A class of infinite-dimensional spaces
Part I: Dimension theory and Alexandroff's Problem

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Abstract. A class of spaces is introduced whose dimension like properties are investigated. The class is large enough to contain all finite dimensional and some infinite dimensional ones. A reasonable dimension theory is worked out, examples and some limitations are noted. Contact is made with Alexandroff's Problem.

In part II, extensions of results in the theory of retracts to the infinite dimensional case will be given.

0. Introduction. In the 1973 Topology Conference at Virginia Polytechnic Institute William Haver [4] defined a covering property for metric spaces which he called *property C* (see the closing section here). He developed this concept in his efforts to prove that the space of piecewise linear homeomorphisms of a compact PL manifold onto itself is an ANR. Haver proved that a countable-dimensional metric space has property C, and that a locally contractible metric space which is a countable union of compact spaces each having property C is an ANR.

In this paper we have reformulated Haver's definition of property C so that it will have meaning in general topological spaces. We then consider property C as it relates to the problem of classifying infinite-dimensional spaces, and we also state (proof to appear elsewhere) an extension theorem which generalizes Haver's results. We are particularly concerned here with developing a satisfactory dimension theory for spaces having property C, and we give examples which indicate some limitations. Also of interest to us is the relation of property C to Alexandroff's Problem (see Nagata [10], p. 162). Vainstein [14] has also studied this rather difficult question.

A word is in order now concerning definitions, conventions, and notations. We follow the terminology found in Kelley, including the absence of blanket assumptions of any separation axioms (but as in Kelley, a paracompact space is regular). We reserve the word *refinement* for covers only and use the term *weak refinement* when dealing with families which may not be covers. Ordinal spaces are written with interval notation. Dimension means covering dimensions as defined in Nagata [10] or Nagami [9].

1. SCN spaces. In order to establish a useful dimension theory for C-spaces, the underlying space usually needs to satisfy a normality condition which we call *strong complete normality.*

1.1. DEFINITION. A family $\{F_\alpha: \alpha \in A\}$ of subsets of a space X is said to be *separated* if for each α in A,

$$\bar{F}_\alpha \cap (\bigcup_{\beta \neq \alpha} F_\beta) = \emptyset \quad \text{and} \quad F_\alpha \cap (\overline{\bigcup_{\beta \neq \alpha} F_\beta}) = \emptyset .$$

1.2. DEFINITION. A space X is *strongly completely normal* (SCN) if for every separated family $\{F_\alpha: \alpha \in A\}$ in X there exists a disjoint open family $\{U_\alpha: \alpha \in A\}$ such that $F_\alpha \subset U_\alpha$ for each α in A.

Recalling that a discrete family is a family in which each point of the space has a neighborhood meeting at most one member of the family, and that a collectionwise normal space is a space in which discrete families can be separated by open sets, it is routine to verify the

1.3. THEOREM. *A space X is* SCN *if and only if it is hereditarily collectionwise normal.*

1.4. COROLLARY. *Strong complete normality is hereditary.*

1.5. COROLLARY. *Metric spaces are* SCN.

But the space or real numbers with the $[a, b)$ topology is a nonmetric SCN space. In view of Theorem 1.3, Steen [13] has proved the following

1.6. THEOREM. *An ordered topological space is* SCN.

Mary Ellen Rudin [12] has given an example of a collectionwise normal Dowker space. It is not known whether this space is hereditarily collectionwise normal. SCN spaces have also been studied by McAuley [8].

2. Dimension theory for C-spaces. Definition and elementary properties. In this section we define a class of topological spaces and establish a dimension theory for this class.

2.1. DEFINITION. A topological space X is said to be a *C-space* (to have *property C*) if for every sequence $\{\mathscr{C}_i\}_{i=1}^\infty$ of open covers of X there exists a sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ of families such that

(1) each $\mathscr{U}_i$ is a pairwise disjoint collection of open sets,

(2) for each i, if $U \in \mathscr{U}_i$, then $U \subset G$ for some member G of $\mathscr{C}_i$, and

(3) the family $\bigcup_{i=1}^\infty \mathscr{U}_i$ is a cover of X.

The sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ is called a *C-refinement* of $\{\mathscr{C}_i\}_{i=1}^\infty$.

The following propopositions are easy consequences of the definition. Their proofs are left to the reader.

2.2. PROPOSITION. *A C-space is screenable (every open cover has a σ-disjoint open refinement).*

2.3. PROPOSITION. *A closed subspace of a C-space is a C-space.*

2.4. PROPOSITION. *If every G_δ subspace of a space X has property C then every subspace of X has property C.*

2.5. PROPOSITION. *If X is a C-space, then so is its one-point compactification.*

2.6. PROPOSITION. *A free union of C-spaces is a C-space.*

The following theorem is of fundamental importance in establishing a dimension theory for C-spaces.

2.7. THEOREM (Countable Sum Theorem). *Let X be an* SCN *space such that $X = \bigcup_{m=1}^\infty X_m$ and each X_m is a C-space (in the relative topology). Then X is a C-space.*

Proof. Let $\{\mathscr{C}_n\}_{n=1}^\infty$ be a sequence of open covers of X. Let $g: N \times N \to N$ be a bijection ($N = \{1, 2, 3, \ldots\}$) and let $\mathscr{C}_{(i,j)}$ denote the cover $\mathscr{C}_{g(i,j)}$. Fix an integer i and consider the family $\{\mathscr{C}_{(i,j)}\}_{j=1}^\infty$ as a sequence of open covers of X_i. Since X_i is a C-space, there is a C-refinement $\{\mathscr{U}_{(i,j)}\}_{j=1}^\infty$ of $\{\mathscr{C}_{(i,j)}\}_{j=1}^\infty$ relative to X_i. Each family $\mathscr{U}_{(i,j)}$ is a disjoint collection of relatively open subsets of X_i, and therefore is a separated family in X. Hence there is a disjoint family $\{F(U): U \in \mathscr{U}_{(i,j)}\}$ of open subsets of X such that $U \subset F(U)$ for each $U \in \mathscr{U}_{(i,j)}$. On the other hand, for each $U \in \mathscr{U}_{(i,j)}$ there is a member $G(U)$ of $\mathscr{C}_{(i,j)}$ which contains U. Let

$$\mathscr{V}_{(i,j)} = \{F(U) \cap G(U): U \in \mathscr{U}_{(i,j)}\} .$$

A C-refinement $\{\mathscr{W}_n\}_{n=1}^\infty$ of $\{\mathscr{C}_n\}_{n=1}^\infty$ is then defined by renumbe ing the $\mathscr{V}_{(i,j)}$: $\mathscr{V}_{(i,j)} = \mathscr{W}_n$ where $n = g(i,j)$. Q.E.D.

The assumption of collectionwise normality cannot be omitted from the hypothesis of the countable sum theorem. Bing's example G is a normal, non-collectionwise normal space which is the union of two subspaces which are relatively discrete. However, this space is not screenable, and hence not a C-space. This example was pointed out by G. M. Reekie.

2.8. COROLLARY. *F_σ subspaces of* SCN *C-spaces have property C.*

2.9. PROPOSITION. *A paracompact space X such that* $\dim X = 0$ *is a C-space.*

Proof. In such spaces, arbitrary open covers have disjoint open refinements which are covers. Q.E.D.

2.10. COROLLARY. *A countable-dimensional hereditarily paracompact space is a C-space.*

The decomposition theorem assures us that a finite-dimensional metric space has property C. In fact, a finite-dimensional paracompact space has property C, but an entirely different approach than that employed in the metric case must be used.

