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# Representation of Baire functions as continuous functions

by

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**Abstract.** Suppose $H$ is a lattice ordered linear space of functions containing the constant functions, $K$ is the set of all pointwise limits of sequences of functions in $H$, and $\varphi$ is a linear lattice homomorphism defined on $K$. Then $\varphi$ preserves pointwise convergence of sequences. Further if $\varphi$ is one-to-one and onto $C(X)$ for some topological space $X$ then $K$ is closed with respect to pointwise convergence.

One of the objects studied in the theory of Baire functions is the Baire system generated by a space of continuous functions, i.e. given a topological space $X$ and $C(X)$ the collection of continuous real valued functions on $X$ the Baire system generated by $C(X)$ is the transfinite sequence $C(X), B_1(X), B_2(X), \ldots, B_\omega(X), \ldots$, where $B_1(X)$ is the set of pointwise limits of sequences of functions in $C(X)$, $B_2(X)$ is the set of pointwise limits of sequences of functions in $B_1(X)$, and in general if $\alpha$ is an ordinal $B_\alpha(X)$ is the set of pointwise limits of sequences of functions drawn from $\bigcup_{\beta<\alpha} B_\beta(X)$. See Mauldin [1] and [2] for a discussion of Baire systems. A question of interest is when can a term be added before $C(X)$, i.e. when does there exist a proper subset $H$ of $C(X)$ such that $C(X)$ is the set of all pointwise limits of sequences of functions in $H$?

Here this question is generalized to the representation of Baire functions as continuous functions. Given a lattice ordered linear space $H$ of functions containing the constant functions and $K$ the set of all pointwise limits of sequences of functions in $H$, when does there exists a one-to-one linear lattice homomorphism $\varphi$ of $K$ or $K^*$ (the set of bounded functions in $K$) onto $C(X)$ for some $X$. It is shown here (Theorem 6) that no such $\varphi$ can exist defined on $K$ unless $K$ is closed with respect to pointwise convergence. Thus if a term can be inserted before $C(X)$ in its Baire system, the sequence is constant from $C(X)$ on.

On the other hand, such a $\varphi$ can always be defined on $K^*$. If $\omega$ denotes the functions in $K^*$ which take on only the values 0 and 1, every function in $K^*$ is the uniform limit of a sequence of functions each of which is a linear combination of the functions in $\omega$ (Theorem 7). The functions in $\omega$ form a Boolean algebra, so by the Stone representation theorem they are isomorphic to the open and closed sets of a totally disconnected compact Hausdorff space $X$. The natural mapping between $K^*$ and

$C(X)$ is a one-to-one linear lattice homomorphism onto $C(X)$. However, if $\varphi$ is additionally required to preserve pointwise convergence of sequences of functions, then the existence of $\varphi$ from $K^*$ onto any $C(X)$ whatsoever implies that $K^*$ is finite dimensional (Theorem 8).

Suppose $X$ is a set, $H$ is a lattice ordered vector space of real valued functions defined on $X$ containing the constant functions, $K$ is the set of all pointwise limits of sequences of functions in $H$, $K^*$ is the set of all bounded functions in $K$, $Y$ is a set, and $L$ is a lattice ordered vector space of real valued functions defined on $Y$. Also $\varphi$ will denote a linear lattice homomorphism from either $K$ or $K^*$ into $L$.

THEOREM 1. *Suppose $f_1, f_2, f_3, \ldots$ is a non-increasing sequence of functions of $K$ which converges pointwise to the zero function. Then $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges pointwise to the zero function.*

