Some remarks on bicommutability

by

M. Artigue, E. Isambert, M. J. Perrin and **A. Zalc** (Paris)

Abstract. Given two languages L and $\mathfrak{L}$ and interpretations from L to $\mathfrak{L}$ and from $\mathfrak{L}$ to L, we define a strong equivalence relation between theories of L and $\mathfrak{L}$: Bicommutability. This definition generalizes the well-known relationship between A_2 and $ZFC^- + V = HC$.

We first give some examples and general properties of that notion, and then apply it to the case of set theory and second order arithmetic with classical interpretations. We give examples of pairs of weak subsystems of set theory and analysis which are bicommutable, and exhibit one of minimal strength. We show that the theory KP of admissible sets has no bicommutable equivalent in second order arithmetic.

Then we try to obtain the same type of results for set theory and third order arithmetic, and show that, in that case, the answer depends on the choosen interpretations ("trees" or "graphs"). This fact is closely related to the satisfaction of different weak forms of the axiom of choice.

We define here the notion of a bicommutable pair of theories which generalizes the known relationship between A_2 and $ZFC^- + V = HC$. We make some general remarks on this relation and then investigate the subsystems of second order arithmetic and set theory which bicommute. We exhibit a pair of those which is "minimal" for the notion of bicommutability. Finally, we examine the case of third order arithmetic.

I. Definitions and generalities

A. Definitions. Let L and $\mathscr{L}$ be two languages (the equality symbols being considered as non logical ones), T a theory in L, $\mathscr{T}$ a theory in $\mathscr{L}$, each of these containing the axioms for equality, such that T is relatively interpretable in $\mathscr{T}$ and $\mathscr{T}$ in T, by fixed interpretations.

With a structure M for L (resp. $\mathscr{M}$ for $\mathscr{L}$) we may therefore associate M° (resp. $\mathscr{M}^+$), the corresponding structure in $\mathscr{L}$ (resp. in L). We shall denote by φ° the translation in L of the formula φ of $\mathscr{L}$ (resp. Φ^+ the translation in $\mathscr{L}$ of the formula Φ in L).

We recall that we have then:

For every model M of T, (resp. $\mathscr{M}$ of $\mathscr{L}$) $M^\circ \models \mathscr{T}$ (resp. $\mathscr{M}^+ \models T$).

Definitions. Let T and $\mathscr{T}$ be two theories which are mutually relatively interpretable by the interpretations $(^\circ, {}^+)$,

(i) T and $\mathcal{T}$ are *bicommutable* for these interpretations iff, for every model M of T (resp. $\mathcal{M}$ of $\mathcal{T}$) there exists an isomorphism j from M onto $M^{\circ +}$ definable in M (resp. k from $\mathcal{M}$ onto $\mathcal{M}^{+\circ}$ definable in $\mathcal{M}$).

Note that isomorphism is understood here by respect to the equality of the languages.

(ii) T and $\mathcal{T}$ are *weakly bicommutable* for ($^\circ$, $^+$) iff, for every model M of T and every model $\mathcal{M}$ of $\mathcal{T}$:

$$M^{\circ +} \equiv M \quad \text{and} \quad \mathcal{M}^{+\circ} \equiv \mathcal{M}.$$

(iii) According to Montague (1), $\mathcal{T}$ and T are bilaterally interpretable for ($^\circ$, $^+$) iff, for every formula Φ of L (resp. φ of $\mathcal{L}$):

$T \vdash \Phi \leftrightarrow \Phi^{+\circ}$ and

$\mathcal{T} \vdash \varphi \leftrightarrow \varphi^{\circ +}$ and

T (resp. $\mathcal{T}$) is a conservative extension of $\mathcal{T}$ (resp. T).

We shall say that T and $\mathcal{T}$ are *bicommutable* (resp. *weakly bicommutable*), if and only if there exist interpretations such that T and $\mathcal{T}$ satisfy (i) (resp. (ii)).

B. Remarks.

Bicommutability is an equivalence relation.

If T and $\mathcal{T}$ bicommute, then they obviously weakly bicommute.

Weak bicommutability is equivalent to bilateral interpretability.

If T and $\mathcal{T}$ bicommute by ($^\circ$, $^+$), let $\Phi(x_1, \ldots, x_n)$ be a formula of L with $x_1, \ldots, x_n$ as free variables. Then, $T \vdash \Phi(x_1, \ldots, x_n) \leftrightarrow \Phi^{+\circ}(j(x_1), \ldots, j(x_n))$.

Suppose we have two pairs of interpretations between L and $\mathcal{L}$ say ($^\circ$, $^+$) and ($'$, *) such that for each of them T and $\mathcal{T}$ bicommute. If for every model M of T we have $M^\circ \approx M'$, then for every model $\mathcal{M}$ of $\mathcal{T}$ we have $\mathcal{M}^+ \approx \mathcal{M}^*$.

The same holds for T and $\mathcal{T}$ weakly bicommutable, replacing isomorphism by elementary equivalence.

C. Examples.

EXAMPLE 1. Let L be the language of first order arithmetic: $(\{0, S, +, \cdot, <, =\})$, $\mathcal{L}$ the language of set theory: $(\{\in, =\})$, T Peano's arithmetic and $\mathcal{T}$ SF+V = HF.

SF is ZF without the axiom of infinity and V = HF is the following axiom:

$\forall x \exists n \exists f$ ["n is a finite ordinal" $\wedge$ "f is a one-one mapping from x onto n"].

Then $\mathcal{T}$ and T bicommute for the obvious interpretations.

EXAMPLE 2. Let L be the vector space's equalitary language with two similarity types denoted by ($^\circ$) and (1) and interpreted respectively by scalars and vectors.

$$L = \{^\circ, 1, +, \cdot, 0, \oplus, \times\}.$$

(1) According to S. Feferman, this definition has been given by Montague in unpublished notes.

Let $\mathcal{L}$ be the affine space's language with three similarity types denoted by ($^\circ$), (1), (2) and interpreted respectively by scalars, vectors and points.

$$\mathcal{L} = \{^\circ, 1, +, \cdot, 0, \oplus, \times, *\}.$$

$*$ is a function symbol with two arguments, the first of type (2), the second of type (1).

T is the vector space's theory and $\mathcal{T}$ is the affine space's theory obtained by adding to T the following axioms:

$\forall X\, X * 0 = X$,

$\forall X \forall v \forall w\, X * (v \oplus w) = (X * v) * w$,

$\forall X \forall Y \exists! v\, X * v = Y$.

T and $\mathcal{T}$ bicommute by the obvious interpretations.

EXAMPLE 3. Let T be a first order theory written in the language L with equality, P a n-place predicate symbol which does not belong to L. Let $\mathcal{L} = L \cup \{P\}$. Let Φ be a formula of L with n free variables.

Then T and $\mathcal{T} = T \cup \{\forall x_1 \ldots \forall x_m (P(x_1, \ldots, x_n) \leftrightarrow \Phi(x_1, \ldots, x_n))\}$ bicommute. For example, taking as before T to be the vector space's theory and P a two-places predicate symbol and Φ (v, w) the formula: $\exists x [x \neq 0 \wedge v = x \times w]$, then $\mathcal{T}$ describes the projective spaces.

EXAMPLE 4. Let $L = \mathcal{L}$ be the language $\{<, a_0, \ldots, a_n, \ldots, b_0, \ldots, b_n, \ldots\}$. T be the theory of dense linear orderings without endpoints with the following ordering of the constants of L: $a_0 < a_1 < \ldots < a_n < \ldots < b_n < \ldots < b_0$.

$\mathcal{T}$ be the theory of linear dense orderings with endpoints, a_0 and b_0 being the endpoints, and the same ordering for the constants of the language.

The interpretations are described as follows:

Start with M a model of T, cut its domain to the interval $[a_0^M, b_0^M]$ to get a model M° of $\mathcal{T}$; a_0^M and b_0^M being the interpretations of a_0 and b_0 in M.

Conversely, given $\mathcal{M}$ a model of $\mathcal{T}$, to get $\mathcal{M}^+$, cut the domain down by taking off the interpretations of a_0 and b_0 in $\mathcal{M}$. Then interpret a_0 in $\mathcal{M}^+$ to be $a_1^{\mathcal{M}}$, a_1 to be $a_2^{\mathcal{M}}, \ldots, a_n$ to be $a_{n+1}^{\mathcal{M}}$ and do the same with the b_i's, the ordering remaining the same.

T and $\mathcal{T}$, with these interpretations weakly bicommute and one can easily see, using the fact that these two theories admit the elimination of quantifiers, that no interpretation can make them bicommute (the isomorphism between M and $M^{\circ +}$ cannot be definable in M).

We obtain as a consequence of this last result the fact that the notion of bicommutability is strictly stronger than that of weak bicommutability.

II. Bicommutable subsystems of analysis and set theory

From now on we shall consider the following languages:

L is the language of second order arithmetic with two similarity types, and the non logical symbols: $0, S, +, \cdot, <, \in, =$

$\mathscr{L}$ is the language of set theory with the non logical symbols: $\in$, $=$

We shall work also in fixed intèrpretations described as follows:

A. Interpretation of L in $\mathscr{L}$. We define Φ^+, the translation of a formula Φ in L, by induction on the complexity of Φ. For Φ atomic, the constant 0, the predicates $y = Sx$, $z = x+y$, $z = x\cdot y$ and $x<y$ will be respectively interpreted by 0 and the usual relations between finite ordinals. The membership relation and the equality will be interpreted as in $\mathscr{L}$. Then:

For $\Phi = \neg\Phi_1$, $\quad \Phi^+ = \neg\Phi_1^+$.

