increasing sequence of functions in ω with infinitely many different terms could be constructed. Denote the least member of λ_x as f_x. Suppose g is a function in ω which is not disjoint from f_x. Then either $\min(f_x, g)$ or $f_x - \min(f_x, g)$ is 1 at x and if $f_x - \min(f_x, g)$ is not the zero function, f_x is not the least element of λ_x. Thus $g \geqslant f_x$. It follows that the collection of all f_x for all x in X is disjoint and therefore finite. Then each function in ω is the sum of a finite number of f_x. Since, by Theorem 7, each function in K^* is the uniform limit of a sequence of functions each of which is a finite linear combination of functions in ω, K^* is finite dimensional.

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Concerning atriodic tree-like continua

by

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Abstract. In this paper it is shown that there is a collection G of atriodic tree-like continua such that if M is a compact metric continuum then there is a member of G which is not a continuous image of M. Thus, there are atriodic tree-like continua which are not weakly chainable.

1. Introduction. In 1934 Z. Waraszkiewicz [6] presented a collection of continua with the property that no compact metric continuum can be mapped onto every member of the collection. Each continuum in the collection is planar, and each contains a simple closed curve.

Russo [5] proved that there is a collection of tree-like continua with the property that no compact metric continuum can be mapped onto every member of the collection. Each continuum in this collection is planar and each contains a simple triod.

In this paper we show that there is a collection of atriodic tree-like continua with the property that no compact metric continuum can be mapped onto every member of the collection. The members of this collection can be embedded in the plane in such a way that they form a collection of mutally exclusive continua.

The proof of Theorem 2 depends heavily on results found in [3], and throughout this paper many references to that paper will be made.

2. Notation and conventions. As in [2] and [3], $T = \{(\varrho, \theta) |\ 0 \leqslant \varrho \leqslant 1$ and $\theta = 0$ $\theta = \frac{1}{2}\pi$, or $\theta = \pi\}$, O denotes $(0, 0)$, A denotes $(1, \frac{1}{2}\pi)$, B denotes $(1, \pi)$, and C denotes $(1, 0)$. (The author incorrectly labelled A, B, and C in [3].)

The mapping f of T onto T is as in [2, pp. 99–100] while r is as in [3, p. 75] and $g = rf$ (i.e. $g = r \circ f$).

Throughout this paper all spaces are metric and the term mapping means continuous function. The two projections of $X \times Y$ onto X and Y, respectively, are denoted by p_1 and p_2 while the projection of a product of sequence of spaces onto the ith factor space is denoted by π_i.

We will use the convention that if each of p and q is a positive integer and $p < q$ then pA/q denotes $(p/q, \frac{1}{2}\pi)$, pB/q denotes $(p/q, \pi)$, and pC/q denotes $(p/q, 0)$.

3. Main Theorems. The proof of the following lemma is essentially the same as the proof provided by Read [4, Lemma p. 236]. A proof is included here only for the sake of completeness.

LEMMA 1. *Suppose M is a compact metric continuum and φ is a mapping of M into T. If α is a subcontinuum of $\varphi(M)$ which is a subset of one of $[OA]$, $[OB]$, and $[OC]$, then some component of $\varphi^{-1}(\alpha)$ is thrown by φ onto α.*

Proof. Suppose α is the arc $[ab]$ of T. Since α is a subset of one of $[OA]$, $[OB]$, and $[OC]$, $(T-\alpha)\cup\{a,b\}$ is the union of two mutually exclusive closed point sets T_1 and T_2 with a in T_1 and b in T_2. If $\varphi\cap(M\times\alpha)$ does not contain a continuum irreducible from $\varphi\cap(M\times\{a\})$ to $\varphi\cap(M\times\{b\})$, then $\varphi\cap(M\times\alpha)$ is the union of two mutually exclusive closed point sets H and K containing $\varphi\cap(M\times\{a\})$ and $\varphi\cap(M\times\{b\})$, respectively. Let $P=H\cup(\varphi\cap p_2^{-1}(T_1))$ and $Q=K\cup(\varphi\cap p_2^{-1}(T_2))$. Then P and Q are mutually exclusive closed sets such that $\varphi=P\cup Q$. This is a contradiction and concludes the proof of Lemma 1.