Proof. (a) Suppose $g_1, g_2, g_3, \ldots$ is a non-increasing sequence of functions of $K$ which converges pointwise to the constant function $-1/r$, where $r$ is a positive integer. Denote by $\mathrm{LS}(H)$ the set of all functions which are the least upper bounds of countable subsets of $H$. For each positive integer $p$, let $\{h_{ip}\}_{i=1}^{\infty}$ be a sequence of points of $H$ converging pointwise to $g_p$. Let $k_{ip} = \max\limits_{j\geqslant i}\{h_{jp}\}$. Then $\{k_{ip}\}_{i=1}^{\infty}$ is a non-increasing sequence of points of $\mathrm{LS}(H)$ converging pointwise to $g_p$. Let $k_i = \min\limits_{p\leqslant i} k_{ip}$. Then $k_1, k_2, k_3, \ldots$ is a non-increasing sequence of functions in $\mathrm{LS}(H)$ converging pointwise to the constant function $-1/r$. Now $\max(0, k_i)$ is in $\mathrm{LS}(H)$ and $\sum\limits_{i=1}^{n}\max(0, k_i)$ is in $\mathrm{LS}(H)$. For each point $x$ of $X$ there is a positive integer $N$ such that $\max(0, k_n(x)) = 0$ for each positive integer $n \geqslant N$.

Thus for each $x$, $\sum\limits_{i=1}^{\infty}\max(0, k_i(x))$ exists. For each positive integer $n$ there exists a non-decreasing sequence $m_{1n}, m_{2n}, m_{3n}, \ldots$ of functions in $H$ converging pointwise to $\sum\limits_{i=1}^{n}\max(0, k_i)$. Let $m_n = \max\limits_{i\leqslant n} m_{ni}$. So that $m_1, m_2, m_3, \ldots$ is a sequence of functions in $H$ converging pointwise to $\sum\limits_{i=1}^{\infty}\max(0, k_i)$. Thus $\sum\limits_{i=1}^{\infty}\max(0, k_i)$ is in $K$ and $\sum\limits_{i=1}^{\infty}\max(0, k_i) \geqslant \sum\limits_{i=1}^{n} g_i$ for each positive integer $n$.

(b) Suppose $r$ is a positive integer. Since

$$\max(f_p - 1/r, 0) = f_p - \min(f_p, 1/r),$$

$$\varphi\Big(\sum_{p=1}^{n}\max(f_p - 1/r, 0)\Big) = \sum_{p=1}^{n}\varphi(f_p) - \varphi(\min(f_p, 1/r)).$$

By part (a) there exists a function $b$ of $K$ such that

$$b \geqslant \sum_{p=1}^{n}\max(f_p - 1/r, 0).$$



Thus

$$\varphi(b) \geqslant \varphi\Big(\sum_{p=1}^{n}\max(f_p - 1/r, 0)\Big) = \sum_{p=1}^{n}\varphi(f_p) - \varphi(\min(f_p, 1/r)).$$

Suppose $l \geqslant 0$ is a lower bound for $\varphi(f_p)$. Then

$$\varphi(b) \geqslant \sum_{p=1}^{n} l - \varphi(\min(f_p, 1/r)) \geqslant \sum_{p=1}^{n} l - \varphi(1/r) = n(l - \varphi(1/r))$$

which implies that $l - \varphi(1/r) \leqslant 0$. It follows that $0 \leqslant l \leqslant 1/r(\varphi(1))$. This implies that $l = 0$ and $\varphi(f_p)$ converges pointwise to zero.

THEOREM 2. *Suppose $\varphi$ is defined either on $K$ or $K^*$. Then $\varphi = \alpha\beta$ where $\beta$ is a linear lattice homomorphism with the property that $\beta(1) = 1$ and $\alpha$ is multiplication by a function in $L$.*

Proof. Suppose $\varphi(1)$ is not 1. Suppose further that $f$ is a function in $K$ and $x$ is a point of $X$ such that $\varphi(f)(x) \neq 0$ but $\varphi(1)(x) = 0$. The function $f$ may be assumed to be non-negative. It can not be true that $f$ is bounded because if there exists a positive integer $n$ such that $n \geqslant f$ then $\varphi(n)(x) = n\varphi(1)(x) \geqslant \varphi(f)(x)$. The sequence $\min(1, f)$, $\min(2, f)$, $\min(3, f), \ldots$ converges pointwise to $f$. By Theorem 1, $\varphi(\min(1, f))$, $\varphi(\min(2, f))$, $\varphi(\min(3, f)), \ldots$ converges pointwise to $\varphi(f)$. But $\varphi(\min(i, f))(x) = 0$ while $\varphi(f)(x) \neq 0$. Thus $\varphi(f)(x) = 0$.

