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Ambiguity and stratification

by

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Abstract. E. Specker has proved that simple type theory with additional axioms expressing typical ambiguity is consistent iff Quine's "New-Foundations" is. His proof is essentially model-theoretic. In this paper, the same result is established using proof theory. It is also shown that there is a recursive procedure that transforms a proof of a stratified formula in a proof in which all formulas are stratified.

1. Let ST denote simple type theory with, as additional axioms, all sentences of the form:

$$A \leftrightarrow A^1, \tag{1.1}$$

where A^1 is obtained from A by raising all types by 1. Specker [2] has proved that ST is consistent iff Quine's NF is. Specker's proof is model-theoretic. The same result will be obtained, here, using proof theory.

Moreover, it is provable that:

(r. p.) there is a recursive procedure for transforming a cut-free derivation $\mathscr{A}$ of a stratified Theorem A of NF (or of a theory all of whose axioms are stratified) into a derivation $\mathscr{B}$, such that

1. $\mathscr{B}$ is a derivation of A, all of whose formulas are stratified;

2. $\mathscr{A}$ and $\mathscr{B}$ are equivalent in the sense that, removing the cuts from $\mathscr{A}$ and $\mathscr{B}$ in the usual way ([1]), one obtains essentially the same derivation.

In fact, the proof of Theorem 2 (below) gives rise to a recursive procedure for obtaining from a cut-free derivation $\mathscr{A}$ of a stratified theorem of the predicate calculus, a derivation $\mathscr{B}$ in type theory with the additional rule:

$$\frac{A}{A^*}. \tag{$*$}$$

In $(*)$ it is understood that A is a theorem and that A^* is as A except, loosely speaking, for the type indices. The details of the proof of (r. p.), although they are clumsy, do not, however, involve any significant difficulties. For this reason, the proof will not be given.

The rule (∗) can be justified as follows. "Forgetting" the types in $\mathscr{B}$, one obtains a stratified derivation. And conversely, it is clear that a stratified derivation can always be converted into a derivation of type theory with (∗).

The rule (∗) seems to express more adequately the idea of typical ambiguity than the schema (1.1). We note, however that the restricted form of (∗):

(1) $$\frac{A}{A^1},$$

already holds in type theory. The rule (∗) of course does not, since, as is easily seen, it entails (1.1). Nevertheless, (∗) is not stronger than the corresponding schema (for any sentence A):

(∗2) $$A \leftrightarrow A^*.$$

Therefore, in order to derive Specker's result from Theorem 2, it is sufficient to show that (1.1) entails (∗2). This will be done in Theorem 1.

2. Preliminary notions. We assign to each n-ary predicate of a first-order language $\mathscr{L}$, a type, that is, an n-tuple $(k_1, \ldots, k_n)$ of natural numbers. This enables us to state the concept of *stratification* for $\mathscr{L}$ as follows:

A formula is *weakly stratified* (*stratified*) if there is an assignment of natural numbers to the occurrences of the variables in that formula, such that the requirements 1, 2 (1, 2, 3) are satisfied:

1. for each atomic sub-formula $R(x_1 \ldots, x_n)$, where R has type $(k_1, \ldots, k_n)$, there is a natural number p such that the number assigned to x_i is $k_i + p$ $(1 \leqslant i \leqslant n)$;

2. the same number is assigned to all the occurrences of a variable bound by the same quantifier;

3. the same number is assigned to each free occurrence of the same variable.

EXAMPLES. Let (1, 2) be the type of ε. Then, $x \in x$ is weakly stratified; $\exists x(x \in x)$ is not weakly stratified.

Next, for any first-order language $\mathscr{L}$, we define an associated language $\mathscr{L}'$, called the typed language (of $\mathscr{L}$), as follows:

1. for each variable x of $\mathscr{L}$ and each natural number i, x^i is a variable of $\mathscr{L}'$;

2. for each n-ary predicate R of $\mathscr{L}$ of type $(k_1, \ldots, k_n)$ and each natural number p, $R(x_1^{k_1+p}, \ldots, x_n^{k_n+p})$ is an atomic formula of $\mathscr{L}'$;

3. if A and B are formulas of $\mathscr{L}'$, so are $A \to B$, $A \wedge B$, $A \vee B$;

4. if A is a formula of $\mathscr{L}'$, so is $\neg A$;

5. if A is a formula of $\mathscr{L}$, so are $\forall x^i A$ and $\exists x^i A$.