For $\Phi = \Phi_1 \vee \Phi_2$, $\quad \Phi^+ = \Phi_1^+ \vee \Phi_2^+$.

For $\Phi = \forall x \Psi$, $\quad \Phi^+ = \forall x(x \in \omega \to \Psi^+)$.

For $\Phi = \forall X \Psi$, $\quad \Phi^+ = \forall x(x \subset \omega \to \Psi^+)$.

$\mathscr{M}^+$ is then $\langle \omega^{\mathscr{M}}, \wp(\omega)^{\mathscr{M}}, 0, S, +, \cdot, <, \in, = \rangle$.

B. Interpretation of $\mathscr{L}$ in L (cf. [17]). The subsystems of analysis that we shall deal with, will always allow us to define a pairing function on integers denoted by: $x, y \mapsto \langle x, y\rangle$. Therefore a set of integers can be seen as a binary relation on integers.

DEFINITIONS. X being a set of integers, we define:

$$\mathrm{Fld}(X)\colon \{x/\exists y(\langle x, y\rangle \in X \vee \langle y, x\rangle \in x)\},$$

$$\begin{aligned}\mathrm{Gr}(X)\colon\ & \forall x \forall y[\forall z(\langle z, x\rangle \in X \leftrightarrow \langle z, y\rangle \in X) \leftrightarrow x = y] \wedge \\ & \wedge \forall Y[\exists x(x \in Y \wedge Y \subset \mathrm{Fld}(X)) \leftrightarrow \exists x \in Y \forall y \in Y \neg(\langle y, x\rangle \in x)] \wedge \\ & \wedge \exists! t[t \in \mathrm{Fld}(X) \wedge \forall u \in \mathrm{Fld}(X)(\langle t, u\rangle \notin X)] \wedge \\ & \wedge (\forall x \in \mathrm{Fld}(X) - \{t\}) \exists s[\mathrm{Sq}(s) \wedge \forall i < \lg(s) - 1 \langle (s)_i, (s)_{i+1}\rangle \in X \wedge \\ & \qquad \wedge (s)_0 = x \wedge (s)_{\lg(s)-1} = t].\end{aligned}$$

If $\langle x, y\rangle \in X$, we may writ: $x \underset{X}{<} y$. If M is a realization for L and X a set of integers in M such that $M \models \mathrm{Gr}(X)$, we shall say that X is a *graph* of M. Such an X is, in M, a well founded and extensional tree with a maximum element from which one can reach each node in a finite decreasing X-chain.

We shall denote its maximum element by $\mathrm{Max}(X)$.

$\mathrm{Gr}(X)$ will be the predicate of relative interpretability.

$\mathscr{I}(X, Y)$ means: "There exists an isomorphism from the binary relation X onto the binary relation Y".

$\mathscr{I}(X, Y)$ will interpret the equality relation.

$Z = X \upharpoonright n$ will be a notation for: $n \in \mathrm{Fld}(X)$ and Z is the restriction of X to the elements of $\mathrm{Fld}(X)$ which are related to n by a finite increasing chain. Notice that:

$$M \models [\mathrm{Gr}(X) \wedge n \in \mathrm{Fld}(X)] \Rightarrow [\mathrm{Gr}(X \upharpoonright n) \wedge \mathrm{Max}(X \upharpoonright n) = n].$$

$\mathscr{E}(X, Y)$: $\exists Z \exists n \exists y\ [y = \mathrm{Max}(Y) \wedge \langle n, y\rangle \in Y \wedge Z = Y \upharpoonright n \wedge \mathscr{I}(X, Y)]$.

$\mathscr{E}(X, Y)$ will interpret the membership relation.

Therefore we have the following inductive definition of the translation φ° of a formula φ in $\mathscr{L}$:

For φ: $x_1 = x_2$, $\quad \varphi^\circ$: $\mathscr{I}(X_1, X_2)$.

For φ: $x_1 \in x_2$, $\quad \varphi^\circ$: $\mathscr{E}(X_1, X_2)$.

For φ: $\neg\Psi$, $\quad \varphi^\circ$: $\neg\Psi^\circ$.

For φ: $\varphi_1 \vee \varphi_2$, $\quad \varphi^\circ$: $\varphi_1^\circ \vee \varphi_2^\circ$.

For φ: $\forall x \varphi$, $\quad \varphi^\circ$: $\forall X[\mathrm{Gr}(X) \to \Psi^\circ(X)]$.

Then M° is the structure $\langle \{X \mid M \models \mathrm{Gr}(X)\}, \mathscr{E}, \mathscr{I}\rangle$ and we have:

$$M \models \varphi^\circ \leftrightarrow M^\circ \models \varphi.$$

From now on, we shall write "bicommutable" instead of "bicommutable for the interpretations defined just above".

C. Subsystems of second order arithmetic and set theory.

1. Subsystems of second order arithmetic. T_1 will be the theory of L containing the following axioms:

Peano's axioms for first order arithmetic with the induction scheme:

$$\forall X[0 \in X \wedge \forall n(n \in X \to n+1 \in X) \to \forall n(n \in X)].$$

Axiom of extensionality.

Arithmetic comprehension scheme (π_∞^0-CA):

$$\forall x_1 \ldots \forall x_k \forall X_1 \ldots \forall X_p \exists X[\forall n(n \in X) \leftrightarrow \Phi(n, x_1, \ldots, X_p)]$$

where Φ is an arithmetic formula and X is not a free variable in Φ.

Σ_1^1 Bar Induction scheme (Σ_1^1-BI$_0$): for all Σ_1^1-formula Φ

$$\forall X[\text{"}X \text{ has no infinite decreasing chain"} \wedge \\ \wedge \left(\forall x \in \mathrm{Fld}(X)\left[\forall y(\langle y, x\rangle \in X \to \Phi(y)) \to \Phi(x)\right] \to \forall x \in \mathrm{Fld}(X)\,\Phi(x)\right)\right].$$

We shall consider extensions of T_1 by adding one or several schemes among the following:

Scheme of choice (AC$_{01}$):

$$\forall x_1 .. \forall x_k \forall X_1 \ldots \forall X_p[\forall n \exists X \Phi(n, X, x_1, \ldots, X_p) \to \exists X \forall n \Phi(n, X^{(n)}, x_1, \ldots, X_p)]$$

where $X^{(n)} = \{m/\langle n, m\rangle \in X\}$.

Comprehension scheme (CA):

$$\forall x_1 \ldots \forall x_k \forall X_1 \ldots \forall X_p \exists X[\forall n(n \in X) \leftrightarrow \Phi(n, x_1, \ldots, X_p)]$$

where X is not a free variable in Φ.

$\mathfrak{F}$ being a class of formulas, $\mathfrak{F}$-AC$_{01}$ and $\mathfrak{F}$-CA are respectively the restrictions of AC$_{01}$ and CA to the formulas in $\mathfrak{F}$.

A_2^- is the theory $T_1 + \mathrm{CA}$.
A_2 is the theory $A_2^- + \mathrm{AC}_{01}$.

Remarks. If T_1^* is the theory T_1 without Σ_1^1-BI_0, we have:

$$T_1^* + \Sigma_k^1\text{-}\mathrm{AC}_{01} \vdash \Delta_k^1\text{-}\mathrm{CA},$$

$$T_1^* + \mathrm{AC}_{01} \vdash \mathrm{CA},$$

$$T_1^* + \Sigma_1^1\text{-}\mathrm{CA} \vdash \Sigma_1^1\text{-}\mathrm{AC}_{01} \text{ (see [8] and [17])},$$

$$T_1^* + \Sigma_1^1\text{-}\mathrm{CA} \vdash \Sigma_1^1\text{-}\mathrm{BI}_0,$$

$$T_1^* + \Delta_2^1\text{-}\mathrm{CA} \vdash \Sigma_2^1\text{-}\mathrm{AC}_{01} \text{ (see [8] and [17])},$$

$$T_1^* + \Sigma_k^1\text{-}\mathrm{CA} \vdash \pi_k^1\text{-}\mathrm{CA} \text{ and } T_1^* + \pi_k^1\text{-}\mathrm{CA} \vdash \Sigma_k^1\text{-}\mathrm{CA},$$

$$T_1^* \vdash \forall X (\text{“}X \text{ well founded”} \leftrightarrow \text{“}X \text{ has no infinite decreasing chain”}) \text{ (see [8])}.$$

2. Subsystems of set theory. E_1 will be the following theory of $\mathscr{L}$:

Extensionality + Pair + Union + Cartesian Product + Transitive closure + Axiom of infinity + Axiom of foundation + Δ_0-Separation.

For the definitions of these axioms and schemes, see [9] and [17].

We shall consider extensions of E_1 by adding one or several axioms and schemes among:

Collapsing (C):

Every extensional and well founded relation is isomorphic to a transitive set.

V = HC:

For all non empty x, there exists a mapping from ω onto x.

Δ_k-Collection (Δ_k-coll):

$$\forall a \forall a_1 \ldots \forall a_n [\forall x \in a \exists y \varphi(x, y, a_1, \ldots, a_n) \rightarrow \exists b \forall x \in a \exists y \in b \varphi(x, y, a_1, \ldots, a_n)]$$

where φ defines a Δ_k-predicate.

Σ_k-Separation (Σ_k-sep):

$$\forall a \forall a_1 \ldots \forall a_n \exists b \forall x [x \in b \leftrightarrow x \in a \wedge \varphi(x, a_1, \ldots, a_n)]$$

where φ is a Σ_k-formula and b is not a free variable in φ.