For each $f$ in $\varphi(K)$ (or $\varphi(K^*)$) let $\gamma(f)(x) = f(x)/\varphi(1)(x)$ unless $\varphi(1)(x) = 0$ in which case $\gamma(f)(x) = 0$. Let $\beta$ be $\gamma\varphi$. Then $\beta$ is a linear lattice homomorphism such that $\beta(1) = 1$. Let $\alpha$ be multiplication by $\varphi(1)$. Thus $\varphi = \alpha\beta$.

THEOREM 3. *Suppose $f_1, f_2, f_3, \ldots$ is a sequence of functions in $K$ which converges pointwise to a function $f$. Then $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges pointwise. Furthermore, if $f$ is in $K$ then $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges pointwise to $\varphi(f)$.*

Proof. Since multiplication by a point of $L$ preserves pointwise convergence it may be assumed because of Theorem 2 that $\varphi(1) = 1$.

(a) Suppose $f_1, f_2, f_3, \ldots$ is non-increasing. Suppose, also, that $y$ is a point of $Y$ such that $\varphi(f_1)(y), \varphi(f_2)(y), \varphi(f_3)(y), \ldots$ does not converge. Then there exists a subsequence $f_{i_1}, f_{i_2}, f_{i_3}, \ldots$ such that $\varphi(f_{i_n})(y) < -(n+1)$. Thus

$$\varphi\big(-(\min(f_{i_n}, -n)+n)\big)(y) > 1.$$

Let $g_n = -(\min(f_{i_n}, -n)+n)$ and $h_n = \min(g_1, g_2, \ldots, g_n)$. Therefore

$$h_1 \geqslant h_2 \geqslant h_3 \geqslant \ldots \geqslant 0$$

and $h_1, h_2, h_3, \ldots$ converges pointwise to 0 but $\varphi(h_i)(y) > 1$ for each positive integer $i$. This contradicts Theorem 1.

(b) Suppose that for each point $x$ of $X$ $\sum\limits_{i=1}^{\infty}|f_i(x) - f_{i+1}(x)|$ is bounded. Then since $\{\sum_{i=1}^{n}|f_i-f_{i+1}|\}_{n=1}^{\infty}$ is a non-decreasing pointwise convergent sequence, by (a)

$$\{\varphi(\sum_{i=1}^{n}|f_i-f_{i+1}|)\}_{n=1}^{\infty} = \{\sum_{i=1}^{n}|\varphi(f_i)-\varphi(f_{i+1})|\}_{i=1}^{\infty}$$

is pointwise convergent.

(c) Suppose that for each point $x$ of $X$ there is a positive integer $N_x$ such that if $n$ is a positive integer greater than $N_x$ then $|f_n(x)|<1$. It is claimed that for each point $y$ of $Y$ there is a positive integer $\mathfrak{N}_x$ such that if $n$ is a positive integer greater than $\mathfrak{N}_x$ then $|\varphi(f_n)(y)|<2$. Suppose not. Then there exists a point $x$ of $X$ and subsequence $f_{i_1}, f_{i_2}, f_{i_3}, \ldots$ such that $|\varphi(f_{i_n})(y)|\geqslant 2$ for each positive integer $n$. Either there exists a subsequence $f_{j_1}, f_{j_2}, f_{j_3}, \ldots$ such that $\varphi(f_{j_n})(y)\geqslant 2$ or there exists a subsequence $f_{j_1}, f_{j_2}, f_{j_3}, \ldots$ such that $\varphi(f_{j_n})(y)\leqslant -2$ for each positive integer $n$. Suppose the former. Let $g_n = \max(f_{j_n}-1, 0)$. Let $k_n = \min_{i\leqslant n}(g_i)$. Thus $\varphi(g_n)(y)\geqslant 1$ and $\varphi(k_n)(y)\geqslant 1$. But $k_1\geqslant k_2\geqslant k_3\geqslant\ldots$ and $k_1, k_2, k_3, \ldots$ converges pointwise to zero. This contradicts Theorem 1.

It also follows that if $f_1, f_2, f_3, \ldots$ converges pointwise to a functi n $K$, then $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges pointwise to $\varphi(f)$.