Denote by G the collection to which the continuum M belongs if and only if there is an inverse limit sequence $\{T_n, f_n\}$ such that (1) M is the inverse limit of the sequence, (2) for each positive integer n T_n is T, and (3) for each n f_n is in $\{f, g\}$. Then G is an uncountable collection of continua in $T\times T\times T\times\ldots$ and G^* (the union of all the members of G) is compact. Denote by ϱ the usual metric for the product space $T\times T\times\ldots$ If φ_1 and φ_2 are mappings of a space X into $T\times T\times\ldots$,

$$d(\varphi_1,\varphi_2)=\text{l.u.b.}\{\varrho(\varphi_1(x),\varphi_2(x))|\ x \text{ is in } X\}$$

is the usual metric for the space of mappings of X into $T\times T\times\ldots$

Recall that a mapping φ of a continuum X onto a continuum Y is *weakly confluent* provided for each subcontinuum K of Y some component of $\varphi^{-1}(K)$ is thrown by φ onto K.

THEOREM 1. *If M is a continuum in G, then every mapping of a compact metric continuum onto M is weakly confluent.*

Proof. Suppose φ is a mapping of a compact metric continuum onto a member M of G, M is the inverse limit of the sequence $\{T_i, f_i^j\}$, and suppose K is a proper subcontinuum of M. By a proof of Read [4, proof of Theorem 4] to show that some component of $\varphi^{-1}(K)$ is thrown by φ onto K it is sufficient to show that there is a positive integer N such that if $n\geqslant N$ then some component of $(\pi_n\varphi)^{-1}(\pi_n(K))$ is thrown by $\pi_n\varphi$ onto $\pi_n(K)$.

First, we show that there is a positive integer N such that if $n\geqslant N$ then $\pi_n(K)$ does not contain O. To see this suppose that if p is a positive integer then there is an integer $q\geqslant p$ such that O is in $\pi_q(K)$. Then, if i is an integer, there is an integer $j\geqslant i+3$ such that O is in $\pi_j(K)$. Further, there is an integer $m\geqslant j+2$ such that O is in $\pi_m(K)$. Since $f_j^m(O)$ is in $\{B, C\}$ and $\pi_j(K)$ is a continuum, $[OB]$ or $[OC]$ is a subset of $\pi_j(K)$. Since $f_i^j[OB]=T$ and $f_i^j[OC]=T$, $\pi_i(K)=T$. Thus, for each i, $\pi_i(K)=T$, which contradicts the assumption that K is a proper subcontinuum of T.

Now, by Lemma 1, if $n\geqslant N$ some component of $(\pi_n\varphi)^{-1}(\pi_n(K))$ is thrown by $\pi_n\varphi$ onto $\pi_n(K)$.

THEOREM 2. *Suppose M_1 and M_2 are two members of G, L is a compact metric continuum, and φ_1 and φ_2 are mapping of L onto M_1 and M_2 respectively. Then $d(\varphi_1,\varphi_2)\geqslant\frac{1}{4}$.*

Proof. Suppose M_1 is the inverse limit of the inverse limit sequence $\{T_i, f_i^j\}$ and M_2 is the inverse limit of the inverse limit sequence $\{T_i, g_i^j\}$. Denote by n a positive integer such that $f_1^n=g_1^n$ while $f_n^{n+1}\neq g_n^{n+1}$ and suppose for the sake of argument that $f_n^{n+1}=f$ while $g_n^{n+1}=g$ (here f_i^i denotes the identity map of T onto T). Suppose L is a compact continuum and φ_1 and φ_2 are mappings of L onto M_1 and M_2 respectively, such that $d(\varphi_1,\varphi_2)<\frac{1}{4}$.