2.11. PROPOSITION (Ostrand [11], p. 213). *Let X be a normal space and $\mathscr{U} = \{U_\alpha: \alpha \in A\}$ a locally finite open cover of X of order $\leqslant n+1$. There exists for each t, $1 \leqslant t \leqslant n+1$ a family $\mathscr{V}_t = \{V_{(t,\alpha)}: \alpha \in A\}$ of pairwise disjoint open sets such that $\mathscr{V}_t$ shrinks $\mathscr{U}$ and $\bigcup_{t=1}^{n+1} \mathscr{V}_t$ covers X.*

2.12. PROPOSITION. *A paracompact space has dimension $\leqslant n$ if and only if every finite sequence $\{\mathscr{C}_i\}_{i=1}^{n+1}$ of open covers of X has a finite C-refinement $\{\mathscr{U}_i\}_{i=1}^{n+1}$.*

Proof. (Necessity) Form the open cover $\mathscr{U} = \{\bigcap_{i=1}^{n+1} G_i \colon G_i \in \mathscr{C}_i, 1 \leqslant i \leqslant n+1\}$. Without loss of generality, this cover is locally finite and is of order $\leqslant n+1$. Let $\{\mathscr{U}_i\}_{i=1}^{n+1}$ be the disjoint open families provided by Ostrand's proposition.

(Sufficiency) Given a finite open cover $\mathscr{C}$, simply set $\mathscr{C}_i = \mathscr{C}$ for each i. The cover $\bigcup_{i=1}^{n+1} \mathscr{U}_i$ is a refinement of order $\leqslant n+1$. Q.E.D.

Property C in Lindelöf spaces. In Lindelöf spaces we obtain a characterization of property C which is analogous to the small inductive characterization of finite dimension in separable metric spaces.

2.13. THEOREM. *Let X be a Lindelöf* SCN *space. The following statements are equivalent.*

(a) *X is a C-space.*

(b) *X has a basis of open sets whose boundaries have property C.*

(c) *Every open cover of X has a refinement consisting of open sets whose boundaries are C.*

Proof. Implications (a)⇒(b) and (b)⇒(c) being clear, we prove that (c)⇒(a). Let $\{\mathscr{C}_i\}_{i=1}^{\infty}$ be a sequence of open covers of X. Let $\{U_j\colon j = 0, 1, 2, \ldots\}$ be a countable refinement of $\mathscr{C}_1$ such that $\operatorname{Bd} U_j$ is a C-space for each j. Define sets V_j inductively by setting $V_0 = U_0$ and $V_m = U_m - \bigcup_{j<m} \overline{U}_j$. The family $\{V_j\colon j = 0, 1, 2, \ldots\}$ is a disjoint open family which refines $\mathscr{C}_i$, and it is not hard to show that the sets V_j cover all of X but at most $\bigcup_{j=0}^{\infty} \operatorname{Bd} U_j$, the latter being a C-space by the countable sum theorem. Now we consider the sequence $\{\mathscr{C}_i\}_{i=2}^{\infty}$ of open covers of $\bigcup_{j=0}^{\infty} \operatorname{Bd} U_j$ and by using the strong collectionwise normality of X as it was used in the proof of the countable sum theorem, we obtain a sequence $\{\mathscr{U}_i\}_{i=2}^{\infty}$ of disjoint open (in X) collections in which members of $\mathscr{U}_i$ are contained in members of $\mathscr{C}_i$ and $\bigcup_{i=2}^{\infty} \mathscr{U}_i$ covers $\bigcup_{j=0}^{\infty} \operatorname{Bd} U_j$. Thus $\{\mathscr{U}_i\}_{i=1}^{\infty}$ C-refines $\{\mathscr{C}_i\}_{i=1}^{\infty}$. Q.E.D.

2.14. EXAMPLE. $[0, \Omega]$ is a C-space. $[0, \Omega]$ is Lindelöf, SCN, and has a basis of open and closed sets (sets with boundary $\emptyset$). Alternatively, note that $\dim [0, \Omega] = 0$.