Let A be a weakly stratified (stratified) formula of $\mathscr{L}$. Then a formula A' of $\mathscr{L}'$ will be called a *typed version* (*stratified typed version*) of A, if A' is obtained from A by substituting (for each variable x) x^i for all occurrences of x to which the number i is assigned in such a way that A be weakly stratified (stratified).

EXAMPLES. $x^1 \in y^2 \wedge y^1 \in z^2$ and $x^1 \in x^2$ are typed versions of $x \in y \wedge y \in z$ and $x \in x$, respectively. $x^1 \in y^2 \wedge y^2 \in z^3$ is a stratified typed version of $x \in y \wedge y \in z$.

Let A be a formula of the typed language $\mathscr{L}'$. A formula B will be called an *ambiguous variant* (*weakly ambiguous variant*) of A, if there is a stratified (weakly stratified) formula C of $\mathscr{L}$ such that A and B are two stratified typed (typed) versions of C. An occurrence of a variable x in C will be said to be *shifting for type i* (*non-shifting for type i*) if x^i is substituted for the occurrence of x in order to obtain A, and if x^{i+1} (x^i) is substituted for the occurrence in order to obtain B.

If B is a weakly ambiguous variant of A which results from A by merely substituting x^{i+1} for some occurrences of x^i, we call B a p.s.v. (*partially shifting variant*) of A.

EXAMPLES. $\forall x^2 \exists y^3 \forall z^1 \exists v^2 (x^2 \in y^3 \to z^1 \in v^2)$ is an ambiguous variant of $\forall x^1 \exists y^2 \forall z^2 \exists v^3 (x^1 \in y^2 \to z^2 \in v^3)$. $x^1 \in x^2 \to x^2 \in x^3$ is a p.s.v. of $x^1 \in x^2 \to x^1 \in x^2$.

Let T be a first-order theory over $\mathscr{L}$ all of whose non-logical axioms are stratified. The *typed version* of T will be the theory T' obtained from T by taking as logical axioms the logical axioms of type theory, and as non-logical axioms the stratified typed versions of the non-logical axioms of T.

Remark. It is to be noted here, that the equality axioms will be considered as non-logical ones. The type of = is assumed to be (1,1). The following axioms for equality are chosen:

$$\forall x (x = x);$$

$$\forall x_1 \ldots x_n y_1 \ldots y_n (x_1 = y_1 \wedge \ldots \wedge x_n = y_n \to R(x_1, \ldots, x_n) \leftrightarrow R(y_1, \ldots, y_n)).$$

All these axioms are stratified.

The *ambiguous typed version* of T is obtained by adding to T' the schema:

(∗2) $$A \leftrightarrow A^*,$$

where A is a stratified typed version of a sentence of $\mathscr{L}$ and A^* is an ambiguous variant of A.

3. Shifting types and ambiguous types. In what follows, let PC denote the predicate calculus (without equality); furthermore, let TT and AT denote the typed version of PC and the ambiguous typed version of PC, respectively.

If A is a formula of the typed language $\mathscr{L}'$, and p an (positive or negative) integer, then A^p will denote the formula obtained from A by "raising all types by p", that is by replacing in A each variable x^i by x^{i+p} (it is understood that, if p is negative, $-p \leqslant i$ for each i occurring in A).

Before stating the first theorem, we define an *elementary formula of* $\mathscr{L}$:

1. an atomic formula is elementary;
2. if A is elementary, then $\neg A$ is elementary;
3. if A and B are elementary, and if there is a variable occurring free in both A and B, then $A \wedge B$, $A \vee B$ and $A \to B$ are elementary;

4. if A is elementary and if x^i occurs free in A, then $\forall x^i A$ and $\exists x^i A$ are elementary.

LEMMA. *If A is a stratified typed version of a formula of $\mathscr{L}$, then A is elementary iff for every ambiguous variant B of A there is an (positive or negative) integer such that B is A^p.*

Since the proof of this lemma is easy, it is omitted.

THEOREM 1. AT *is equivalent to* TT *which have been added as axioms the sentences of the form:*

$$A \leftrightarrow A^1, \tag{1.1}$$

where A is elementary.

Proof. 1. Needless to say, if A is a sentence, then $A \leftrightarrow A^1$ is provable in AT.