(d) Suppose there is a point $y$ of $Y$ such that $\varphi(f_1)(y), \varphi(f_2)(y), \varphi$ $y), \ldots$ does not converge. There is a positive number $\varepsilon$ such that if $N$ is a positive integer there are two positive integers $m$ and $n$ greater than $N$ such that $|\varphi(f_m)(y)-\varphi(f_n)(y)|>\varepsilon$.

By Theorem 1 of Tucker [3] there is a function $g$ which uniformly approximates $f$ within $\frac{1}{16}\varepsilon$ such that there exists a sequence $g_1, g_2, g_3, \ldots$ of functions in $K$ converging pointwise to $g$ with the property that for each point $x$ of $X$

$$\sum_{i=1}^{\infty}|g_{i+1}(x)-g_i(x)|$$

is bounded.

Since $\varphi(g_1), \varphi(g_2), \varphi(g_3), \ldots$ must converge pointwise by (b), there is a positive integer $N_1$ such that if $m$ and $n$ are two integers greater than $N_1$, then

$$|\varphi(g_m)(y)-\varphi(g_n)(y)|<\tfrac{1}{4}\varepsilon .$$

Since $f_1-g_1, f_2-g_2, f_3-g_3, \ldots$ is a sequence of functions in $K$ such that for each point $x$ of $X$ there exists a positive integer $N$ with the property that if $n$ is an integer greater than $N$ then $|(f_n-g_n)(x)|<\frac{1}{8}\varepsilon$ then by (c) there exists a positive integer $N_2$ such that if $n$ is an integer greater than $N_2$ then $|\varphi(f_n-g_n)(y)|<\frac{1}{4}\varepsilon$. Let $m$ and $n$ be two positive integers greater than $N_1+N_2$. Then

$$|\varphi(f_n)(y)-\varphi(f_m)(y)| = |\varphi(f_n-g_n)(y)+\varphi(g_m-f_m)(y)+\varphi(g_n-g_m)(y)|$$
$$<\tfrac{1}{4}\varepsilon+\tfrac{1}{4}\varepsilon+\tfrac{1}{4}\varepsilon<\varepsilon .$$

Thus $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges pointwise.



THEOREM 4. *Suppose $Y$ is a topological space, $\varphi(1)(y)\neq 0$ for each point $y$ in $Y$, and each function in $L$ is continuous. Then if $f_1, f_2, f_3, \ldots$ is a sequence of functions in $K$ which converges pointwise to a function $f$, the sequence $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges pointwise to a continuous function.*

Proof. Since multiplication by a continuous function preserves continuity, it may be assumed because of Theorem 2 that $\varphi(1) = 1$.

Since $f_1, f_2, f_3, \ldots$ converges pointwise, by Theorem 3, $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges pointwise to a function $l$.

First, suppose $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ is non-increasing and $l$ takes on only the values 0 and 1. There is a sequence $k_1\geqslant k_2\geqslant k_3\geqslant\ldots$ of functions in $LS(\varphi(H))$ converging pointwise to $l$. The functions $k_i$ may be taken to have values between 0 and 1. Replace $k_i$ by $\max(2k_i, 1)-1$. Thus for each $y$ in $Y$ either $k_i(y) = 1$ for every $i$ or there exists a positive integer $i$ such that $k_i(y) = 0$.

Let $\alpha = l^{-1}(1)$ and $\beta_i = k_i^{-1}(1)$, so that $\alpha = \bigcap \beta_i$.

For each positive integer $i$, let $h_{i1}, h_{i2}, h_{i3}, \ldots$ be a non-decreasing sequence of points of $\varphi(H)$ converging pointwise to $k_i$. The values of the $h_{ij}$'s may be taken to be between 0 and 1. Replace $h_{ij}$ by $\min(2h_{ij}, 1)$. Let $\gamma_{ij} = h_{ij}^{-1}(1)$ and thus $\beta_i = \bigcup_{j=1}^{\infty}\gamma_{ij}$.