Z^- is Zermelo's set theory without the power set axiom.

ZFC^- is Zermelo Fraenkel set theory without power set axiom and with the following scheme of choice:

$$\forall a \forall a_1 \ldots \forall a_n [\forall x \in a \exists y \varphi(x, y, a_1, \ldots, a_n) \rightarrow \exists b \forall x \in a \exists ! y \in b \varphi(x, y, a_1, \ldots, a_n)].$$

Remarks. We have:

$$E_1 + \Delta_k\text{-coll} \vdash \Delta_k\text{-sep},$$

$$E_1 + \Sigma_1\text{-sep} + \Delta_0\text{-coll} \vdash (C) \text{ (see [3])},$$

$$E_1 + V = HC \vdash AC.$$

D. Theorem 1.

Theorem 1. *The following pairs of theories are bicommutable*:

	L	$\mathfrak{L}$
1	A_2	$ZFC^- + V = HC$ (see [8] and [16])
2	A_2^-	$Z^- + V = HC + \Delta_0$-coll (see [10])
3 (1)	$T_1 + \Sigma_2^1$-CA	$E_1 + \Delta_0$-coll $+ \Sigma_1$-sep $+ V = HC$
4	$T_1 + \Sigma_2^1$-AC_{01}	$E_1 + \Delta_0$-coll $+ (C) + V = HC$

We give here proof for 3 and 4.

Lemma II.1. *There exist formulas $I_1(X, Y)$ and $I_2(X, Y)$ respectively π_1^1 and Σ_1^1 such that*:

$$T_1 + \Sigma_1^1\text{-}\mathrm{AC}_{01} \vdash \mathrm{Gr}(Y) \rightarrow (\forall X [(\mathrm{Gr}(X) \wedge \mathscr{I}(X, Y) \leftrightarrow I_1(X, Y)] \wedge \wedge \forall X [\mathrm{Gr}(X) \wedge \mathscr{I}(X, Y) \leftrightarrow I_2(X, Y)]).$$

Proof. We define the formula $E(T, X_1, X_2)$ by:

$$[T \subset \mathrm{Fld}(X_1) \times \mathrm{Fld}(X_2)] \wedge \forall s \in \mathrm{Fld}(X_1) \forall t \in \mathrm{Fld}(X_2) [\langle s, t\rangle \in T \leftrightarrow \text{“}T \cap (\mathrm{Fld}(X_{1 \underset{<}{\restriction} s}) \times \mathrm{Fld}(X_{2 \underset{<}{\restriction} t})) \text{ is an isomorphism of } X_{1 \underset{<}{\restriction} s} \text{ onto } X_{2 \underset{<}{\restriction} t}\text{”}] \wedge [\forall x ! y \langle x, y\rangle \in T].$$

Where $X_{\underset{<}{\restriction} x}$ is the restriction of X to $\mathrm{Fld}(X_{\restriction x}) - \{x\}$.

$E(T, X_1, X_2)$ is an arithmetic formula meaning that T is a maximal isomorphism between a transitive subset of X_1 and a transitive subset of X_2. We claim that:

$$T_1 + \Sigma_1^1\text{-}\mathrm{AC}_{01} \vdash \forall X_1 \forall X_2 [\mathrm{Gr}(X_1) \wedge \mathrm{Gr}(X_2) \rightarrow \exists ! T E(T, X_1, X_2)].$$

The uniqueness is easily proved in T_1, pointing out the fact that, if X is an extensional well founded relation, there is no non-trivial automorphism of X.

The existence of such a T is proved by using Σ_1^1-BI_0 with the formula:

$$\Phi(x) = \exists T \exists X [X = X_{1 \restriction x} \wedge E(T, X, X_2)].$$

(1) W. Marek mentions us the two following corollaries of results 3 and 4:
Let $L_\alpha \models V = HC$ then
(i) α is non projectible $\leftrightarrow L_\alpha \cap \wp(\omega) \models \Sigma_2^1$-CA,
(ii) α is recursively inaccessible $\leftrightarrow L_\alpha \cap \wp(\omega) \models \Delta_2^1$-CA
(see the paper of Marek, same issue).

$I_1(X, Y)$ and $I_2(X, Y)$ will be respectively the formulas:

$$\forall T\exists s\exists t[E(T, X, Y)\to s = \mathrm{Max}(X)\wedge t = \mathrm{Max}(Y)\wedge\langle s, t\rangle\in T],$$

$$\exists T\exists s\exists t[E(T, X, Y)\wedge s = \mathrm{Max}(X)\wedge t = \mathrm{Max}(Y)\wedge\langle s, t\rangle\in T\wedge \wedge\forall y\in\mathrm{Fld}(Y)\exists x\in\mathrm{Fld}(X)(\langle x, y\rangle\in T)].$$

COROLLARY II.1. *Let M be a model of $T_1+\Sigma^1_1$-AC_{01} and φ be a formula of $\mathscr{L}$, then*:

If φ is Δ_0 in M°, then φ° is Δ^1_1 in M.

If φ is Σ_1 in M°, then φ° is Σ^1_2 in M.

LEMMA II.2. *Let $\mathscr{M}$ be a model of $E_1+(\mathrm{C})$ and Φ a formula of L, then*:

If Φ is Σ^1_2 in $\mathscr{M}^+$, Φ^+ is Σ_1 in $\mathscr{M}$.

Proof. As Φ is Σ^1_2 in $\mathscr{M}^+$, there exists a primitive recursive predicate P such that:

$$\mathscr{M}^+\models\Phi(n)\leftrightarrow\exists f\forall g\exists m P(n, f, \overline{g(m)}).$$

Let $T(n, f) = \{s\in\omega/\forall i<\lg(s)\neg P(n, f, \langle(s)_0\dots(s)_i\rangle)\}$ and $t\prec^* s$ denote Church–Kleene linear ordering on finite sequences of integers. One can easily see that in $\mathscr{M}$, Φ^+ is equivalent to:

$$\exists f\exists a\exists\varphi[a \text{ is a transitive set }\wedge\varphi\text{ is an isomorphism from }(a, \in)\text{ onto } (T(n, f), \prec^*)].$$

LEMMA II.3 (Patching lemma). *In a model M of $T_1+\Sigma^1_1$-AC_{01}, let Y be a set of integers such that, for every n, $Y^{(n)}$ is a graph. Then there exists a graph Z satisfying*:

$$\forall n[\langle n, \mathrm{Max}(Z)\rangle\in Z\leftrightarrow\exists m\mathscr{I}(Y^{(m)}, Z_{\restriction n})]\wedge\forall m\exists n[\langle n, \mathrm{Max}(Z)\rangle\in Z\wedge \wedge\mathscr{I}(Y^{(m)}, Z_{\restriction n})].$$

Proof. One can assume that the graphs $Y^{(n)}$ are pairwise disjoint; otherwise one replaces them by: $Y'^{(n)} = Y^{(n)}\times\{n\}$. One then defines a mapping f whose domain is $\bigcup_n Y^{(n)}$ by:

$$f(y) = \mu x[\exists m\exists n(y\in Y^{(n)}\wedge x\in Y^{(m)}\wedge\mathscr{I}(Y^{(n)}_{\restriction y}, Y^{(m)}_{\restriction x}))].$$

According to Lemma II.1, f has a Δ^1_1-definition; so it is a set in M by Δ^1_1-CA (consequence of Σ^1_1-AC_{01}).

The graph Z is defined by:

$$\forall x\forall x'[\langle x, x'\rangle\in Z\leftrightarrow\exists y\exists y'\exists n[f(y) = x\wedge f(y') = x'\wedge\langle y, y'\rangle\in Y^{(n)}]\vee \vee(y = \mathrm{Max}(Y^{(n)})\wedge f(y) = x\wedge x' = a)]$$

where a is an integer not in $\bigcup_n Y^{(n)}$. This definition of Z is π^0_∞, so Z is a set in M. One verifies that it satisfies the required conditions.

COROLLARY II.2.

(a) *If $M\models T_1$, then $M^\circ\models$ Extensionality + Foundation + Union + Infinity.*

(b) *If $M\models T_1+\Sigma^1_1$-AC_{01}, then $M^\circ\models$ Pair + Cartesian Product + Δ_0-sep + (C) + + V = HC.*

(c) *If $M\models T_1+\Sigma^1_2$-AC_{01}, then $M^\circ\models\Delta_0$-coll.*

Proof. (a) is easy to prove using methods similar to those of [16] and [17].

(b) The axioms of pair, cartesian product and V = HC are direct consequences of Lemma II.3.

Δ_0-sep is obvious after Corollary II.1.

Let us show that M° satisfies (C):

Let r be a well-founded extensional relation in M°, therefore a graph of M whose elements are represented by pairs $[X, Y]$ (the relation $Z = [X, Y]$ being defined by:

$$\exists u\exists v\exists t\big|\forall x\forall y(\langle x, y\rangle\in Z\leftrightarrow\langle x, y\rangle\in X\vee\langle x, y\rangle\in Y\vee \vee(x = \mathrm{Max}(X)\wedge y = u)\vee(x = \mathrm{Max}(X)\wedge y = v)\vee \vee(x = \mathrm{Max}(Y)\wedge y = v)\vee(x = u\wedge y = t)\vee(x = v\wedge y = t))\big|,$$

Z is a graph representing (X, Y) of M°).