2.15. EXAMPLE. $[0, \Omega)$ is not a C-space since it is not screenable. Thus the requirement that X be Lindelöf cannot be omitted from Theorem 2.13.

Property C in paracompact spaces. As in all good dimension theories, the property is a local one. We prove:

2.16. LOCAL THEOREM. *Let X be a paracompact* SCN *space. Then X has property C if and only if every point of X has a neighborhood which has property C.*

Proof. If X is C, then X is a neighborhood of each of its points which is a C-space.

For the converse, first take note that every open cover of a paracompact space has a σ-discrete closed refinement. Now let $\mathscr{U}$ be a cover of X by neighborhoods which are C-spaces. Let $\mathscr{V} = \bigcup \mathscr{V}_i$, $i \in N$, be a σ-discrete closed refinement of with each $\mathscr{V}_i$ a discrete family of closed sets, all of which are C-spaces. We know that the union of the members of each $\mathscr{V}_i$ is a C-space, and hence X, as a countable union of these, is a C-space. Q.E.D.

This result leads to the

2.17. SUM THEOREM. *Let X be a paracompact* SCN *space. If $\{F_\alpha\colon \alpha \in A\}$ is a locally countable cover of X by C-spaces, then X is a C-space.*

Proof. For each x in X, let $N(x)$ be a closed neighborhood of x meeting at most countably many F_α. As a closed subset of a countable union of C-spaces in an SCN space, $N(x)$ is also a C-space. The local theorem provides the last line of the proof. Q.E.D.

2.18. Remarks. The hypothesis that X be paracompact is needed in both results above. Consider the regular SCN space $[0, \Omega)$. If $\alpha < \Omega$, then $[0, \alpha]$ is an open countable neighborhood of α and hence is a C-space, but $[0, \Omega)$ itself is not a C-space.

Furthermore if for each $\alpha < \Omega$ we let α^* be the least limit ordinal strictly greater than α, $\{[\alpha, \alpha^*)\colon \alpha < \Omega\}$ is a locally countable cover of $[0, \Omega)$ by C-spaces since each $[\alpha, \alpha^*)$ is countable. Unfortunately, as before, $[0, \Omega)$ is not C.

2.19. THEOREM. *Let X be a regular* SCN *space. The following are equivalent.*

(a) *X is a paracompact C-space.*

(b) *Every open cover of X has an open σ-discrete (σ-locally finite) refinement by open sets whose boundaries are C-spaces.*

(c) *Every open cover of X has an open locally finite refinement by sets whose boundaries are C-spaces.*

Proof. It is clear that (a) implies (b) and that (a) implies (c) and (c) implies the second part of (b). Thus we now assume that every open cover of X has a σ-locally finite refinement by open sets whose boundaries are C-spaces and prove (a). From this position it is well known that X is paracompact, so we proceed to prove property C. Let $\{\mathscr{C}_i\}_{i=1}^{\infty}$ be a sequence of open covers of X. Consider $\mathscr{C}_1$ and let $\mathscr{W} = \bigcup_{m=1}^{\infty} \mathscr{W}_m$ be a σ-locally finite refinement with the boundary of each W in $\mathscr{W}_m$ a C-space. Next well order each $\mathscr{W}_m$ and then well order $\mathscr{W}$ so that if $V, W \in \mathscr{W}$ then $V \leqslant W$ in case (1) $W \in \mathscr{W}_j$ and $V \in \mathscr{W}_k$ and $j<k$ or (2) W, V are in $\mathscr{W}_m$ and $V \leqslant W$ in the ordering in $\mathscr{W}_m$. For each W in $\mathscr{W}$, define

$$V(W) = W - \bigcup \{\overline{U}\colon U \in \mathscr{W} \text{ and } U < W\}.$$