Since each function in $\varphi(K)$ is continuous, $\alpha$ is closed. Let $y$ be a point of $\alpha$. Then $y$ belongs to $\bigcap_{i=1}^{\infty}\bigcap_{j=1}^{\infty}\gamma_{ij}$, which in that, for each positive integer $i$, $y$ belongs to $\bigcup_{j=1}^{\infty}\gamma_{ij}$. Therefore there exists a s ice $j_1, j_2, j_3, \ldots$ of positive integers such that $y$ belongs to $\gamma_{ij_i}$ for each positive integer $i$.

Let $t = \min(h_{ij_i})$. Since each $h_{ij_i}\geqslant 0$ there is a function $v_{ij_i}$ in $\varphi^{-1}(h_{ij_i})$ such that $v_{ij_i}$ is in $H$ and $v_{ij_i}\geqslant 0$. Let $w = \min(v_{ij_i})$. Then $w$ is in $K$ and by Theorem 3 $\varphi(w) = t$. Thus $t$ is continuous. Suppose $p$ is a point of $Y$ such that $t(p)\neq 0$. This implies that for every positive integer $i$ $h_{ij_i}(p)>0$ and thus $k_i(p)>0$. It follows that $k_i(p) = 1$ for each positive integer $i$ and $l(p) = 1$.

Since $t$ is continuous and no value of $t$ exceeds 1, $t^{-1}((0, 1])$ is open. Also $t^{-1}((0, 1])$ is a subset of $\alpha$ containing $y$. Therefore $\alpha$ is open and $l$ is continuous.

Now suppose $\varphi(f_1)\geqslant\varphi(f_2)\geqslant\varphi(f_3)\geqslant\ldots$ and that $l$ takes on only values between 0 and 1. It may be assumed that the values of $\varphi(f_i)$ are between 0 and 1. Suppose $r$ is a positive integer. Now

$$\varphi(f_i) = \sum_{p=0}^{r-1}\min\big(\max(\varphi(f_i), p/r), p+1/r\big)-p/r$$
$$= \varphi\sum_{p=0}^{r-1}\min\big(\max(f_i, p/r), p+1/r-p/r\big).$$

Thus $\Big\{\varphi\Big(\sum_{p=0}^{r-1}\max\Big(i\big(\min\big(\max(f_i, p/r), p+1/r\big)-p+1/r\big), -1/r\Big)+1/r\Big)\Big\}_{i=1}^{\infty}$ is a non-increasing sequence of functions in $\varphi(K)$ which converges pointwise to a function $m$

which takes on only a finite number of values and which approximates $l$ uniformly within $1/r$.

Suppose $m$ is not continuous at $y$. Then, since

$$\min\big(\max(m, m(y)-1/r), m(y)+1/r\big)-m(y)$$
$$= \max\big(\min(m-m(y), 1/r), 0\big)+\max\big(\min(m-m(y), 0), -1/r\big),$$

one of $\max\big(\min(m-m(y), 1/r), 0\big)$ or $\max\big(\min(m-m(x), 0), -1/r\big)$ fails to be continuous at $y$. But both are functions which take on only two values and are the pointwise limit of a non-increasing sequence of functions in $\varphi(K)$. Thus by an above argument both are continuous and $m$ is continuous. It follows that $l$ is continuous.

Consequently any function, not necessarily bounded, which is the pointwise limit of a non-increasing sequence of functions in $\varphi(K)$ is also continuous.

By Theorem 1 of Tucker [3], any function which is the pointwise limit of a sequence of functions in $\varphi(K)$ can be uniformly approximated by the difference of two functions each of which is the pointwise limit of a non-increasing sequence of functions in $\varphi(K)$ and is therefore continuous.

If $\varphi$ is defined only on $K^*$, the statement that *$\varphi$ preserves pointwise convergence* means that if $f_1, f_2, f_3, \ldots$ is a pointwise convergent sequence of functions in $K^*$ then $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges pointwise and further that if $f_1, f_2, f_3, \ldots$ converges to a function $f$ in $K^*$ then $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges to $\varphi(f)$.