2. It is provable by induction that every formula is equivalent to a conjunctions (and also to a disjunction of conjunctions) of elementary formulas.

Now, let A be a stratified typed version of a sentence of $\mathscr{L}$ and A^* an ambiguous variant of A. Then A is equivalent to a conjunction of disjunctions of elementary stratified typed versions of sentences of $\mathscr{L}$: $\bigwedge\!\!\bigwedge_{1 \leqslant i \leqslant n} \bigvee\!\!\bigvee_{1 \leqslant i \leqslant k_i} A_{ij}$. Furthermore, A^* is equivalent to $\bigwedge\!\!\bigwedge_{1 \leqslant i \leqslant n} \bigvee\!\!\bigvee_{1 \leqslant i \leqslant k_i} A^*_{ij}$, where A^*_{ij} is an ambiguous variant of A_{ij}. Hence by the above lemma, A^*_{ij} is $A^{p_{ij}}_{ij}$ for some p_{ij}. Therefore, $A \leftrightarrow A^*$ is provable within our theory.

4. Ambiguous types and stratification. From now on, we use Gentzen's L formulation [1] of PC and TT. The initial sequents will be those of the form $A \vdash A$, where A is an atomic, instead, as in [1], of an arbitrary formula. The introduction rule for the antecedent (succedent) of the connective c is denoted by $cA(cC)$.

Notions such is stratification, typed version etc..., defined for formulas, can be extended to sequents if one associates to a sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ the formula $A_1 \wedge \ldots \wedge A_n \to B_1 \vee \ldots \vee B_m$. It is easily shown that one can replace the scheme (*2) of AT by the rule $\dfrac{A}{A^*}$, where A is a formula and A^* is an ambiguous variant of A. So, by addition to a Gentzen-type formulation of TT, of the rule

$$\frac{\Gamma \vdash \Delta}{\Gamma^* \vdash \Delta^*}, \tag{*}$$

where $\Gamma^* \vdash \Delta^*$ is an ambiguous variant of $\Gamma \vdash \Delta$, we obtain a Gentzen-type formulation of AT.

LEMMA 1. *Let E'' be a* p.s.v. *of E' and let $\Gamma, E' \vdash \Delta$ and $\vdash E''$ be provable in* TT. *Let x be a variable and k a natural number such that no x^j $(j \neq k)$ occur free in Γ, Δ. Then, there exist formulas F', F'' such that:*

1. *F'' is a* p.s.v. *of F',*
2. *no x^j $(j \neq k)$ occur free in F',*
3. *$\Gamma, F' \vdash \Delta$ and $\vdash F''$ are provable in* TT.

Proof. We use induction on the number n of the j's $(j \neq k)$, such that x^j does not occur free in E'.

If $n = 0$, there is nothing to prove.

If $n = p+1$, there are two cases.

Case 1. There is a j $(j > k)$ such that x^j occurs free in E'. Let r be the greatest such j. Furthermore, let E be the weakly stratified formula whose E' and E'' are typed variants. We suppose that there are both shifting and non-shifting free occurrences of x for type r in E (all other cases are simplifications of this one).

Now, let v and w be two variables not occurring in E, and let D be the result of the substitution of w for all shifting free occurrences of x for type r, and of v for all non-shifting free occurrences of x for type r.

(We shall denote by $A(y_1, \ldots, y_n/x_1 \ldots x_n)$ the formula obtained from A by the simultaneous substitution of $y_1, \ldots, y_n$ for $x_1, \ldots, x_n$ in A. We suppose such substitution to be carried out subject to the habitual restrictions for the avoidance of the clash of bound and free variables.)

Let D' and D'' be two typed variants of D such that D' $(x^r x^r/w^r v^r)$ is E' and D'' $(x^{r+1} x^r/w^{r+1} v^r)$ is E''. Since r is the greatest j $(j \neq k)$ such that x^j occurs free in E', x^{r+1} does not occur free in E'. So, one obtains:

$$\dfrac{\dfrac{\Gamma, E' \vdash \Delta}{\Gamma, \forall w^r D'(x^r/v^r) \vdash \Delta}}{\Gamma, \exists v^r \forall w^r D' \vdash \Delta} \quad \text{and} \quad \dfrac{\dfrac{\vdash E''}{\vdash \forall w^{r+1} D''(x^r/v^r)}}{\vdash \exists v^r \forall w^{r+1} D''}.$$

The formula $\exists v^r \forall w^r D'$ contains less than n free variables x^j $(j \neq k)$, and it is clear that $\exists v^r \forall w^{r+1} D''$ is a p.s.v. of $\exists v^r \forall w^r D'$. We are, therefore, able to apply the inductive hypothesis.