We associate with r the following relation S' on the integers:

$$\forall x\forall y[\langle x, y\rangle\in S'\leftrightarrow\exists u[\langle u, \mathrm{Max}(r)\rangle\in r\wedge r_{\restriction u} = [r_{\restriction x}, r_{\restriction y}]]].$$

Then we define a graph S in M by:

$$\forall x\forall y[\langle x, y\rangle\in S\leftrightarrow\langle x, y\rangle\in S'\vee(x\in\mathrm{Fld}(S')\wedge y = \mathrm{Max}(r))].$$

In M°, S is a transitive set and $(S, \in)$ is isomorphic to the relation r. There exists a set Y in M such that:

$$\forall x\in\mathrm{Fld}(S)\, Y^{(x)} = [r_{\restriction x}, S_{\restriction x}]$$

and, Patching Lemma provides us with a graph Z which is in M° an isomorphism from the relation r onto the transitive set S.

(c) Assume $M^\circ\models\forall x\in a\exists y\varphi(x, y)$, φ being a Δ_0-formula, a is a graph in M and:

$$M\models\forall n[\langle n, \mathrm{Max}(a)\rangle\in a\to\exists Y[\mathrm{Gr}(Y)\wedge\varphi^\circ(a_{\restriction n}, Y)]].$$

$\mathrm{Gr}(Y)\wedge\varphi^\circ(a_{\restriction n}, Y)$ is a Σ^1_2-formula, by Σ^1_2-AC_{01} we have:

$$M\models\exists Y\forall n[\langle n, \mathrm{Max}(a)\rangle\in a\to[\mathrm{Gr}(Y^{(n)})\wedge\varphi^\circ(a_{\restriction n}, Y^{(n)})]].$$

Again, by the Patching Lemma, we obtain a set b in M° such that:

$$M^\circ\models\forall x\in a\exists y\in b\varphi(x, y).$$

LEMMA II.4.

(a) *If $\mathscr{M}\models E_1+\Delta_1$-sep, then $\mathscr{M}^+\models$ Peano's Axioms + Extensionality + Δ^1_1-CA.*

(b) *If $\mathscr{M}\models E_1+\Delta_0$-coll+V = HC, then $\mathscr{M}^+\models T^*_1+\Sigma^1_1$-$\mathrm{AC}_{01}$.*

(c) *If* $\mathcal{M} \models E_1 + \Delta_0$-coll$+V = $ HC$+$(C), *then* $\mathcal{M}^+ \models T_1 + \Sigma_2^1$-AC$_{01}$.

Proof. (a) is easy to prove.

(b) Assume $\mathcal{M}^+ \models \forall n \exists X \Phi(n, X)$ with Φ a π_∞°-formula. Φ^+ is a Δ_0-formula and:

$$\mathcal{M} \models \forall n \in \omega \exists x \in \omega \Phi^+(n, x)\,.$$

By Δ_0-coll:

$$\mathcal{M} \models \exists a \forall n \in \omega \exists x \in a [x \subset \omega \wedge \Phi^+(n, x)]\,.$$

As $\mathcal{M} \models V = $ HC, there exists in $\mathcal{M}$ a mapping f from ω onto a. So

$$\mathcal{M} \models \forall n \in \omega \exists m \in \omega \big[f(m) \subset \omega \wedge \Phi^+(n, f(m))\big]\,.$$

By Δ_0-sep, the following set X is in $\mathcal{M}$:

$$X = \{\langle n, p\rangle / \exists m \in \omega \big[p \in f(m) \wedge \Phi^+(n, f(m)) \wedge \forall q < m \neg \Phi^+(n, f(q))\big]\}\,.$$

Finally $\mathcal{M}^+ \models \forall n \Phi(n, X^{(n)})$.

(c) Suppose $\mathcal{M}^+ \models \forall n \exists X \Phi(n, X)$ with $\Phi \Delta_2^1$ in $\mathcal{M}^+$; as $\mathcal{M}$ satisfies (C), Φ^+ is Δ_1 in $\mathcal{M}$ (Lemma II.2). Then, using Δ_0-coll and Δ_1-sep we prove as above that:

$$\mathcal{M}^+ \models \exists X \forall n \Phi(n, X^{(n)})\,.$$

Proof of Theorem 1. (a) Let us show that: $T_2 = T_1 + \Sigma_2^1$-AC$_{01}$ and $E_2 = E_1 + \Delta_0$-coll$+$(C)$+V = $ HC are bicommutable.

In view of the preceding lemmas, it is enough to show that if $M \models T_2$ (resp. $\mathcal{M} \models E_2$), then $M^{\circ +} \approx M$ (resp. $\mathcal{M}^{+\circ} \approx \mathcal{M}$) with definable isomorphisms.

To define the isomorphism from M onto $M^{\circ +}$, we associate to every set of integers in M X the graphs Z such that: $M \models \mathcal{I}(Z, \hat{X})$ where

$$\forall x \forall y \langle x, y\rangle \in \hat{X} \leftrightarrow \exists n \exists m \exists p [(x = \langle 0, n\rangle \wedge y = \langle 0, m\rangle \wedge n < m \wedge m \leqslant p \wedge p \in X) \vee \vee (x = \langle 0, p\rangle \wedge p \in X \wedge y = \langle 1, 0\rangle)]\,,$$

$\hat{X}$ can be considered as a canonical representation of X in M°.

Conversely, let $\mathcal{M}$ satisfying E_2. Take a belonging to $\mathcal{M}$. There exists an injective map φ from TC$(\{a\})$ into ω. This mapping induces on the integers a relation r isomorphic to $(\mathrm{TC}(\{a\}), \in)$. r is a graph in $\mathcal{M}^+$ and $\varphi(a) = \mathrm{Max}(r)$.

Then, the isomorphism onto $\mathcal{M}^{+\circ}$ is given associating to each a in $\mathcal{M}$ all the relations r on ω such that:

$$\mathcal{M} \models \exists \varphi(\text{“}\varphi \text{ is an injective mapping from } \mathrm{TC}(\{a\}) \text{ into } \omega\text{”} \wedge \wedge r = \{\langle \varphi(x), \varphi(y)\rangle / x, y \in \mathrm{TC}(\{a\}) \wedge x \in y\})\,.$$

As $\mathcal{M}$ satisfies (C), every graph A of $\mathcal{M}^+$ is isomorphic in $\mathcal{M}$ to a transitive set b by a mapping Ψ and:

$$b = \mathrm{TC}\big(\{\Psi(\mathrm{Max}(A))\}\big)\,.$$

So A is associated to $\Psi(\mathrm{Max}(A))$ in the above isomorphism.

(b) In order to show that $T_1 + \Sigma_2^1$-CA and $E_1 + \Delta_0$-coll$+\Sigma_1$-sep$+V = $ HC bicommute it is enough to verify that, if $M \models T_1 + \Sigma_2^1$-CA, then $M^\circ \models \Sigma_1$-sep and, if $\mathcal{M} \models E_1 + \Sigma_1$-sep, then $\mathcal{M}^+ \models \Sigma_2^1$-CA. These results are easy consequences of Corollary II.1 and Lemma II.2, noting that:

$$E_1 + \Sigma_1\text{-sep} + \Delta_0\text{-coll} + V = \mathrm{HC} \vdash (\mathrm{C})\,.$$

(c) In order to prove that A_2^- and $Z^- + V = $ HC$+\Delta_0$-coll are bicommutable, we recall that:

$$A_2^- \vdash T_1 + \Sigma_2^1\text{-AC}_{01} \quad \text{and} \quad Z^- + V = \mathrm{HC} + \Delta_0\text{-coll} \vdash E_1 + (\mathrm{C}) + V = \mathrm{HC}\,.$$

It is then easy to state that, if $M \models A_2^-$, then $M^\circ \models$ Sep and, if $\mathcal{M} \models Z^- + \Delta_0$-coll$+$ $+V = $ HC, then $\mathcal{M}^+ \models$ CA.

This result has been proved by W. Marek, using different methods.

E. Minimality. The results of Theorem 1 give us pairs of bicommutable theories of decreasing strength. The following theorem shows that the fourth case of Theorem 1 gives a pair of theories which are, in a way, minimal for the bicommutability.

THEOREM 2. *Let S be a theory of $\mathcal{L}$, and T a theory of L, such that*:

$$S \vdash E_1 + \Delta_1\text{-sep} \quad \textit{and} \quad T \vdash T_1 + \Sigma_1^1\text{-AC}_{01}\,.$$

If S and T are weakly bicommutable (*through the interpretations* $^\circ$ *and* $^+$) *then*:

$$S \vdash E_1 + \Delta_0\text{-coll} + (\mathrm{C}) + V = \mathrm{HC} \quad \textit{and} \quad T \vdash T_1 + \Sigma_2^1\text{-AC}_{01}\,;$$

moreover S and T are bicommutable.

Proof. Let $\mathcal{M}$ be a model of S. Then, by hypothesis, $\mathcal{M}^+ \models T_1 + \Sigma_1^1$-AC$_{01}$. Therefore (by Corollary II.2) $\mathcal{M}^{+\circ} \models$ (C)$+V = $ HC, and so does $\mathcal{M}$, for $\mathcal{M} \equiv \mathcal{M}^{+\circ}$. According to Lemma II.2, if Φ is Δ_2^1 in T, Φ^+ is Δ_1 in S. So, if M is a model of T $M^\circ \models S$ and $M^{\circ +} \models \Delta_2^1$-CA which is equivalent to Σ_2^1-AC$_{01}$. Therefore $M \models \Sigma_2^1$-AC$_{01}$ as well as $T \vdash T_1 + \Sigma_2^1$-AC$_{01}$, Corollary II.2 shows that $S \vdash \Delta_0$-coll. Finally, as in the proof of Theorem 1, one shows that S and T are strongly bicommutable.