THEOREM 5. *Suppose $Y$ is a topological space, each function in $L$ is continuous, $\varphi$ is assumed to be defined only on $K^*$, $\varphi(1)(y) \neq 0$ for each point $y$ of $Y$, and $\varphi$ preserves pointwise convergence. Then if $f_1, f_2, f_3, \ldots$ is a sequence of functions in $K^*$ which converges pointwise to a function $f$, the sequence $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges pointwise to a continuous function.*

Proof. This follows from an argument similar to that for Theorem 4.

THEOREM 6. *Suppose $Y$ is a topological space, $L$ is the set of all continuous functions on $Y$, and $\varphi$ is a one-to-one mapping of $K$ onto $L$. Then $K$ is closed with respect to pointwise convergence.*

Proof. Since $L$ is the set of all continuous functions on $Y$, $L$ contains the constant function 1. It was shown in Theorem 2 that $\varphi(1)$ is not zero at a point $y$ of $Y$ unless every function in $L$ is zero there also. Therefore $\varphi(1)$ is not zero and the hypothesis of Theorem 4 is satisfied. Further $\varphi$ followed by multiplication by $1/\varphi(1)$ is a one-to-one linear lattice homomorphism of $K$ onto $L$. Thus $\varphi(1)$ may be assumed to be the constant function 1.

The space $L$ is the set of all pointwise limits of the functions in $\varphi(H)$. By Theorem 2, $\varphi^{-1}$ preserves pointwise convergence and $K$ is closed with respect to pointwise convergence since $L$ is.

Theorem 6 is not true if the requirement that $\varphi$ is one-to-one is dropped, e.g. let $L$ be the real numbers and pick a particular point $x$ of $X$ and let $\varphi(f) = f(x)$.

THEOREM 7. *If $f$ is a function in $K$ it can be uniformly approximated by a function $g$ in $K$ which has the property that any bounded subset of its range is finite.*



Proof. Suppose $\varepsilon>0$. By Theorem 1 of Tucker [3] there exists a function $k$ which approximates $f$ uniformly within $\frac{1}{2}\varepsilon$ such that $k$ is the difference of two functions $k_1$ and $k_2$ in $\mathrm{US}(H)$ each of which is bounded above.

Suppose $k_1$ is bounded. Its values may be assumed to be between 0 and $-1$. Let $h_1 \geqslant h_2 \geqslant h_3 \geqslant \ldots$ be a sequence of functions in $H$ converging pointwise to $k_1$. The values of $h_i$ may be assumed to be between 0 and $-1$.

Let $r$ be a positive integer such that $1/r<\varepsilon/4$. Then

$$k_1 = \sum_{p=0}^{r-1} \min\big(\max(k_1, -(p+1)/r), -p/r\big)+p/r$$

and

$$h_i = \sum_{p=0}^{r=1} \min\big(\max(h_{i'}-(p+1)/r), -p/r\big)+p/r .$$

Let

$$q_i = \sum_{p=0}^{r-1} \max\Big(i\big(\min\big(\max(h_i, -(p+1)/r), -p/r\big)+p/r\big), -1/r\Big) .$$

Then $q_1, q_2, q_3, \ldots$ is a non-increasing sequence of functions in $H$ converging pointwise to a function $t_1$ in $\mathrm{LS}(H)$ which only takes on values which are multiples of $1/r$ and which approximates $k_1$ uniformly within $1/r$. The method of construction may be extended to the case where $k_1$ is not bounded. A function $t_2$ corresponding to $k_2$ may also be constructed and $t_1-t_2$ is the desired function $g$.

It follows from this theorem that if $\omega$ is the collection of all functions in $K$ which take on only the values 0 and 1, then every bounded function in $K$ may be uniformly approximated by finite linear combinations of functions in $\omega$.

THEOREM 8. *Suppose $Y$ is a topological space, $L$ is the set of all continuous functions on $Y$, and $\varphi$ is a one-to-one mapping of $K^*$ onto $L$ that preserves pointwise convergence. Then $K^*$ is finite dimensional.*

Proof. As shown in the argument for Theorem 6, $\varphi(1)(y) \neq 0$ for each point $y$ in $Y$. Thus $\varphi$ followed by multiplication by $1/\varphi(1)$ is a one-to-one linear lattice homomorphism of $K^*$ onto $L$ that preserves pointwise convergence. Therefore it may be assumed that $\varphi(1) = 1$.