Case 2. No x^j $(j > k)$ occur free in E'. Let r be the smallest of the j's such that x^j is free in E'. Assume, furthermore, the same situation as in the first case, and define D, D' and D'' as above. We, then, have the following:

$$\dfrac{\dfrac{\Gamma, E' \vdash \Delta}{\Gamma, \forall v^r D'(x^r/w^r) \vdash \Delta}}{\Gamma, \exists w^r \forall v^r D' \vdash \Delta} \quad \text{and} \quad \dfrac{\dfrac{\vdash E''}{\vdash \forall v^r D''(x^{r+1}/w^{r+1})}}{\vdash \exists w^{r+1} \forall v^r D''}.$$

We, again, apply the inductive hypothesis, and this, then, ends the proof.

LEMMA 2. *Let $\Gamma \vdash \Delta$ be a weakly stratified provable sequent of* PC, *and let $\Gamma' \vdash \Delta'$ be a typed version of $\Gamma \vdash \Delta$. Then, $\Gamma' \vdash \Delta'$ is provable in* TT, *or there are two formulas F' and F'' such that F'' is a* p.s.v. *of F', and such that $\Gamma', F' \vdash \Delta'$ and $\vdash F''$ are provable sequents of* TT.

Proof. Let $\mathscr{A}$ be a *cut-free* derivation of $\Gamma \vdash \Delta$. We prove the lemma by induction on the length of $\mathscr{A}$.

Case 1. $\mathscr{A}$ is an initial sequent: $R(x_1, \dots, x_n) \vdash R(x_1, \dots, x_n)$. Let $\Gamma' \vdash \Delta'$ be $R(x_1^{r_1}, \dots, x_n^{r_n}) \vdash R(x_1^{s_1}, \dots, x_n^{s_n})$.

If $r_i = s_i$ $(1 \leqslant i \leqslant n)$, then $\Gamma' \vdash \Delta'$ is already provable in TT. If $r_i < s_i$, one sets F' identical to

$$R(x_1^{r_1}, \dots, x_n^{r_n}) \to R(x_1^{r_1+1}, \dots, x_n^{r_n+1}) \wedge \dots \wedge R(x_1^{s_1-1}, \dots, x_n^{s_n-1}) \to R(x_1^{s_1}, \dots, x_n^{s_n}),$$

and F'' identical to

$$R(x_1^{r_1+1}, \dots, x_n^{r_n+1}) \to R(x_1^{r_1+1}, \dots, x_n^{r_n+1}) \wedge \dots \wedge R(x_1^{s_1}, \dots, x_n^{s_n}) \to R(x_1^{s_1}, \dots, x_n^{s_n}).$$

If $s_i < r_i$, one puts F' identical to

$$R(x_1^{r_1}, \dots, x_n^{r_n}) \to R(x_1^{r_1-1}, \dots, x_n^{r_n-1}) \wedge \dots \wedge R(x_1^{s_1+1}, \dots, x_n^{s_n+1}) \to R(x_1^{s_1}, \dots, x_n^{s_n}),$$

and F'' identical to

$$R(x_1^{r_1}, \dots, x_n^{r_n}) \to R(x_1^{r_1}, \dots, x_n^{r_n}) \wedge \dots \wedge R(x_1^{s_1+1}, \dots, x_n^{s_n+1}) \to R(x_1^{s_1+1}, \dots, x_n^{s_n+1}).$$

Case 2. The last inference of $\mathscr{A}$ is allowed by $\wedge A$, $\wedge C$, $\vee A$, $\vee C$, $\to A$, $\to C$, $\neg A$, $\neg C$, $\exists C$, $\forall A$ or a structural rule.