Because $\mathscr{W}$ is well ordered in such a way that every initial segment is closure preserving, $\{V(W)\colon W\in\mathscr{W}\}$ is a weak open refinement of $\mathscr{C}_1$ and

$$X-\bigcup\{V(W)\colon W\in\mathscr{W}\}$$

is contained in $\bigcup\{\mathrm{Bd}\,W\colon W\in\mathscr{W}\}$. This last union is, in a trivial way, given as a locally countable union of C-spaces; hence by the Sum Theorem it is a C-space. Now $\mathscr{C}_2, \mathscr{C}_3, \ldots$ forms a sequence of covers of this union of boundaries and hence a C-refinement of these covers together with $\{V(W)\colon W\in\mathscr{W}\}$ forms a C-refinement of $\{\mathscr{C}_i\}_{i=1}^{\infty}$. Q.E.D.

Note that the Sum Theorem is employed here in a manner similar to that in which the Countable Sum Theorem is used in Theorem 2.13.

Property C in metric spaces. Using the standard metrization theorems, characterizations of metric C-spaces are easily produced. For example:

2.20. THEOREM. *The following are equivalent conditions on X.*

(a) *X is a metrizable C-space.*

(b) *X is a regular T_1 space with a σ-locally finite (σ-discrete) base of open sets whose boundaries are C.*

The spirit of dimension theory appears next in the form of large inductive dimension.

2.21. THEOREM. *Let X be a metric space. X is a C-space if and only if for every disjoint pair $\{A, B\}$ of closed sets in X, there exists an open set U such that $A\subset U\subset X-B$ and $\mathrm{Bd}\,U$ is a C-space.*

Proof. In the nontrivial direction let $\mathscr{B}=\bigcup_{i=1}^{\infty}\mathscr{U}_i$ be a σ-discrete base for X. Each $U\in\mathscr{U}_i$ can be written as $U=\bigcup_{j=1}^{\infty}F(U,j)$ with each $F(U,j)$ closed in X because X is perfectly normal. By assumption there exists open sets $V(U,j)$ with $F(U,j)\subset V(U,j)\subset U$ with $\mathrm{Bd}\,V(U,j)$ a C-space. It is clear that

$$\mathscr{V}_{(i,j)}=\{V(U,j)\colon U\in\mathscr{U}_i\}$$

is a discrete family and that $\{\mathscr{V}_{(i,j)}\colon i,j\in N\}$ is a σ-discrete base of open sets with C boundaries. From above X is a C-space. Q.E.D.

Products of C-spaces. The last effort in our dimension theory is to produce a product theorem. It is clear that if $X\times Y$ is a C-space then the factors X and Y must also be C-spaces. Unfortunately the converse of this observation is false in general.

2.22. EXAMPLE. Let X be the set of real numbers with the topology generated by the set of right half-open intervals. Each such interval is both open and closed, hence X is a regular Lindelöf space with $\mathrm{ind}\,X=0$. Standard arguments show that X is SCN, and we know these conditions imply that X is a C-space. Next we show that $X\times X$ is not a C-space by showing that $X\times X$ is not screenable. To this end, let $\mathscr{C}$ be the cover of $X\times X$ given by $\{(x,y)\colon y\neq -x\}\cup\{[x,-x+1)\times[x,-x+1)\colon x\in X\}$. Now suppose a screening exists for this sequence, say $\{\mathscr{U}_i\colon i\in N\}$. Each neighborhood of a point $(x,-x)$ must contain a set of the form $[x,-x+\varepsilon)\times[x,-x+\varepsilon)$ which we call an ε-square at x. Since the members of $\mathscr{U}_i$ are pairwise disjoint, each $U\in\mathscr{U}_i$ can contain at most one ε-square with a given vertex $(x,-x)$. It follows that for a fixed $\varepsilon>0$, there are at most countably many ε-squares refining $\mathscr{U}_i$, and therefore as ε ranges through $1,\frac{1}{2},\frac{1}{3},\ldots$, $\mathscr{U}_i$ contains only countable many members meeting the line $y=-x$. As i ranges through N, $\bigcup\{\mathscr{U}_i\colon i\in N\}$ can cover only a countable subset of the line $y=-x$. Evidently $X\times X$ is not screenable and hence not a C-space.

