Proof. Assume $f(I)$ is nondegenerate. By Theorem 5′ $f(I)$ is atriodic. Since $f(I)$ is also locally connected, $f(I)$ is an arc or a simple closed curve.

To show that a simple closed curve can be obtained as the weakly confluent image of I, and also to show that the weakly confluent image of an arc-like continuum need not be unicoherent, consider the following example.

EXAMPLE 3. A weakly confluent map from $I = [0, 1]$ onto the unit circle, J, in the plane.

If $\theta \in I$, let $f(\theta) = e^{4\pi i\theta}$. Clearly f is a weakly confluent map from I onto J.

COROLLARY 8. *If f is a weakly confluent map defined on a simple closed curve J, then $f(J)$ is an arc, a simple closed curve or a point.*

EXAMPLE 4. A confluent map from the unit circle, J, in the plane onto $[-1, 1]$.

If (x, y) is in J, let $f((x, y)) = x$. Clearly f is a confluent map from J onto $[-1, 1]$.

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A generalization of right simple semigroups

by

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Abstract. An element s in a semigroup S is called a *right simple* element if $sS = S$. This paper developes the notion of right simple elements, and uses it to generalize right simple semigroups.

A non-right simple semigroup with right simple elements is called a *right simple element* semigroup and denoted as RSE. The subset of S of right simple elements is denoted by R, and the non-right simple elements by N. If R is a right simple subsemigroup (right group, subgroup) of S, then S is called a *partial* right simple semigroup (*partial* right group, *partial* group) and denoted by PRS (PRG, PG). While a PRS semigroup is by definition an RSE semigroup, the converse is shown to be false.

The structure of RSE semigroups is determined, and a decomposition found for R. The existence of a maximum right ideal is found to be a necessary, but not sufficient, condition for right simple elements to exist. A partial converse is given.

The structure theorems are then applied to RSE semigroups possessing other properties, such as the descending chain condition on right ideals of N, finiteness, or left (right) cancellativity. It is shown that, if S is a RSE and left simple (left cancellative), then S is a PG (PRG). For right cancellativity, the development parallels that of the Baer–Levi Theory.

1. Introduction. Recall that a semigroup S is called right simple if for all s in S, $sS = S$. An element x in a semigroup S will be called a *right simple element* if $xS = S$. Note that a semigroup is right simple if and only if each of its elements is a right simple element.

This paper uses the concept of right simple elements to generalize right simple semigroups. In particular, semigroups containing right simple elements are investigated, with some of the results obtained analogous to those obtained for right simple semigroups. Throughout the paper, a semigroup containing right simple elements will be called a *right simple element* semigroup, and it will be denoted by RSE. The class of RSE semigroups therefore contains the class of right simple semigroups.

In Section 2, various structure theorems are presented for RSE semigroups. The results of Section 2 are then used in Section 3 to discuss homomorphisms on RSE semigroups, and to extend group homomorphism results. In Section 4, applications of Sections 2 and 3 are developed for RSE semigroups where other conditions such as right (left) cancellativity, left simplicity, finiteness, are also present. Examples appear in various parts of the paper.

The subset of S of right simple elements will be denoted by R, and the non-right simple elements by N. Additionally, if $N \neq \square$ and R is a right simple subsemigroup (right group) of S, then S will be called a *partial* right simple semigroup (*partial* right group), and will be denoted by PRS (PRG). Note that a PRS semigroup is by definition a RSE semigroup. The converse will be shown to be false.

The basic definitions and notations are that of [1]. Also, the symbol ■ denotes the end of a proof, $\setminus$ denotes set difference, $|X|$ denotes the cardinality of the set X, and $\subset$ denotes a proper subset.

2. Structure theorems for right simple element semigroups. We consider first the structure of RSE semigroups, and then apply the results in the remaining sections of the paper.

Before proceeding further, some examples are provided. Let $S = R \cup N$, where R and N are distinct semigroups, and define $rn = nr = n$ for all r in R, n in N. If R is a right simple semigroup, a right group, or a group, then S is a PRS, PRG, or a partial group respectively.