We now give an example of a theory of $\mathcal{L}$ which does not bicommute (by ($^\circ$, $^+$)) with any theory of L.

THEOREM 3. *No subsystem of A_2 weakly bicommutes by* ($^\circ$, $^+$) *with* KP$+V = $ HC.

Proof. We shall display two models $\mathcal{M}$ and $\mathcal{N}$ of KP$+V = $ HC such that: $\mathcal{M}^{+\circ} \not\equiv \mathcal{M}$ and $\mathcal{N}^{+\circ} \not\equiv \mathcal{N}$. The difference between $\mathcal{M}$ and $\mathcal{N}$ is that $\mathcal{M}^+$ will be a model of a weak subsystem of analysis (namely $T_1 + \Sigma_1^1$-AC$_{01} + \neg \Sigma_1^1$-CA) while $\mathcal{N}^+$ will be a model of full second order arithmetic.

EXAMPLE 1. Let $\mathcal{N}'$ be a model of ZF such that ω is standard in $\mathcal{N}'$ but $\aleph_1$ is not (see for example [6]), HC$^{\mathcal{N}'}$, the set of hereditarily countable sets of $\mathcal{N}'$, is a transitive model of ZFC$^-$$+V = $ HC. Let $\mathcal{N}$ be its standard part. Then $\mathcal{N} \models$ KP$+V = $ HC (To show that $\mathcal{N} \models \Delta_0$-coll, let $\varphi(x, y)$ be a Δ_0-formula of $\mathcal{L}$

such that $\mathscr{N} \models \forall x \in a \exists y \varphi(x, y)$. We apply in $\mathrm{HC}^{\mathscr{N}'}$ the reflexion principle to the formula

$$\varphi(x, y) \wedge \forall z [rk(z) < rk(y) \rightarrow \neg \varphi(x, z)] .$$

Moreover $\mathscr{N} \models \neg(\mathrm{C})$: for, let α be a countable non standard ordinal of $\mathscr{N}'$; there exists a subset X of ω representing in $\mathscr{N}'$ a well ordering of type α; Then $X \in |\mathscr{N}|$ because ω is standard in $\mathscr{N}'$, but X is not isomorphic to any transitive set of $\mathscr{N}$ (because $\alpha \notin |\mathscr{N}|$). Moreover, $\mathscr{N}^+ = \mathscr{N}'^+$, so $\mathscr{N}^+ \models A_2$. On the other hand, $\mathscr{N}^{+\circ} = \mathrm{HC}^{\mathscr{N}'}$, which satisfies (C). So we have $\mathscr{N}^{+\circ} \not\equiv \mathscr{N}$.

EXAMPLE 2. ω_1^{CK} is the least non-recursive ordinal. It is well known (see for example [12]) that: $L_{\omega_1^{\mathrm{CK}}} \models \mathrm{KP} + \mathrm{V} = \mathrm{HC}$, and that the subsets of ω in $L_{\omega_1^{\mathrm{CK}}}$ are exactly the hyperarithmetical subsets of ω.

Take $\mathscr{M} = L_{\omega_1^{\mathrm{CK}}}$. Then $\mathscr{M}^+ \models T_1 + \Sigma_1^1\text{-AC}_{01} + \neg \Sigma_1^1\text{-CA}$ (see [8]). The ordinals of $\mathscr{M}$ are isomorphic to the graphs of the type $\leqslant_{\restriction \{y/y \leqslant a\}}$ for $a \in 0$ (with the notations of [5] and [7]). On the other hand, for all $a \in 0^*$, $\leqslant_{\restriction \{y/y \leqslant a\}}$ is a pseudo-well ordering (see [5]), so it is a linear ordered graph of $\mathscr{M}^+$, defining an ordinal in $\mathscr{M}^{+\circ}$. That remark shows that $\mathscr{M}^{+\circ}$ is not isomorphic to $\mathscr{M}$, for in $L_{\omega_1^{\mathrm{CK}}}$ every two well-orderings are comparable which is not true for pseudo-well orderings in $\mathscr{M}^+ = \mathrm{HA}$ (see [5]).

Remark. The existence of two pseudo-well orderings in HA which are not hyperarithmetically comparable allows us to show that, in HA, there do not exist formulas $I_1(X, Y)$ and $I_2(X, Y)$ satisfying the conditions of Lemma II.1. Therefore the notion of isomorphism between graphs is not Δ_1^1 in the theory $T_1^* + \Sigma_1^1\text{-AC}_{01}$. That is why we need $\Sigma_1^1\text{-BI}_0$ in T_1.

Let us prove that Lemma II.1 is not true in $\mathscr{M}^+$:

Harrisson has proved the following theorem in [5]: If $a \in O^* - O$ and if S is a Σ_1^1 subset of ω such that $O \subseteq S \subseteq O^*$, there exists in S an integer b such that: $\leqslant_{\restriction \{y/y \leqslant a\}}$ and $\leqslant_{\restriction \{y/y \leqslant b\}}$ are not hyperarithmetically comparable.

Let A (resp. B) be a subset of ω representing $\leqslant_{\restriction \{y/y \leqslant a\}}$ (resp. $\leqslant_{\restriction \{y/y \leqslant b\}}$). We remark that A and B are graphs of HA. Put

$$X = \{n \in \mathrm{Fld}(A) / \forall m \in \mathrm{Fld}(B) \neg \mathscr{I}(A_{\restriction n}, B_{\restriction m})\} .$$

If Lemma II.1 were true in HA, X would be hyperarithmetical, therefore it would belong to the model HA. X is not empty, otherwise, in HA, A would be isomorphic to an initial section of B; let n_0 be a minimal element of X for A. No integer n satisfies $\langle n, n_0 \rangle \in A \wedge \mathscr{I}(A_{\restriction n}, B)$ in HA. So, by Δ_1^1-CA there exists m_0 which, for B, is a minimal element of the subset Y defined by:

$$Y = \{m \in \mathrm{Fld}(B) / \forall n [\langle n, n_0 \rangle \in A \rightarrow \neg \mathscr{I}(A_{\restriction n}, B_{\restriction m})]\} .$$

A and B are linear orderings, we should get: $\mathrm{HA} \models \mathscr{I}(A_{\restriction n_0}, B_{\restriction m_0})$, which is not compatible with the definition of n_0.

Remark. The results proved by W. Marek in [10], permit an alternative proof of the following fact: $(L_{\omega_1^{\mathrm{CK}}})^{+\circ} \not\equiv L_{\omega_1^{\mathrm{CK}}}$.

Let us mention also the theorem proved by Gandy in [4]:

THEOREM. *No theory in L bicommutes with* $\mathrm{ZF}^- + \mathrm{V} = \mathrm{HC}$.

III. Third order arithmetic

Now we treat the case of third order arithmetic and set theory, and we shall show that the notion of bicommutability for some theories of those languages depends on the interpretation. Note that P. Zbierski, in [16] using a slightly different interpretation of the language of arithmetic in set theory, studied the correspondence between β-models of A_n and standard models of set theory.

A. Definitions and preliminaries.

1. Languages and axioms. L^2 is the language of third order arithmetic, with 3 similarity types; we shall use $m, n, \ldots$ for first-order variables, $x, y, \ldots \alpha, \beta, \ldots$ for the second order and $X, Y, \ldots$ for the third order. First order objects are named integers, second order objects are named reals and third order objects sets of reals.

The theories we consider will always allow us to define a pairing function on the integers (resp. on the reals), noted $n, m \rightarrow \langle n, m \rangle$ (resp. $x, y \rightarrow \langle x, y \rangle$) and an injective mapping from the countable sequences of reals into the reals: if x is a notation for a sequence (u_n) we shall denote $u_n = (x)_n$.

We define

$$X^{(x)} = \{y / \langle x, y \rangle \in X\} \quad \text{and} \quad \mathrm{Fld}(X) = \{x / \exists y [\langle x, y \rangle \in X \vee \langle y, x \rangle \in X]\} .$$

$\mathfrak{F}$ being a class of formulas of L^2, we define the following schemes of axioms:

$\mathfrak{F}$-AC_{12} is the scheme

$$\forall x_1 x_2 \ldots x_k [\forall x \exists X \Phi(x, X) \rightarrow \exists Y \forall x \Phi(x, Y^{(x)})]$$

for all formulas $\Phi \in \mathfrak{F}$.

$\mathfrak{F}$-BI_1 is the scheme

$$\forall X \{\{\forall x \exists i \langle (x)_i, (x)_{i+1} \rangle \notin X\} \wedge (\forall x \in \mathrm{Fld}(X)) \{\forall x [\forall y (\langle y, x \rangle \in X \rightarrow \Phi(y)) \rightarrow \Phi(x)]\} \rightarrow (\forall x \in \mathrm{Fld}(X)) \Phi(x)\}$$

for all formulas $\Phi \in \mathfrak{F}$.

