D} = \{D_i\colon i \in \omega\}$ of dense open sets so that $\bigcap \mathscr{D}$ is empty.

Oxtoby [16] showed that the continuum hypothesis implies that there is a barely Baire space. More recently, P. E. Cohen improved this to an absolute result. Cohen's

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(1) The title was suggested by Judy Roitman; Eric van Douwen informs us that he had already used the phrase in a different sense.