PROPOSITION. *If S_1 and S_2 are right simple element semigroups, then $S_1 \times S_2$ is also a right simple element semigroup.*

The proof is straightforward and, moreover, in S, $R = R_1 \times R_2$.

THEOREM 2.1. *If S is a right simple element semigroup, then*:

(1) *The set R of right simple elements is a subsemigroup.*

(2) *The set $S\setminus R$, if non-empty, is the maximum right ideal of S, and is prime.*

Proof. Suppose that S is not right simple, and denote the proper subset $S\setminus R$ of S by N. Then N is a right ideal of S, since $a \in N$, $s \in S$, and $as \notin N$ imply that $S = asS \subseteq aS$, contradicting that $a \in N$. Suppose next that I is a right ideal of S and that I is not contained in N. For x in $I\setminus N$, $xS = S$ implies that $S = xS \subseteq I$, i.e., $I = S$. Thus N is the maximum right ideal of S.

Since $aS = bS = S$ for a, b in R, then $abS = aS = S$ implies that R is a subsemigroup of S. It follows then that when $N \neq \square$, N is a prime ideal of S. ■

Remark 2.2. The above theorem is originally due to H. B. Grimble, [3], and also appears as Exercise 7, p. 40, in [1]. In that exercise, Grimble calls a right simple element a universal left divisor, and accordingly, $S\setminus R$ is called the universal maximal right ideal. The development there is in the ideal theory direction, as compared to a generalization of right simple concepts as presented in this paper.

The following shows that the existence of a maximum right ideal is not enough for S to be a RSE semigroup. Let S be the semigroup defined as follows:

$\cdot$	a	b	c
a	a	b	a
b	b	a	b
c	a	b	a

The unique maximal right ideal of S is $\{a, b\}$, but c is not a right simple element of S, i.e., $\{a, b\}$ is not prime.

The next theorem is then a converse for Theorem 2.1.

THEOREM 2.3. *If a semigroup S has a unique proper maximal right ideal A such that $S\setminus A$ contains more than one element, then the set (subsemigroup) of right simple elements of S is precisely $S\setminus A$.*

Proof. Since A is a right ideal of S, no element of A can be a right simple element of S. Thus if $s \in S\setminus A$, then by hypothesis, $sS \subseteq A$ or $sS = S$. If $sS \subseteq A$, it follows then that $A \cup \{s\}$ is a right ideal of S properly containing A, and hence it must be that $A \cup \{s\} = S$. But this contradicts the fact that $|S\setminus A| > 1$. Hence s in $S\setminus A$ implies that s is a right simple element and therefore $N = A$ and $R = S\setminus A$. ■

We next consider the structure of R and its relationship to N and S.

THEOREM 2.4. *If S is a right simple element semigroup, with R the set of right simple elements and $N = S\setminus R \neq \square$, then for each r in R, $rN = N$ or $rN = S$. Moreover, if $rR \neq R$, then $rN = S$.*

Proof. If $r \in R$, then $rS = S$ implies that $rR \cup rN = R \cup N$, where the union on the right is disjoint. By Theorem 2.1, $rR \subseteq R$ and hence $N \subseteq rN$. However, Theorem 2.1 also implies that $rN \subseteq N$ or $rN = S$. Thus $rN = N$ or $rN = S$.

If $rR \neq R$, then $rR \subset R$ implies that $N \subset rN$, i.e., $rN = S$. ■

The preceding theorem thus yields a decomposition of R, viz.,

$$B = \{r \in R\colon rN = S\} \quad \text{and} \quad C = \{r \in R\colon rN = N\}.$$

The next theorem relates B and C to the structure of R and N, and finds necessary and sufficient conditions in order that N be a maximum ideal of S.

THEOREM 2.5. *Let S be a right simple element semigroup, denoting the right simple elements by R. For $N = S\setminus R \neq \square$, define*

$$B = \{r \in R\colon rN = S\} \quad \textit{and} \quad C = \{r \in R\colon rN = N\}.$$

Then $R = B \cup C$, and the following conditions are equivalent:

(1) $B = \square$.