On the other hand, we are able to show the following

2.23. THEOREM. *Let X be a C-space such that every open cover of X has a precise open F_σ refinement. Let Y be a compact C-space such that*

(a) *$X\times Y$ is* SCN.

(b) *There is a basis $\mathscr{B}$ for Y such that for all $B\in\mathscr{B}$ $X\times\mathrm{Bd}\,B$ is a C-space. Then $X\times Y$ is a C-space.*

Remark. Both G_δ spaces and paracompact spaces satisfy the covering condition specified on X. Thus when X and Y are metric C-spaces all that is required is that Y be compact and satisfy (b) an "ind" condition. In particular, the hypotheses are satisfied when X is metric and $Y=I$, the unit interval.

Proof. We can assume that $\{\mathscr{C}_i\}_{i=1}^{\infty}$ is a sequence of open covers of $X\times Y$ all of whose elements are of the form $V\times B$ where V is open in X and $B\in\mathscr{B}$. We call such sets boxes and the factors V and B, x-sides and y-sides respectively. Now for a given $n\in N$ and x in X, $\{x\}\times Y$ can be covered by finitely many boxes from $\mathscr{C}_n$. We fix such a cover for each x and let $U(x,n)$ be the intersection of the x-sides. The collection $\mathscr{D}_n=\{U(x,n)\colon x\in X\}$ is an open cover of X and hence there is a C-refinement $\{\mathscr{U}_n\colon n \text{ odd}\}$ for the sequence $\{\mathscr{D}_n\colon n\text{ odd}\}$. Without loss of generality we can suppose that each element U of $\mathscr{U}_n$ is an F_σ set for every n. Consequently if $B\in\mathscr{B}$ then $U\times\mathrm{Bd}\,B$ is a C-space because of the countable sum theorem and condition (b). To exhibit a C-refinement consider $U\in\mathscr{U}_n$ and choose some x in X with $U\subset U(x,n)$. Now $\{x\}\times Y$ is covered by finitely many boxes previously fixed, say $\{V_i\times B_i\}_{i=1}^{m}$, in such a way that

$$\bigcup\{U\times B_i\colon 1\leqslant i\leqslant m\}\subset\bigcup\{V_i\times B_i\colon 1\leqslant i\leqslant m\}.$$

Next set $W_1=U\times B_1$ and for $x\geqslant 2$, set

$$W_k=U\times B_k-\bigcup_{j<k}\overline{U\times B_j}.$$

Some routine set containments show that

$$U\times Y-\bigcup\{W_k\colon 1\leqslant k\leqslant m\}\subset\bigcup\{U\times\mathrm{Bd}\,B_j\colon 1\leqslant j\leqslant m\},$$

this last union being a C-space. Note the collection of W's weakly refine $\mathscr{C}_n$. Letting U now vary through $\mathscr{U}_n$, the entire collection $\mathscr{W}_n$ of W's produced, will be a weak refinement of $\mathscr{C}_n$ covering all of $\bigcup\{U\times Y\colon U\in\mathscr{U}_n\}$ except for a C-subspace in $X\times Y$. Thus, since $\bigcup\{\mathscr{U}_n\colon n \text{ odd}\}$ covers X, the family $\bigcup\{\mathscr{W}_n\colon n \text{ odd}\}$ covers all of $X\times Y$ except for at most a C-space. Using $\{\mathscr{C}_n\colon n \text{ even}\}$ as a sequence of covers for the residual C-space it is easy to fill out the $\mathscr{W}$-sequence to $\{\mathscr{W}_n\colon n\in N\}$ achieving the required refinement. Q.E.D.

As a result of the product theorem we obtain the following catalogue of product C-spaces.