PC_2 (principle of choice for the reals) is the axiom

$$\forall X [\text{“}X \text{ is an equivalence relation on the reals”} \rightarrow \exists Y \forall x \exists! y (\langle x, y \rangle \in X \wedge y \in Y)] .$$

For the language $\mathscr{L}$, we shall use the axioms defined previously, adding the following ones:

$$\mathrm{AC}(\beth_1, \beth_2)\colon \forall f\{[\text{“}f \text{ is a function of domain } \wp(\omega)\text{”} \wedge \forall x \in \wp(\omega)(f(x) \neq \emptyset \wedge \wedge f(x) \subset \wp(\omega))] \to \exists g\, [\text{“}g \text{ is a function of domain } \wp(\omega)\text{”} \wedge \forall x \in \wp(\omega)(g(x) \in f(x))]\}.$$

Axiom of strong collapsing (C*)

$$\forall r\,[\text{“}r \text{ is a well-founded relation”} \to \exists b \exists f\,[\text{“}b \text{ is a transitive set”} \wedge \wedge \text{“}f \text{ is an homomorphism from } r \text{ onto } (b, \in)\text{”}].$$

$$\mathrm{V} = \mathrm{H}\beth_1\colon \forall x \exists f \text{ “}f \text{ maps } \wp(\omega) \text{ onto } x\text{”}$$

$$\Sigma_1 - I(\in)\colon \forall x[(\forall y \in x)\varphi(y) \to \varphi(x)] \to \forall x \varphi(x) \quad \text{for all } \Sigma_1\text{-formulas } \varphi.$$

2. Interpretations.

(i) *Interpretation of L^2 in $\mathscr{L}$.* To the interpretation + defined previously for L, we add the condition:

$$\text{if } \Phi \text{ is } \forall X \Psi(X) \text{ then } \Phi^+ \text{ is } \forall x(x \subset \wp(\omega) \to \Psi^+(x)).$$

As before, it induces a mapping $\mathscr{M} \to \mathscr{M}^+$ from the structures of $\mathscr{L}$ to those of L^2, such that $\mathscr{M} \models \Phi^+ \Leftrightarrow \mathscr{M}^+ \models \Phi$ for all sentences Φ of L^2.

(ii) *Two different interpretations of $\mathscr{L}$ in L^2.* $\mathrm{Tr}(X)$ is the formula:

$$\forall x \exists i\, [\langle (x)_i, (x)_{i+1}\rangle \notin X] \wedge (\exists! z \in \mathrm{Fld}(X))[(\forall x \in \mathrm{Fld}(X) - \{z\}) (\exists s)(\text{“}s \text{ is a finite sequence of reals”} \wedge (\forall i < \lg(s))\langle (s)_i, (s)_{i+1}\rangle \in X \wedge \wedge (s)_0 = x \wedge (s)_{\lg(s)-1} = z)].$$

Similarly with the case of L, $\mathrm{Gr}(X)$ is the formula:

$$\mathrm{Tr}(X) \wedge (\forall x \in \mathrm{Fld}(X))(\forall y \in \mathrm{Fld}(X))[\forall z(\langle z, x\rangle \in X \leftrightarrow \langle z, y\rangle \in X) \to x = y].$$

Let M be a structure for L; if $M \models \mathrm{Tr}(X)$ (resp. $\mathrm{Gr}(X)$) we shall call X a *tree* (resp. a *graph*) of M. The unique maximal element of a tree (or a graph) X will be denoted by $\mathrm{Max}(X)$. If $x \in \mathrm{Fld}(X)$, we define $X_{\restriction x}$ as before.

The interpretation $\varphi \to \varphi^\circ$ from $\mathscr{L}$ into L^2 is defined as before, using the new definition of the predicate Gr; $\mathscr{I}$ and $\mathscr{E}$ will again denote the predicates interpreting equality and membership. Thence we get a mapping $M \to M^\circ$ from the structures of L^2 to those of $\mathscr{L}$.

Let F, $\mathscr{H}$, $\mathscr{F}$ be the following formulas:

$$F(X_1, X_2, T)\colon \mathrm{Tr}(X_1) \wedge \mathrm{Tr}(X_2) \wedge \forall x \forall y \{\langle x, y\rangle \in T \leftrightarrow [x \in \mathrm{Fld}(X_1) \wedge \wedge y \in \mathrm{Fld}(X_2) \wedge \forall s[\langle s, x\rangle \in X_1 \to \exists t(\langle t, y\rangle \in X_2 \wedge \langle s, t\rangle \in T)] \wedge \wedge \forall t[\langle t, y\rangle \in X_2 \to \exists s(\langle s, x\rangle \in X_1 \wedge \langle s, t\rangle \in T)]]\},$$

$$\mathscr{H}(X_1, X_2)\colon \exists T[F(X_1, X_2, T) \wedge \langle \mathrm{Max}(X_1), \mathrm{Max}(X_2)\rangle \in T],$$

$$\mathscr{F}(X_1, X_2)\colon \exists x[\langle x, \mathrm{Max}(X_2)\rangle \in X_2 \wedge \mathscr{H}(X_1, X_{2\restriction x})].$$

We now define the interpretation $\varphi \to \varphi^*$ by induction on the length of the formula φ in $\mathscr{L}$.

If φ is $x_1 = x_2$ φ^* is $\mathscr{H}(X_1, X_2)$.
If φ is $x_1 \in x_2$ φ^* is $\mathscr{F}(X_1, X_2)$.
If φ is $\psi \vee \chi$ φ^* is $\psi^* \vee \chi^*$.
If φ is $\neg\psi$ φ^* is $\neg\psi^*$.
If φ is $\exists x \psi(x)$ φ^* is $\exists X[\mathrm{Tr}(X) \wedge \psi^*(X)]$.

This interpretation induces a mapping $M \to M^*$ from the structures of L^2 to those of $\mathscr{L}$.

From now on, we shall write that T and $\mathscr{T}$, two theories of L^2 and $\mathscr{L}$ bicommute through trees (resp. through graphs) if the pair of interpretations $({}^+, {}^*)$ (resp. $({}^+, {}^\circ)$) makes them bicommute.

B. Results of bicommutability. T_3 will be the theory of L^2 containing the following axioms:

Peano's axioms for first order.
Extensionality for second and third order.
Δ_0^2-CA and Σ_1^2-BI_1.
E_1 is the theory defined in II.

LEMMA III.1. *If M is a model of T_3, there exists a graph R of M such that:*

$$M^* \models R = \wp(\omega) \quad \textit{and} \quad M^\circ \models R = \wp(\omega).$$

Proof. The graph R is defined by:

$$\forall x, y\{\langle x, y\rangle \in R \leftrightarrow \exists n \exists m[x = \langle\{0\}, \{n\}\rangle \wedge y = \langle\{0\}, \{m\}\rangle \wedge n < m] \vee \vee \exists n \exists z[x = \langle\{0\}, \{n\}\rangle \wedge y = \langle\{1\}, z\rangle \wedge n \in z] \vee \vee \exists z[x = \langle\{1\}, z\rangle \wedge y = \langle\{2\}, \{0\}\rangle]\}.$$

LEMMA III.2. *In $T_3 + \Sigma_1^2$-AC_{12} we have:*

(a) $\forall X_1 \forall X_2 [\mathrm{Tr}(X_1) \wedge \mathrm{Tr}(X_2) \to \exists! T F(X_1, X_2, T)]$,

(b) $\forall X_1 \forall X_2 [\mathrm{Gr}(X_1) \wedge \mathrm{Gr}(X_2) \to (\mathscr{H}(X_1, X_2) \leftrightarrow \mathscr{I}(X_1, X_2))]$.

Proof. The proof of (a) is analogous to that of Lemma II.1.

The proof of (b) is easy, pointing out the fact that, if X_1 and X_2 are graphs, and if we have $F(X_1, X_2, T)$, then T is an isomorphism between a transitive part of X_1 and a transitive part of X_2, and that, conversely, if T is an isomorphism between X_1 and X_2 we have $F(X_1, X_2, T)$.

COROLLARY III.1. *Let M be a model of $T_3 + \Sigma_1^2$-AC_{12}. The predicates $\mathscr{H}$ and $\mathscr{F}$ are Δ_1^2 on M. If φ is Δ_0 on M^* (resp. M°), φ^* (resp. φ°) is Δ_1^2 on M.*

If φ is Σ_k $(k\geqslant 1)$ on M^* (resp. M°), φ^* (resp. φ°) is Σ_k^2 on M.

LEMMA III.3. $E_1+\mathrm{AC}(\beth_1,\beth_2)+\mathrm{V}=\mathrm{H}\beth_1\vdash \mathrm{AC}(\beth_1)$.

Proof. In a model of $E_1+\mathrm{AC}(\beth_1,\beth_2)+\mathrm{V}=\mathrm{H}\beth_1$, let f be a function of domain $\wp(\omega)$ such that $(\forall x\in\wp(\omega))f(x)\neq\varnothing$. There exists a mapping g from $\wp(\omega)$ onto $\bigcup\mathrm{Rge}(f)$. We define a function h from $\wp(\omega)$ into $\wp\wp(\omega)$ by $h(x)=\{y/g(y)\in f(x)\}$. Then, we have $(\forall x\in\wp(\omega))h(x)\neq\varnothing$ and, by $\mathrm{AC}(\beth_1,\beth_2)$, there exists a function j of domain $\wp(\omega)$ satisfying $(\forall x\in\wp(\omega))j(x)\in h(x)$. Put $k=g_0 j$. One can easily see that $(\forall x\in\wp(\omega))k(x)\in f(x)$.