(2) *N is the (unique) maximum ideal of S.*

(3) *R is a right simple subsemigroup of S such that $C \neq \square$.*

Proof. That $R = B \cup C$ follows directly from Theorem 2.4. (1)→(2): If $B = \square$, then $R = C$ and hence for all s in S, $sN \subseteq N$. By Theorem 2.1, N is a right ideal of S, and therefore N is an ideal of S. By a similar argument as used in Theorem 2.1, N is the maximum ideal of S.

(2)→(3): For a, b in R, there exists x in S such that $ax = b$. Since N is an ideal of S, $ax \notin N$ implies $x \notin N$. Thus $x \in R$ and R is a right simple subsemigroup of S. Next, if $C = \square$, then there exists an r in R such that $rN = S$, contradicting that N is an ideal of S.

(3)→(1): Proceeding indirectly, suppose B is not empty. Leb $b \in B$ and $r \in R$. By Theorem 2.4, $rN = S$ or $rN = N$ implies that $brN = bS = S$ or $brN = bN = S$. Thus $br \in B$, and B is a right ideal of R. Since R is right simple, $B = R$ and $C = \Box$, a contradiction. ■

The above Theorem 2.5 also says that a RSE semigroup with N a non-empty ideal of S is a PRS semigroup. Also, in light of Theorem 2.5, it is logical to consider the situation where $B \neq \Box$. When this is the case, the next result demonstrates that the structure of S is very complex. In particular, B is infinite, S is factorizable [9] in terms of N and left magnifying elements [4] from B, and there exists an infinite descending chain of right ideals of S in N for each element in B.

THEOREM 2.6. *Let S be a right simple element semigroup, with R, N, B and C as in Theorem 2.5.*

If $B \neq \Box$, then:

(1) *B is an idempotent-free infinite ideal of R.*

(2) *S is factorizable.*

(3) *For each b in B, there exists $N_i \subset N$, for $i = 1, 2, 3$, such that the N_i are necessarily distinct and $bN_1 = N$, $bN_2 = B$, and $bN_3 = C$.*

(4) *For each b in B there exists an infinite descending chain of right ideals of S, N_j, $j = 1, 2, 3, \ldots$, such that*

$$N = N_0 \supset N_1 \supset N_2 \supset \ldots, \tag{2.7}$$

and for all $j = 1, 2, 3, \ldots$, $bN_{j+1} = N_j$ and $b^j N_j = N$.

If $C \neq \Box$, then:

(5) *C is a right simple subsemigroup of R.*

(6) *For each $c \in C$, $cR = R$ and $cB = B$.*

(7) *If B is also non-empty, then B is a proper ideal of R.*

Proof. (1) Consider the proof of (3)→(1) in Theorem 2.5 and note that for r in R and b in B, $rbN = rS = S$ implies that $rb \in B$. Thus, B is an ideal of R.

Next let E_B denote the set of idempotents of B and let $e \in E_B$. For any n in N there exists x in S such that $ex = n$. Thus $en = n$, and therefore $eN = N$. But then $e \in C$, contradicting that $e \in B$. Hence $E_B = \Box$. Clearly, if B is finite, $E_B \neq \Box$, i.e., a contradiction. Thus B is an idempotent-free infinite ideal of R.

(2) Any b in B is left magnifying element. By [9; Theorem 4.1, p 532] S is factorizable.

(3) This follows directly from the definition of B; when $C = \Box$, let $N_3 = \Box$.

(4) Let $b \in B$ and let $N_1 = \{a \in N: ba \in N\}$. Since $bN = S \neq N$, then $N_1 \neq N$. If $s \in S$ and $a \in N_1$, then $as \in N$ and $bN_1 s \subseteq Ns \subseteq N$ imply that bas $\in N$. Therefore, $as \in N_1$, and N_1 is a right ideal of S in N.

Now let $n \in N$. Since $bN = S$, there exists $x \in N$ such that $bx = n$. Evidently $x \in N$, and hence by definition, $x \in N_1$. Thus $N \subseteq bN_1$, and since $bN_1 \subseteq N$, then $bN_1 = N$.