2.24 Corollary. *Suppose that X is a C-space for which every open cover has precise open F_σ refinement. For any space Y such that $X\times Y$ is* SCN, *$X\times Y$ is a C-space provided*

(1) *Y is a countable union of discrete subspaces, or*

(2) *Y is σ-compact and* $\operatorname{ind} Y = 0$, *or*

(3) *Y is a locally compact finite-dimensional metric space.*

Proof. The first conclusion follows directly from the Sum Theorem for SCN C-spaces. In the second case if $Y=\bigcup\{Y_n\colon n\in N\}$ with each Y_n compact, we see that $\operatorname{ind} Y_n = 0$ for each n and hence we can use the Product Theorem and the Sum Theorem again. Finally for the third case, use of the Sum Theorem shows that $X\times Y$ is a C-space whenever Y is a closed subset of R^n (i.e. a finite-dimensional locally compact separable metric space) since it holds for $[0,1]$ (by the Product Theorem). From Dugundji ([2], p. 241) we have that every locally compact paracompact space is a free union of σ-compact locally compact subspaces and hence that Y is a free union of locally compact finite-dimensional separable metric spaces, say $Y=\bigcup_\alpha Y_\alpha$. But $X\times Y=\bigcup_\alpha X\times Y_\alpha$ and thus $X\times Y$ is a C-space. Q.E.D.

3. Property C and Alexandroff's Problem.

3.1. Definition. A space X is said to be *weakly infinite-dimensional in the sense of Alexandroff (in the sense of Smirnov)* if for every sequence $\{(A_i, B_i)\}_{i=1}^\infty$ of disjoint pairs of closed sets there exists a sequence $\{U_i\}_{i=1}^\infty$ of open sets such that $A_i\subset U_i \subset X-B_i$ for each i and $\bigcap_{i=1}^\infty \operatorname{Bd} U_i=\emptyset$ (respectively, $\bigcap_{i=1}^k \operatorname{Bd} U_i=\emptyset$ for some positive integer k). Weak infinite-dimensionality will always be understood to be in the sense Alexandroff unless stated explicitly otherwise.

Note that in compact spaces, the two notions of weak infinite-dimensionality are equivalent.

3.2. Theorem. *A normal C-space is weakly infinite-dimensional.*

Proof. Assume, to the contrary, that a normal C-space X is not weakly infinite-dimensional. Then there exists a sequence $\{(A_i, B_i)\}_{i=1}^\infty$ of disjoint pairs of closed sets such that whenever $\{U_i\}_{i=1}^\infty$ is a sequence of open sets with $A_i\subset U_i\subset X-B_i$ for all i, then $\bigcap_{i=1}^\infty \operatorname{Bd} U_i\neq\emptyset$. For each i we define a cover $\mathscr{C}_i$ of X as follows: let V_i be an open set containing B_i such that $\overline{V}_i\cap A_i=\emptyset$ and set $\mathscr{C}_i=\{V_i, X-B_i\}$. Let $\{\mathscr{U}_i\}_{i=1}^\infty$ be a C-refinement of $\{\mathscr{C}_i\}_{i=1}^\infty$. For each i define sets U_i, W_i by

$$U_i=\bigcup\{A\in\mathscr{U}_i\colon A\subset X-B_i\}\cup(X-\overline{V}_i)$$

and

$$W_i=\bigcup\{A\in\mathscr{U}_i\colon A\cap B_i\neq\emptyset\}.$$

Now we observe:

(1) U_i is an open set with $A_i\subset U_i\subset X-B_i$.

(2) W_i is an open set with $W_i\subset V_i$; if $A\in\mathscr{U}_i$ and $A\cap B_i\neq\emptyset$ then $A\not\subset X-B_i$ and so by the C-refinement property, $A\subset V_i$.