We now turn to [3] for several observations. By the proof of Lemma 3 of that paper, there is a continuum W in $T\times T$ such that if (x, y) is in W then the distance from $f_1^n(x)$ to $f_1^n(y)$ is not less than $\frac{1}{2}$. Moreover, W has property L [3, pp. 73–74] and thus contains continua

$$D_1=\langle OB, O\tfrac{1}{2}A\rangle\cup\langle OC, O\tfrac{1}{2}A\rangle,\quad D_2=\langle OC, OA\rangle,$$
$$D_3=\langle OC, OB\rangle\cup\langle OC, O\tfrac{1}{2}A\rangle,\quad D_4=\langle OB, OC\rangle\cup\langle OB, O\tfrac{1}{2}A\rangle \quad\text{and}$$
$$D_5=\langle OB, OA\rangle\cup\langle OB, OC\rangle.$$

That D_1, D_2, D_4 and D_5 are continua may be verified by noting the definition of property L. Also, recall that $\langle t, u\rangle^{-1}=\langle u, t\rangle$.

The following property of the continua $D_1, D_2, \ldots, D_5$ is used many times in the remainder of the proof of Theorem 2:

LEMMA 2. *If D is one of the continua $D_1, D_2, \ldots, D_5, D_1^{-1}, D_2^{-1}, \ldots, D_5^{-1}$ and J is a subcontinuum of L such that $\pi_n\varphi_1(J)$ is a subset of $p_1(D)$ and $\pi_n\varphi_2(J)$ is $p_2(D)$ (similarly, $\pi_n\varphi_1(J)$ is $p_1(D)$ and $\pi_n\varphi_2(J)$ is a subset of $p_2(D)$), then J contains a point x such that $\varrho(\varphi_1(x),\varphi_2(x))\geqslant\frac{1}{4}$.*

Proof. Note that D is a subcontinuum of the product of two arcs which projects onto both factors. If Δ is the diagonal of $J\times J$, then $E=(\pi_n\varphi_1\times\pi_n\varphi_2)(\Delta)$ is a continuum in that product of arcs. Since $\pi_n\varphi_1(J)$ is a subset of $p_1(D)$ and $\pi_n\varphi_2(J)$ is $p_2(D)$, there is a point belonging both to D and to E. If x is a point of J such that $(\pi_n\varphi_1(x), \pi_n\varphi_2(x))$ is in D, then the distance from $\pi_1\varphi_1(x)$ to $\pi_1\varphi_2(x)$ is not less than $\frac{1}{2}$. Thus, $\varrho(\varphi_1(x),\varphi_2(x))\geqslant\frac{1}{4}$. This establishes Lemma 2.

Denote by α an arc in M_1 with $\pi_{n+1}(\alpha)=[OC]\cup[O\frac{1}{2}B]$. Note that α is the union of two arcs α_1 and α_2 such that $\pi_n(\alpha_1)=[OB]\cup[OC]$, $\pi_n(\alpha_2)=[OB]\cup[O\frac{1}{2}A]$ and α_1 and α_2 intersect only in a common end point. By Theorem 1, there is a subcontinuum K of L such that $\varphi_1(K)$ is α.

We first show that:

(1) $\pi_{n+1}\varphi_2(K)$ contains a point of $(\frac{1}{2}CC]$.

Suppose H is a subcontinuum of K thrown by φ_1 onto α_1. Then $\pi_n\varphi_1(H)$ is $[OB]\cup[OC]$. Not both O and $\frac{1}{2}A$ belong to $\pi_n\varphi_2(H)$ for suppose both O and $\frac{1}{2}A$ belong to $\pi_n\varphi_2(H)$. By Lemma 1 there is a subcontinuum J of H such that $\pi_n\varphi_2(J)=[O\frac{1}{2}A]$. Since $\pi_n\varphi_1(J)$ is a subset of $p_1(D_1)$ and $\pi_n\varphi_2(J)$ is $p_2(D_2)$, by Lemma 2 there is a point x of J such that $\varrho(\varphi_1(x),\varphi_2(x))\geqslant\frac{1}{4}$. Then $d(\varphi_1,\varphi_2)\geqslant\frac{1}{4}$

which contradicts our assumption that $d(\varphi_1, \varphi_2) < \frac{1}{4}$. Therefore, $\pi_n\varphi_2(H)$ is a subset of one of $(OA]$ and $[OB] \cup [OC] \cup [O\frac{1}{2}A)$.