To generalize, we proceed in an inductive manner. Define

$$N_{j+1} = \{a \in N_j: ba \in N_j\} \quad \text{for} \quad j = 1, 2, 3, \ldots,$$

and note that $N_j \subseteq N_{j-1}$ for all j. Suppose that $bN_j = N_{j-1}$ for some j. By the definition of N_{j+1}, $bN_{j+1} \subseteq N_j$. If $n \in N_j$, then n is also in N_{j-1}, and therefore $bN_j = N_{j-1}$ implies that there exists an x in N_j such that $bx = n$. Thus x and bx in N_j imply that $x \in N_{j+1}$ and therefore $N_j \subseteq bN_{j+1}$, i.e., $bN_{j+1} = N_j$.

Next, if for some j, $N_{j+1} = N_j$, then $N_j = bN_{j+1} = bN_j = N_{j-1}$. But then $bN_j = bN_{j-1} = N_{j-2}$ implies that $N_j = N_{j-1}$. Continuing in this manner produces $N_1 = N_0 = N$, a contradiction.

Lastly, let N_j be a right ideal of S, $a \in N_{j+1}$, and $s \in S$. Since $a \in N_{j+1}$, then $a \in N_j$ and therefore as $\in N_j$. Also, bas $\in N_{js} \subseteq N_j$. Therefore as $\in N_{j+1}$ and N_{j+1} is a right ideal of S in N_j. It follows directly that for $j = 1, 2, 3, \ldots$, $b^j N_j = N$, and hence part (4) is established.

(5) If $c, d \in C$, then $cdN = cN = N$ implies C is a subsemigroup of S. For c, d in C there exists an x in S such that $cx = d$. Where $x \in N$, then cx in $cN = N$ contradicts that $d \in C$. Similarly, if $x \in B$, then $xN = S$ implies that $S = cS = cxN = dN = N$, which is a contradiction. Thus $x \in C$, and therefore C is a right simple subsemigroup of R.

(6) Let $c \in C$ and consider the following: $S = c(N \cup R) = cN \cup cR = N \cup cR = N \cup R$. It follows that $cR = R$, since $cR \subseteq R$ and N and R are disjoint. By a similar argument on B, $cB = B$.

(7) This follows directly from (1). ■

The following illustrates Theorem 2.6 and gives a non-trivial example of a semigroup which is a RSE, but not a PRS semigroup. A trivial example is any right simple semigroup.

EXAMPLE 2.8. Let S be the bicyclic semigroup generated by p and q with $dq = 1$, $qp \neq 1$. It is straightforward to show that $R = \{1, p, p^2, \ldots\}$, with $C = \{1\}$ and $B = \{p, p^2, \ldots\}$, and that $N = \{q^i p^j: i > 0, j \geqslant 0\}$, where i, j are integers. For any $p^k \in B$, the sets $N_j = \{q^{m+(j-1)k} p^n: m > k, n \geqslant 0\}$, for $j = 0, 1, 2, \ldots$, satisfy the conditions of Theorem 2.6. In particular, since $m > k$ implies that $m + (j-1)k > jk$, it then follows that $(p^k)^j N_j = N$.

Lastly, since $pR \neq R$, R is not right simple and therefore S is a RSE semigroup which is not a PRS semigroup.

The last result of this section generalizes the fact that in a right simple semigroup S, all of S is an $\mathcal{R}$-class under Green's $\mathcal{R}$ relation.

THEOREM 2.9. *If S is a right simple element semigroup, then the set of right simple elements is an $\mathcal{R}$-class of S. Moreover, each right simple element is also a right regular element.*

Proof. If $a, b \in R$, then $aS = bS = S$ implies $a\mathcal{R}b$. Thus R is contained in some $\mathcal{R}$-class. Conversely, if $x\mathcal{R}a$ such that $a \in R$, then $xS^1 = aS^1 = S$. If $N = \Box$,

$x \in R$. If $N \neq \square$, then $x \notin N$ since x in N implies that $\{x\} \cup xS \subseteq N$, contradicting $xS^1 = S$. Thus in either case $x \in R$.