THEOREM 4. (a) *Put $E_3=E_1+\Sigma_1\text{-}I(\epsilon)+\Delta_1\text{-sep}+\Delta_0\text{-coll}+(\mathrm{C}^*)+\mathrm{AC}(\beth_1,\beth_2)+\mathrm{V}=\mathrm{H}\beth_1+\exists x(x=\wp(\omega))$ and $T_4=T_3+\Sigma_1^2\text{-AC}_{12}$. Then E_3 and T_4 bicommute through trees.*

(b) *The theories $T_4+\mathrm{PC}$ and $E_3+\mathrm{AC}$ bicommute through trees.*

Proof. It has been proved in [17] that:

$$M\vDash T_4\Rightarrow M^*\vDash E_1+\Delta_1\text{-sep}+\Delta_0\text{-coll}+\exists x(x=\wp(\omega))$$

and that:

$$M\vDash E_3\Rightarrow M^+\vDash T_4\,.$$

One proves, by methods analogous to those employed in the proof of Corollary II.2, that if $M\vDash T_3$, then

$$M^*\vDash(\mathrm{C}^*)+\mathrm{V}=\mathrm{H}\beth_1+\Sigma_1\text{-}I(\in)\,.$$

Let us show now that, if $M\vDash T_4$, then $M^*\vDash\mathrm{AC}(\beth_1,\beth_2)$:

In M^*, let f be a function of domain $\wp(\omega)$ such that $\forall x\in\wp(\omega)(f(x)\neq\varnothing)$. As the formula of $\mathscr{L}$, $z=(x,y)$ is Δ_0, there exists a Δ_1^2 formula of L, $C(X,Y,Z)$, such that if X, Y, Z are trees of M, then

$$M\vDash C(X,Y,Z)\Leftrightarrow M^*\vDash Z=(X,Y)\,.$$

The function f is a tree of M such that

$$\begin{aligned}M\vDash\forall z\in\mathrm{Fld}(f)\{\langle z,\mathrm{Max}(f)\rangle\in f\to\exists X\exists Y[(\mathrm{Tr}(X)\wedge\mathrm{Tr}(Y)\wedge C(X,Y,f_{\restriction z}))\wedge\\ \wedge\exists!x(x\in\mathrm{Fld}(R)\wedge\langle x,\mathrm{Max}(R)\rangle\in R\wedge\mathscr{H}(x,R_{\restriction x}))\wedge\exists t(\langle t,\mathrm{Max}(Y)\rangle\in Y)]\}\wedge\\ \wedge\forall x\in\mathrm{Fld}(R)\{\langle x,\mathrm{Max}(R)\rangle\in R\to\exists t\exists X\exists Y\exists z[\mathrm{Tr}(X)\wedge\mathrm{Tr}(Y)\wedge C(X,Y,f_{\restriction z})\wedge\\ \wedge\mathscr{H}(X,R_{\restriction x})\wedge\langle t,\mathrm{Max}(Y)\rangle\in Y\wedge\langle z,\mathrm{Max}(f)\rangle\in f]\}\end{aligned}$$

where R is the graph defined in Lemma III.1.

We apply $\Sigma_1^2\text{-AC}_{12}$ to the second member of the above conjunction and obtain

$$\begin{aligned}\exists Z\forall x\in\mathrm{Fld}(R)\{\langle x,\mathrm{Max}(R)\rangle\in R\to\exists t[Z^{(x)}=f_{\restriction t}\wedge\exists X\exists Y\exists z(\mathrm{Tr}(X)\wedge\mathrm{Tr}(Y)\wedge\\ \wedge C(X,Y,f_{\restriction z})\wedge\mathscr{H}(X,R_{\restriction x})\wedge\mathscr{F}(Z^{(x)},Y)\wedge\langle z,\mathrm{Max}(f)\rangle\in f)]\}\,.\end{aligned}$$

Now we define a tree G in M by:

$$\begin{aligned}\forall s\forall t\{\langle s,t\rangle\in G\leftrightarrow\exists x\exists u\exists v\{[s=\langle\{0\},u\rangle\wedge t=\langle\{0\},v\rangle\wedge\langle u,v\rangle\in R_{\restriction x}]\vee\\ \vee[s=\langle\{1\},u\rangle\wedge t=\langle\{1\},v\rangle\wedge\langle u,v\rangle\in Z^{(x)}]\vee[s=\langle\{0\},x\rangle\wedge t=\langle\{2\},x\rangle]\vee\\ \vee[s=\langle\{1\},\mathrm{Max}(Z^{(x)})\rangle\wedge t=\langle\{3\},x\rangle]\vee[s=\langle\{0\},x\rangle\wedge t=\langle\{3\},x\rangle]\vee\\ \vee[s=\langle\{2\},x\rangle\wedge t=\langle\{4\},x\rangle]\vee[s=\langle\{4\},x\rangle\wedge t=\langle\{5\},\mathrm{Max}(f)\rangle]\}\}\,.\end{aligned}$$

G is constructed so that $M\vDash\forall X[\mathscr{F}(X,G)\leftrightarrow\exists x\,C(R_{\restriction x},Z^{(x)},X)]$ and in M^*, we have "G is a function of domain $\wp(\omega)$" and $\forall x\in\wp(\omega)\,G(x)\in f(x)$.

To define the isomorphism between M and M^{*+}:

With every integer n of M, we associate the trees Y such that $M\vDash\mathscr{H}(R_{\restriction\langle\{0\},\{n\}\rangle},Y)$.

With every real x of M, we associate the trees Y such that $M\vDash\mathscr{H}(R_{\restriction\langle\{1\},x\rangle},Y)$.

With every set of reals X of M, we associate the trees Y such that $M\vDash\mathscr{H}(\hat{X},Y)$, where $\hat{X}$ is defined as follows:

$$\forall s\forall t\{\langle s,t\rangle\in\hat{X}\leftrightarrow\exists x\in X[(\langle s,t\rangle\in R_{\restriction\langle\{1\},x\rangle})\vee(s=\langle\{1\},x\rangle\wedge t=\mathrm{Max}(R))]\}\,.$$

We define the isomorphism between $\mathscr{M}$ and $\mathscr{M}^{+*}$:

Let a be an element of $\mathscr{M}$, f a mapping from $\wp(\omega)$ onto $\mathrm{TC}(\{a\})$ and x_0 such that $f(x_0)=a$. $A(a,f,x_0)$ is the tree defined as follows:

$$\begin{aligned}\forall s\forall t\{\langle s,t\rangle\in A(a,f,x_0)\leftrightarrow[s\in\wp(\omega)\wedge t\in\wp(\omega)\wedge[(f(s)\in f(t)\wedge f(t)\in\mathrm{TC}(a))\wedge\\ \vee(f(s)\in a\wedge t=x_0)]]\}\,.\end{aligned}$$

One checks that, if f and g are two mappings from $\wp(\omega)$ onto $\mathrm{TC}(\{a\})$ and if $g(x_1)=f(x_0)=a$, then we have $\mathscr{H}[A(a,f,x_0),A(a,g,x_1)]$. The isomorphism is given by associating with every a in $\mathscr{M}$, the trees Y of $\mathscr{M}^+$ such that:

$$\exists f\exists x_0[\text{"}f\text{ maps }\wp(\omega)\text{ onto }\mathrm{TC}(\{a\})\text{"}\wedge f(x_0)=a\wedge\mathscr{H}(Y,A(a,f,x_0))]\,.$$

For the second part of the theorem, take M a model of $T_4+\mathrm{PC}$. We show that $M^*\vDash\mathrm{AC}$.

Let a be a non-empty set of M^*, such that $\forall x\in a(x\neq\varnothing)$ a is a tree of M and we have:

$$M\vDash\exists x\langle x,\mathrm{Max}(a)\rangle\in a\wedge\forall x[\langle x,\mathrm{Max}(a)\rangle\in a\to\exists y\langle y,x\rangle\in a]\,.$$

Then we define, on a subset of $\mathrm{Fld}(a)$, the following equivalence relation

$$\begin{aligned}S(x,y)\text{ iff }\exists u\exists v[\langle u,\mathrm{Max}(a)\rangle\in a\wedge\langle v,\mathrm{Max}(a)\rangle\in a\wedge\mathscr{H}(a_{\restriction u},a_{\restriction v})\wedge\\ \wedge\langle x,u\rangle\in a\wedge\langle y,v\rangle\in a]\,.\end{aligned}$$

By Δ_1^2-CA, S can be considered as a set of reals of M. Applying PC, we easily display a tree b of M such that

$$M^*\vDash\forall x\in a\exists!y\,y\in b\cap x\,.$$

Conversely, it is obvious that, if $\mathscr{M}\vDash E_3+\mathrm{AC}$, then $\mathscr{M}^+\vDash\mathrm{PC}$.

C. Comparison between bicommutability through trees and through graphs. Similarly, we could have defined a notion of bicommutability through trees for theories of L and $\mathcal{L}$. In fact, as soon as T contains T_1 and $\mathcal{T}$ contains E_1, it is equivalent for T and $\mathcal{T}$ to bicommute through graphs or through trees.

Now we give some results which show that those notions of bicommutability are not equivalent for L^2 and $\mathcal{L}$.

We recall that the properties of the bicommutability through graphs are essentially based upon the validity of Patching Lemma (PL):

$$\forall Y\{\forall x\,\mathrm{Gr}(Y^{(x)})\to\exists Z[\mathrm{Gr}(Z)\wedge\forall s(\langle s,\mathrm{Max}(Z)\rangle\in Z\to\exists x\,\mathcal{I}(Z_{\restriction s},Y^{(x)}))\wedge$$
$$\wedge\forall x\exists s(\langle s,\mathrm{Max}(Z)\rangle\in Z\wedge\mathcal{I}(Z_{\restriction x},Y^{(x)}))]\}.$$

LEMMA III.4. $T_4+\mathrm{PL}\vdash\forall X[\mathrm{Tr}(X)\to\exists Y(\mathrm{Gr}(Y)\wedge\mathcal{H}(X,Y))]$.