We show $\pi_n\varphi_2(H)$ is not a subset of $(OA]$ for suppose $\pi_n\varphi_2(H)$ is a subset of $(OA]$. By Lemma 1 H contains a subcontinuum J thrown by $\pi_n\varphi_1$ onto $[OC]$. Then, $\pi_n\varphi_1(J)$ is $p_1(D_2)$ and $\pi_n\varphi_2(J)$ is a subset of $p_2(D_2)$. As before, Lemma 2 leads to the contradiction that $d(\varphi_1, \varphi_2) \geqslant \frac{1}{4}$. Thus, $\pi_n\varphi_2(H)$ is a subset of $[OB] \cup [OC] \cup \cup [O\frac{1}{2}A)$.

Now we show $\pi_n\varphi_2(H)$ intersects both $(OB]$ and $(OC]$. Suppose $\pi_n\varphi_2(H)$ does not contain a point of $(OC]$. There is a subcontinuum J of H thrown by $\pi_n\varphi_1$ onto $[OC]$. Then $\pi_n\varphi_1(J)$ is $p_1(D_3)$ and $\pi_n\varphi_2(J)$ is a subset of D_3 which, as before, yields the contradiction that $d(\varphi_1, \varphi_2) \geqslant \frac{1}{4}$. Suppose $\pi_n\varphi_2(H)$ does not contain a point of $(OB]$. There is a subcontinuum J of H such that $\pi_n\varphi_1(J)$ is $[OB]$. Then $\pi_n\varphi_1(J)$ is $p_1(D_4)$ and $\pi_n\varphi_2(J)$ is a subset of $p_2(D_4)$ which leads to a contradiction.

Since $\pi_n\varphi_2(H)$ intersects $(OB]$ and $(OC]$, $\pi_n\varphi_2(H)$, is a subset of $[OB] \cup [OC] \cup \cup [O\frac{1}{2}A)$ and $g_n^{n+1} = g$, we conclude that $\pi_{n+1}\varphi_2(H)$ contains a point of $(\frac{1}{2}CC]$. Therefore, $\pi_{n+1}\varphi_2(K)$ contains a point of $(\frac{1}{2}CC]$.

Next we show that:

(2) $\pi_{n+1}\varphi_2(K)$ contains a point of $(\frac{2}{3}BB]$ or $(\frac{3}{4}AA]$.

Suppose H' is a subcontinuum of K thrown by φ_1 onto α_2. Then $\pi_n\varphi_1(H')$ is $[OB] \cup [O\frac{1}{2}A]$. Not both O and C belong to $\pi_n\varphi_2(H')$ for suppose both O and C belong to $\pi_n\varphi_2(H')$. By Lemma 1 there is a subcontinuum J of H' thrown by $\pi_n\varphi_2$ onto $[OC]$. Then $\pi_n\varphi_1(J)$ is a subset of $p_1(D_3^{-1})$ and $\pi_n\varphi_2(J)$ is $p_2(D_3^{-1})$. This yields the contradiction that $d(\varphi_1, \varphi_2) \geqslant \frac{1}{4}$. Thus, $\pi_n\varphi_2(H')$ is a subset of one of $(OC]$ and $T - \{C\}$.

We show $\pi_n\varphi_2(H')$ is not a subset of $(OC]$. In fact, we show $\pi_n\varphi_2(H')$ is not a subset of $[OB] \cup [OC]$. Since H' contains a continuum J thrown by $\pi_n\varphi_1$ onto $[O\frac{1}{2}A]$, if $\pi_n\varphi_2(H')$ is a subset of $[OB] \cup [OC]$, then $\pi_n\varphi_1(J)$ is $p_1(D_1^{-1})$ and $\pi_n\varphi_2(J)$ is a subset of $p_2(D_1^{-1})$. Again, this yields a contradiction.

Next, we show $\pi_n\varphi_2(H')$ contains a point of $(OB]$. By Lemma 1, H' contains a subcontinuum J thrown by $\pi_n\varphi_1$ onto $[OB]$. If $\pi_n\varphi_2(H')$ does not contain a point of $(OB]$, then $\pi_n\varphi_1(J)$ is $p_1(D_5)$ and $\pi_n\varphi_2(J)$ is a subset of $p_2(D_5)$. This yields a contradiction.