For r in R, there exists x in S such that $r^2 x = r$, i.e., r is right regular. ■

The converse to the second part of the above theorem is an open question.

3. Homomorphism on right simple element semigroups. In [8], R. R. Stoll describes group homomorphs of a semigroup. Since a group is left and right simple, it is natural to seek a generalization of Stoll's work by factoring a group homomorphism into a right simple and a left simple homomorphism. While the work in [6] pursues this notion for semigroups in general, this section extends Stoll's results for group images with zero, to right simple images with zero. In particular, we investigate right simple (right simple with zero) homomorphisms on RSE semigroups. Moreover some of the results consider the preservation of the right simple elements in a one-to-one manner under the homomorphism, and the effect the homomorphism has on N. A lemma is needed.

LEMMA 3.1 [6; Theorem 2.7]. *If γ is a right simple homomorphism on a semigroup S, then γ maps every right ideal of S onto $S\gamma$.*

A difficulty related to the concept of preserving the right simple elements under a right simple homomorphism is illustrated by the next result. Additionally, it shows that the *maximum* right simple homomorph may preserve none of the right simple elements and may in fact preserve (isomorphically) some of the non-right simple elements.

THEOREM 3.2. *There exists a class of semigroups having right simple elements such that the maximum right simple homomorph of the semigroup does not separate any of the right simple elements. Moreover, there exists a subset of the non-right simple elements which is isomorphic to the maximum right simple image.*

Proof. Let G be a group with identity e, and let $\mathscr{T}$ denote the full transformation semigroup on $X = \{1, 2, 3\}$, such that G and $\mathscr{T}$ are disjoint. Define S as $G \cup \mathscr{T}$ with multiplication in S given by the following. For g, h in G and s, t in $\mathscr{T}$, let $gh \in G$, $st \in \mathscr{T}$, and $gt = tg = g$ in G, i.e., G is an ideal of S. Recalling that the symmetric group G_3 is contained in $\mathscr{T}$, it can be verified that $G_3 = R$, and hence that $N = G \cup (\mathscr{T} \setminus G_3)$.

It is evident that S is a regular semigroup, so let E_s denote the set of idempotents of S and let $V(c)$ denote the set of inverses of c in S. Recall that S is called conventional if $cE_s c' \subseteq E_s$ for each c in S and each $c' \in V(c)$. By [5; Proposition (2), p. 398], $\mathscr{T}$ is a conventional semigroup, and it follows that $cE_s c' \subseteq E_s$ for each c in S and c' in $V(c)$, i.e., S is a conventional semigroup.

The minimum group congruence on S will also be the minimum right simple congruence on S, since Lemma 3.1 implies that G will be mapped onto the homomorph of S. By [5; Theorem 3.1, p. 396], the minimum group congruence on S is given by:

$$\beta = \{(a, b) \in S \times S\colon\ iai = ibi \text{ for some } i \in E_S\}.$$

This reduces in the case of S to:

$$\beta = \{(a, b) \in S \times S\colon\ ea = eb\}. \tag{3.3}$$

It follows from equation (3.3) that the induced homomorphism, $\beta^{\natural}$, maps all of R to $e\beta$, and maps G isomorphically onto S/β.

Thus the right simple elements of S are collapsed under $\beta^{\natural}$, and G, a proper subset of N, maps isomorphically onto the minimum right simple image of S. ■

Remark 3.4. In the previous theorem, G is a group ideal of S and so S is also a homogroup. Therefore by [6; Cor. 3.22] it can also be shown that the minimum group congruence on S is given by β, and that $G \cong S/\beta$.

The next result shows that some RSE semigroups possess a right simple with zero homomorph.