Take, for example, $\to A$. That is, $\mathscr{A}$ ends in:

$$\frac{\Sigma \vdash A, \Delta \qquad \Sigma, B \vdash \Delta}{\Sigma, A \to B \vdash \Delta}.$$

Since Σ, $A \to B \vdash \Delta$ is weakly stratified, so are $\Sigma \vdash A$, Δ and Σ, $B \vdash \Delta$. $\Sigma' \vdash A', \Delta'$ and $\Sigma', B' \vdash \Delta'$ are typed versions of $\Sigma \vdash A, \Delta$ and $\Sigma, B \vdash \Delta$, respectively. Suppose that neither $\Sigma' \vdash A', \Delta'$ nor $\Sigma', B' \vdash \Delta'$ are provable in TT (the other cases are immediate). Then, there are formulas E', E'', D', D'' such that $\Sigma', E' \vdash A', \Delta'$; $\Sigma', B', D' \vdash \Delta'$; $\vdash E''$ and $\vdash D''$ are provable in TT. Now, we have only to let F' be $E' \wedge D'$ and F'' be $E'' \wedge D''$, and we are done.

Case 3. The last inference of $\mathscr{A}$ is an instance of $\forall C$ or $\exists A$. Let us take $\forall C$, for example. $\mathscr{A}$ ends in:

$$\frac{\Gamma \vdash A, \Pi}{\Gamma \vdash \forall y A(y/x), \Pi}.$$

$\Gamma' \vdash A', \Pi'$ is a typed version of $\Gamma \vdash A, \Pi$. If x does not occur free in A, or if $\Gamma' \vdash A', \Pi'$ is already provable in TT, then, the result is obvious. Otherwise, we know that there are two formulas E' and E'' such that E'' is a p.s.v. of E' and that $\Gamma', E' \vdash A', \Pi''$ and $\vdash E''$ are provable in TT. Let k be the type given to x in A'. By the restriction on $\forall C$, no x^j $(j \neq k)$ occur free in Γ', Π', A'. Hence, by Lemma 1, we may suppose that no x^j $(j \neq k)$ occur free in E'.

If there are shifting and non-shifting free occurrences of x for type k in E (E as in Lemma 1), then we define D, D', D'' as in Lemma 1. We have, therefore,

$$\cfrac{\cfrac{\Gamma', E' \vdash A', \Pi'}{\Gamma', \forall w^k D'(x^k/v^k) \vdash A', \Pi'}}{\Gamma', \forall v^k \forall w^k D' \vdash A', \Pi'} \quad \text{and} \quad \cfrac{\cfrac{\vdash E''}{\vdash \forall w^{k+1} D''(x^k/v^k)}}{\vdash \forall v^k \forall w^{k+1} D''}.$$

We now put F' identical to $\forall v^k \forall w^k D'$, and F'' identical to $\forall v^k \forall w^{k+1} D''$. Other cases are treated in a similar fashion.

THEOREM 2. *If $\Gamma \vdash \Delta$ is a stratified sequent provable in* PC *and if $\Gamma' \vdash \Delta'$ is a stratified typed version of $\Gamma \vdash \Delta$, then $\Gamma' \vdash \Delta'$ is provable in* AT.

Proof. Suppose that $\Gamma' \vdash \Delta'$ is not already provable in TT. Then, by Lemma 2 there are formulas F' and F'' such that F'' is a p.s.v. of F' and $\Gamma', F' \vdash \Delta'$ and $\vdash F''$ are provable in TT. Since $\Gamma' \vdash \Delta'$ is a stratified typed version, one can (by Lemma 1) suppose that F'' is an ambiguous variant of F'. So, using the rule (*) and the cut-rule, one obtains:

$$\frac{\Gamma', F' \vdash \Delta' \qquad \cfrac{\vdash F''}{\vdash F'}}{\Gamma' \vdash \Delta'}.$$

COROLLARY 1. *Let T be a theory all of whose non-logical axioms are stratified, and let T' be the ambiguous typed version of T. Then, if A is a stratified formula provable in T and if A' is a stratified typed version of A, it follows that A' is provable in T'.*

Proof. Since A is provable in T, there are stratified closed formulas $B_1, \dots, B_n$ such that $B_1, \dots, B_n \vdash A$ is provable in PC. Therefore, $B_1', \dots, B_n' \vdash A'$ is provable in AT (Theorem 2) for any stratified typed versions $B_1', \dots, B_n'$ of $B_1, \dots, B_n$. That is, A' is provable in T'.

COROLLARY 2 (Specker). NF *is consistent iff* ST *is.*

Proof. Simple type theory with the schema (*2) is the ambiguous typed version of NF. So, by Corollary 1 and Theorem 1, if ST is consistent, then NF is consistent.

The converse is obvious since, by "forgetting" the types, every derivation in ST gives rise to a derivation in NF.

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