Since $\pi_n\varphi_2(H')$ is not a subset of $[OB] \cup [OC]$, C is not in $\pi_n\varphi_2(H')$, $\pi_n\varphi_2(H')$ contains a point of $(OB]$, and $g_n^{n+1} = g$, we conclude that $\pi_{n+1}\varphi_2(H')$ contains a point of $(\frac{2}{3}BB]$ or $(\frac{3}{4}AA]$. Therefore, $\pi_{n+1}\varphi_2(K)$ contains a point of $(\frac{2}{3}BB]$ or $(\frac{3}{4}AA]$.

By (1) and (2) $\pi_{n+1}\varphi_2(K)$ contains $[O\frac{1}{2}B]$ or $[O\frac{3}{8}A]$. In either case K contains a continuum K' thrown by $\pi_n\varphi_2$ onto $[OC] \cup [O\frac{1}{2}A]$.

We show that

(3) $\pi_n\varphi_1(K')$ does not contain both O and B.

If both O and B belong to $\pi_n\varphi_1(K')$, then K' contains a subcontinuum J thrown by $\pi_n\varphi_1$ onto $[OB]$. Since $\pi_n\varphi_1(J)$ is $p_1(D_4)$ and $\pi_n\varphi_2(J)$ is $p_2(D_4)$, we obtain a contradiction.

Since $\pi_n\varphi_1(K)$ is $[OB] \cup [OC] \cup [O\frac{1}{2}A]$ and $\pi_n\varphi_1(K')$ does not contain both O and B, $\pi_n\varphi_1(K')$ is a subset of $(OB]$ or $[OC] \cup [O\frac{1}{2}A] \cup [OB)$.

We show that $\pi_n\varphi_1(K')$ is not a subset of $(OB]$. In fact, $\pi_n\varphi_1(K')$ is not a subset of $[OB] \cup [OC]$. There is a subcontinuum J of K' thrown by $\pi_n\varphi_2$ onto $[O\frac{1}{2}A]$. If $\pi_n\varphi_1(K')$ is a subset of $[OB] \cup [OC]$, then $\pi_n\varphi_1(J)$ is a subset of $p_1(D_1)$ and $\pi_n\varphi_2(J)$ is $p_2(D_1)$. This yields a contradiction. Therefore, $\pi_n\varphi_1(K')$ intersects $(O\frac{1}{2}A]$.

Finally, $\pi_n\varphi_1(K')$ contains a point of $(OC]$. There is a subcontinuum J of K' thrown by $\pi_n\varphi_2$ onto $[OC]$. If $\pi_n\varphi_1(K')$ does not contain a point of $(OC]$, then $\pi_n\varphi_1(J)$ is a subset of $p_1(D_3^{-1})$ and $\pi_n\varphi_2(J)$ is $p_2(D_3^{-1})$. This yields a contradiction.

Since $\pi_n\varphi_1(K')$ intersects both $(O\frac{1}{2}A]$ and $(OC]$, $f_n^{n+1} = f$, and $\pi_{n+1}\varphi_1(K')$ is a subset of $[OC] \cup [O\frac{1}{2}B]$, we conclude that $\pi_{n+1}\varphi_1(K')$ contains a point of $(\frac{1}{2}CC]$ and a point of $(\frac{1}{3}B\frac{1}{2}B]$. This yields that $[O\frac{1}{2}C]$ is a subset of $\pi_{n+1}\varphi_1(K')$, and, thus, both O and B are in $\pi_n\varphi_1(K')$. This contradicts (3).

THEOREM 3. *If M is a compact metric continuum and G_0 is a subcollection of G such that M can be mapped onto every member of G_0, then G_0 is countable.*

Proof. The space mappings of M into G^* is a separable metric space [1]. If a compact metric continuum can be mapped onto every member of an uncountable subcollection of G, then by Theorem 2 that separable metric space contains an uncountable discrete set.

4. Remark. Uncountably many members of G are not a continuous image of the pseudo-arc (i.e. not weakly chainable). It would be interesting to know if some member of G is weakly chainable.

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