(3) $U_i\cap W_i=\emptyset$. This follows from the disjointness of the family $\mathscr{U}_i$. Because $\bigcup_{i=1}^\infty\mathscr{U}_i$ is a cover of X, we have

$$X=\bigcup_{i=1}^\infty U_i\cup\bigcup_{i=1}^\infty W_i.$$

Since X is not weakly infinite-dimensional, $\bigcap_{i=1}^\infty \operatorname{Bd} U_i\neq\emptyset$, let $x\in\bigcap_{i=1}^\infty \operatorname{Bd} U_i$. Then $x\notin\bigcup_{i=1}^\infty U_i$ since each U_i is open. Hence for some j, $x\in W_j$ and thus $\operatorname{Bd} U_j\cap W_j\neq\emptyset$. Since W_j is open, $U_j\cap W_j\neq\emptyset$ a contradiction to (3). Hence X must be weakly infinite-dimensional. Q.E.D.

3.3. Corollary. *Hilbert Cube I^∞ is not a C-space.*

Proof. Nagata [10], pp. 164–165, shows that Hilbert Cube is not weakly infinite-dimensional. Q.E.D.

The question of whether the concepts of countable dimensionally and weak infinite dimensionality are equivalent in compact metric space is known as *Alexandroff's Problem* [1]. We have shown that in metric spaces,

$$\text{countable dimensionality} \Rightarrow \text{Property } C \Rightarrow \text{weak infinite dimensionality}$$

The reversibility of either implication is an open question.

4. Applications. The principal application of the foregoing theory is the following general extension theorem. The proof involves techniques using nerves of carefully chosen covers and will be presented in a separate paper.

4.1. Theorem. *Let Y be a metrizable locally contractible space. Suppose that X is a metrizable space, A a closed subspace of X such that* $\operatorname{Bd} A$ *has property C, and $f\colon A\to Y$ a map. Then there exists a neighborhood U of A in X and a continuous extension $F\colon U\to Y$ of f. If Y is also contractible, we may take $U=X$.*

4.2. COROLLARY. *A locally contractible metrizable countable-dimensional space is an* ANR.

This theorem generalizes results of Haver [3], [4]. The theory of C-spaces and these methods can also be developed to yield some somewhat specialized results about the existence of near-selections and selections. See [4], p. 5, and [5].

5. Comments and open problems. The results in this paper indicate that the theory of C-spaces is a dimension theory for a large class of spaces. The examples show that for the most part propositions are as general as can be expected. But there are obvious questions and difficulties arising in nice surroundings.

For one, a dimension concept on a space X, should be inhereted by all subspaces. By example this is not true if $X = [0, \Omega]$. The situation is unknown for metric spaces and this uncertainty could be resolved either way. Metric C-spaces, being more general than finite- or countable-dimensional spaces but less general than weakly infinite-dimensional spaces, are found between a class in which dimension is hereditary and one in which dimension shares some of the characteristics of property C. See Vainstein [14].

PROBLEM 1. Is property C hereditary in metrizable C-spaces?

The difficulty is in part revealed by a retreat to Haver's [4] original definition of C-space which we paraphrase: A metric space (X, d) is a "C" space if for each sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ of positive numbers there is a C-refinement (in the sense of this paper) for the sequence whose nth cover is the open cover of X by ε_n-neighborhoods at each point. It is a trivial observation that this property is hereditary to all metric subspaces. The authors have generalized this definition in the natural way and defined the concept on uniform spaces. The uniformly C (UC) concept is hereditary to subspaces with the induced uniformity. However, $[0, \Omega]$ rises again with its unique uniformity yielding a UC space, implying that $[0, \Omega)$ is UC but not C.

PROBLEM 2. Is there a metrizable space X with metrics D and d so that (X, D) is UC but (X, d) is not UC? (*A* negative answer settles Alexandroff's Problem.)

Another difficulty arises in attempting to achieve a product theorem. The product of two countable-dimensional spaces is again countable-dimensional. By example this is not true for general C-spaces.

PROBLEM 3. Is the product of a metrizable C-space and the set of irrational real numbers again a C-space?

PROBLEM 4. Is the product of any two metrizable C-spaces again a C-space?

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