Proof. Let M be a model of $T_4+\mathrm{PL}$ and A be a tree of M; by induction on the well-founded relation A, we construct a graph B verifying $M\models\mathcal{H}(A,B)$. Let $\Phi(x)$ be the formula

$$x\in\mathrm{Fld}(A)\wedge\exists C[\mathrm{Gr}(C)\wedge\mathcal{H}(C,A_{\restriction x})].$$

Suppose there exists x in $\mathrm{Fld}(A)$ such that $M\models\neg\Phi(x)$. Then by Σ_1^2-BI_1 there exists an element x_0 in $\mathrm{Fld}(A)$ such that

$$M\models\neg\Phi(x_0)\wedge\forall y[\langle y,x_0\rangle\in A\to\Phi(y)].$$

By Σ_1^2-AC_{12}, there exists Y such that

$$\forall y[\langle y,x_0\rangle\in A\to(\mathrm{Gr}(Y^{(y)})\wedge\mathcal{H}(Y^{(y)},A_{\restriction y}))].$$

One may take, for example, $Y^{(y)}=\{0\}$ if $\langle y,x_0\rangle\notin A$.

Applying Patching Lemma to Y, we obtain a graph Z verifying $\mathcal{H}(Z,A_{\restriction x_0})$ which contradicts the definition of x_0.

So, $M\models\forall x\in\mathrm{Fld}(A)\Phi(x)$ and therefore $M\models\Phi(\mathrm{Max}(A))$ which proves the existence of the graph B.

THEOREM 5. *Let T and $\mathcal{T}$ be theories of L^2 and $\mathcal{L}$ respectively, such that T and $\mathcal{T}$ bicommute through trees, $T\supset T_4$ and $\mathcal{T}\supset E_3$. Then the followings are equivalent:*

(i) $\mathcal{T}\vdash\mathrm{AC}$,

(ii) $T\vdash\mathrm{PC}$,

(iii) $T\vdash\mathrm{PL}$,

(iv) *T and $\mathcal{T}$ bicommute through graphs.*

Proof. The equivalence of (i) and (ii) is an immediate consequence of Theorem 4 (b).

(ii)⇒(iii): Let, in M, Y be a set of reals such that $\forall x\,\mathrm{Gr}(Y^{(x)})$. We define the following equivalence relation S on the reals:

$$S(s,t)\ \text{iff}\ \exists y\exists z[s\in\mathrm{Fld}(Y^{(y)})\wedge t\in\mathrm{Fld}(Y^{(z)})\wedge\mathcal{I}(Y^{(y)}_{\restriction s},Y^{(z)}_{\restriction t})].$$

By PC, there exists a set X having exactly one element in each equivalence class of S. We may assume, for example, that $O\notin X$. We define a graph Z as follows:

$$\forall x\forall y\{\langle x,y\rangle\in Z\leftrightarrow[[y=0\wedge x\in X\wedge\exists z S(x,\mathrm{Max}(Y^{(z)})]\vee$$
$$\vee[x\in X\wedge y\in X\wedge\exists z\exists z'(x\in\mathrm{Fld}(Y^{(z)})\wedge y\in\mathrm{Fld}(Y^{(z')})\wedge\mathcal{E}(Y^{(z)}_{\restriction x},Y^{(z')}_{\restriction y})]]\}.$$

One verifies that Z is the desired "patched" graph.

(iii)⇒(iv): Suppose $T\vdash T_4+\mathrm{PL}$. Let M be a model of T. By Lemma III.4, there exists at least one graph in each equivalence class of the relation $\mathcal{H}$ defined on the trees of M, and according to Lemma III.2 the graph of a given class for $\mathcal{H}$ constitutes an equivalence class for $\mathcal{I}$. Therefore M° and M^* are isomorphic relatively to the equality relations $\mathcal{H}$ and $\mathcal{I}$. As T and $\mathcal{T}$ bicommute by the interpretations $(^+,{}^*)$ the same is true for the interpretations $(^+,{}^\circ)$.

(iv)⇒(i): Actually, we show that if $T\supset T_4$ and $\mathcal{T}\supset E_3$ and if they bicommute through graphs, then $\mathcal{T}\vdash\mathrm{AC}$. Let $\mathcal{M}$ be a model of $\mathcal{T}$; there exists a model M of T such that $\mathcal{M}$ is isomorphic to M°. Let a be an element of $\mathcal{M}$ such that

$$\mathcal{M}\models a\neq\emptyset\wedge\forall x\in a(x\neq\emptyset)$$

and A be a graph of M corresponding to a by the above isomorphism. Then we have

$$M\models\forall x\in\mathrm{Fld}(A)[\langle x,\mathrm{Max}(A)\rangle\in A\to\exists Y\exists y(\langle y,x\rangle\in A\wedge Y=\{y\})].$$

By Σ_1^2-AC_{12},

$$M\models\exists Z\forall x\in\mathrm{Fld}(A)[\langle x,\mathrm{Max}(A)\rangle\in A\to\exists y(Z^{(x)}=\{y\}\wedge\langle y,x\rangle\in A)].$$

So there exists X in M such that

$$M\models\forall x\in\mathrm{Fld}(A)[\langle x,\mathrm{Max}(A)\rangle\in A\to\exists!y(y\in X\wedge\langle y,x\rangle\in A)].$$

Then, we define a graph B as follows:

$$\forall s\forall t\{\langle s,t\rangle\in B\leftrightarrow[s\in X\wedge t=\mathrm{Max}(A)\wedge\exists x[\langle x,\mathrm{Max}(A)\rangle\in A\wedge\langle s,x\rangle\in A]]\vee$$
$$\vee\exists y\exists x[y\in X\wedge\langle x,\mathrm{Max}(A)\rangle\in A\wedge\langle y,x\rangle\in A\wedge\langle s,t\rangle\in A_{\restriction y}]\}.$$

If b is the element of $\mathcal{M}$ which corresponds to B, we have

$$\mathcal{M}\models\forall x\in a\exists y(x\cap b=\{y\}).$$

Remark. The last part of the preceding proof shows in fact that T_4 and E_3 are minimal theories for the bicommutability through graphs, in the sense of II.C. We have not been able to give a similar result for the bicommutability through trees.

COROLLARY III.2. *There exists a pair of theories of L^2 and $\mathcal{L}$ which bicommute through trees and do not bicommute through graphs.*

Proof. Those theories are respectively $T_4+\neg\mathrm{PC}$ and $E_3+\neg\mathrm{AC}$. Their consistency is a consequence of the following result, proved by J. K. Truss [15]:

If ZF is consistent, so is the theory: $ZF+AC(\beth_1, \beth_2)+\neg CR$ where CR is the following weak axiom of choice:

$$\forall r \subset \wp\wp(\omega)[\forall x \in r \forall y \in r(x = y \vee x \cap y = \emptyset)$$
$$\to \exists a \subset \wp(\omega) \forall x \in r(x \neq \emptyset \to \exists t\, x \cap a = \{t\})].$$

If $\mathcal{M}$ is a model of that theory, $\mathcal{M}^+$ is a model of $T_4+\neg PC$.

Post-script. Independently, W. Marek, using the interpretation of second order arithmetic in set theory given here and in [11], gets some results on ω-models of second order arithmetic corresponding to sets L_α, for some countable α. These results are in print in Fund. Math. [10].

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Some comments on the paper by Artigue, Isambert, Perrin, and Zalc

by

W. Marek (Warszawa)

DEFINITION. (a) $\{J_\alpha\}_{\alpha \in On}$ *is the Jensen splitting of constructible sets into a hierarchy.*

(b) J_α *is projectible iff there is a* 1-1 $\Sigma_1^{J_\alpha}$ *function on* J_α *into some* $x \in J_\alpha$.

The following is well known although no proof of it has appeared.

THEOREM 1 (Kripke). *The following conditions are equivalent for admissible* J_α $(\alpha > \omega)$:

(a) J_α *is non-projectible,*

(b) J_α *satisfies the* Σ_1*-separation scheme,*

(c) J_α *possesses a cofinal tower of* Σ_1*-elementary transitive subsystems.*

Using the following proposition of Artigue, Isambert, Perrin and Zalc.

PROPOSITION. *Theories* Σ_2^1-CA *and* KP+Σ_1*-separation* + V = HC *are bicommutable by means of well-founded trees and restrictions to* $\wp(\omega)$.

We find that

THEOREM 2. *If* $J_\alpha \models V = HC$, *then the following conditions are equivalent:*

(a) $J_\alpha \cap \wp(\omega) \models \Sigma_2^1$-CA,

(b) J_α *is non-projectible,*

(c) J_α *satisfies the* Σ_1*-separation scheme,*

(d) J_α *possesses a cofinal tower of* Σ_1*-elementary transitive subsystems.*

We notice that (d) implies that $J_\alpha \cap \wp(\omega)$ is a β-model. As a corollary we find that

If $J_\alpha \cap \wp(\omega) \models \Sigma_2^1$-CA, then $J_\alpha \cap \wp(\omega)$ possesses a cofinal tower of Σ_2^1-elementary subsystems (each satisfying thus Δ_2^1-CA).

Using another proposition of Artigue, Isambert, Perrin and Zalc, namely

PROPOSITION. *Theories* Δ_2^1-CA *and* KP+"*Mostowski Contraction lemma*" +V = HC *are bicommutable.*

We get