Let $\omega$ be the collection of all functions in $K^*$ that take on only the values 0 and 1. If there exists an infinite pairwise disjoint subcollection of $\omega$, an unbounded non-decreasing pointwise convergent sequence $f_1, f_2, f_3, \ldots$ of points of $K^*$ may be constructed. This implies $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ is pointwise convergent. As $\varphi^{-1}$ is order preserving $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ is unbounded. But by Theorem 5 $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ must converge to a continuous function. This is a contradiction as all continuous functions on $Y$ are bounded. Thus any pairwise disjoint subcollection of $\omega$ must be finite.

Any non-increasing sequence of functions in $\omega$ must contain only finitely many different terms. Suppose $x$ is a point of $X$. Let $\lambda_x$ be the collection of all functions in $\omega$ whose value at $x$ is 1. There must be a least element of $\lambda_x$, otherwise a non-increasing sequence of functions in $\omega$ with infinitely many different terms could be constructed. Denote the least member of $\lambda_x$ as $f_x$. Suppose $g$ is a function in $\omega$ which is not disjoint from $f_x$. Then either $\min(f_x, g)$ or $f_x - \min(f_x, g)$ is 1 at $x$ and if $f_x - \min(f_x, g)$ is not the zero function, $f_x$ is not the least element of $\lambda_x$. Thus $g \geqslant f_x$. It follows that the collection of all $f_x$ for all $x$ in $X$ is disjoint and therefore finite. Then each function in $\omega$ is the sum of a finite number of $f_x$. Since, by Theorem 7, each function in $K^*$ is the uniform limit of a sequence of functions each of which is a finite linear combination of functions in $\omega$, $K^*$ is finite dimensional.

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# Concerning atriodic tree-like continua

by

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**Abstract.** In this paper it is shown that there is a collection $G$ of atriodic tree-like continua such that if $M$ is a compact metric continuum then there is a member of $G$ which is not a continuous image of $M$. Thus, there are atriodic tree-like continua which are not weakly chainable.

**1. Introduction.** In 1934 Z. Waraszkiewicz [6] presented a collection of continua with the property that no compact metric continuum can be mapped onto every member of the collection. Each continuum in the collection is planar, and each contains a simple closed curve.

Russo [5] proved that there is a collection of tree-like continua with the property that no compact metric continuum can be mapped onto every member of the collection. Each continuum in this collection is planar and each contains a simple triod.

In this paper we show that there is a collection of atriodic tree-like continua with the property that no compact metric continuum can be mapped onto every member of the collection. The members of this collection can be embedded in the plane in such a way that they form a collection of mutally exclusive continua.

The proof of Theorem 2 depends heavily on results found in [3], and throughout this paper many references to that paper will be made.

**2. Notation and conventions.** As in [2] and [3], $T = \{(\varrho, \theta) |\ 0 \leqslant \varrho \leqslant 1$ and $\theta = 0$ $\theta = \frac{1}{2}\pi$, or $\theta = \pi\}$, $O$ denotes $(0, 0)$, $A$ denotes $(1, \frac{1}{2}\pi)$, $B$ denotes $(1, \pi)$, and $C$ denotes $(1, 0)$. (The author incorrectly labelled $A$, $B$, and $C$ in [3].)

The mapping $f$ of $T$ onto $T$ is as in [2, pp. 99–100] while $r$ is as in [3, p. 75] and $g = rf$ (i.e. $g = r \circ f$).

Throughout this paper all spaces are metric and the term mapping means continuous function. The two projections of $X \times Y$ onto $X$ and $Y$, respectively, are denoted by $p_1$ and $p_2$ while the projection of a product of sequence of spaces onto the $i$th factor space is denoted by $\pi_i$.

We will use the convention that if each of $p$ and $q$ is a positive integer and $p < q$ then $pA/q$ denotes $(p/q, \frac{1}{2}\pi)$, $pB/q$ denotes $(p/q, \pi)$, and $pC/q$ denotes $(p/q, 0)$.

**3. Main Theorems.** The proof of the following lemma is essentially the same as the proof provided by Read [4, Lemma p. 236]. A proof is included here only for the sake of completeness.