THEOREM 3.5. *Let S be a right simple element semigroup. Denote the set of right simple elements by R and let $N = S \setminus R$. If N is an ideal of S, then*:

(1) $S/N \cong R \cup \{0\}$,

(2) *S/N is a right simple image with zero.*

Proof. If $N = \square$, the results follow trivially. If $N \neq \square$, then the Rees factor semigroup is isomorphic to $R \cup \{0\}$, and (1) is true. Also, by Theorem 2.5, R is a right simple subsemigroup and (2) is true. ■

While the next theorem is stated for right 0-simple homomorphs, it also applies to right simple with zero homomorphs. Additionally the following results generalize the group with zero results of Stoll, [8; Theorem 1, p. 476]. Note that in [8; Theorem 2, p. 476] Stoll calls $(0)\gamma^{-1}$ the residue of $(e)\gamma^{-1}$, where e is the identity of the group image. By residue, Stoll means, in the notation of [2; § 10.2, p. 182], that the left and right residues of $(e)\gamma^{-1}$ are equal. Recall that the right residue of a subset H of S is $W_H = \{s \in S\colon\ s^{[-1]}H = \square\}$, where $s^{[-1]}H = \{x \in S\colon\ sx \in H\}$.

THEOREM 3.6. *Let S be as in Theorem 3.5, and let γ be any non-trivial right 0-simple homomorphism on S, and denote $(0)\gamma^{-1}$ by I. Then*:

(1) *I is a prime ideal of S contained in N.*

(2) *I is a proper subset of N if and only if $N\gamma = S\gamma$.*

(3) *I is the right residue of $S \setminus I$.*

Proof. For any $x \in I$ and $s \in S$, it follows that $(xs)\gamma = (sx)\gamma = 0$, and therefore I is an ideal of S. By Theorem 2.5, $I \subseteq N$, so let $xy \in I$ for $x, y \in S$ and assume that $x, y \notin I$. Then xS and yS are right ideals of S such that $xS \nsubseteq I$ and $yS \nsubseteq I$, i.e., were $xS \subset I$, then under γ there exists $u \in S$ such that $xu\gamma = x\gamma$. But then $xu \in I$ would imply that $x\gamma = xu\gamma = 0$, contradicting that $x \notin I$.

Since $(xS)\gamma = (yS)\gamma = S\gamma$, $S\gamma = (xS)\gamma = (xyS)\gamma \subseteq (IS)\gamma = 0$ is a contradiction. Hence x or $y \in I$, and I is a prime ideal of S.

In (2), $N\gamma \cdot S\gamma = (NS)\gamma \subseteq N\gamma$ implies that $N\gamma$ is a right ideal of $S\gamma$ and therefore $N\gamma = S\gamma$ or $N\gamma = 0$. If $I \neq N$ then $I\gamma = 0$ and $N\gamma = S\gamma$. Conversely, if $I = N$ then $N\gamma = 0$.

To prove (3), let $H = S\setminus I$. If $i \in I$ and $i^{[-1]}H \neq \square$, then there exists $a \in i^{[-1]}H$ such that $ia \in H$. But this contradicts part (1) since $ia \in I$. Thus $i^{[-1]}H = \square$ and $I \subseteq W_H$.

Conversely, suppose that $w \in W_H$ and let $h \in H$, i.e., $h\gamma \neq 0$. If $w \notin I$, then there exists $x \in S$ such that $wx\gamma = h\gamma$. Since $h\gamma \neq 0$, then $wx \in H$ and therefore $x \in w^{[-1]}H$, contradicting that $w^{[-1]}H = \square$. Thus $W_H \subseteq I$ and $I = W_H$. ■

It can be shown additionally that $I \subseteq {}_HW$, the left dual of W_H. In order that $I = {}_HW$, γ would also need to be left simple, i.e., a group with zero homomorphism as in Stoll's paper.

In the following, γ is restricted to being a right simple homomorphism as compared to right simple with zero. By Lemma 3.1 and Theorem 2.5, it is evident that for a right simple homomorphism γ on S, $N\gamma = S\gamma$, i.e., elements of N may not be collapsed to a single element under γ as they are under a right 0-simple homomorphism. If γ also preserves R in a one-to-one manner, then we have the following result.

THEOREM 3.7. *Let S be a right simple element semigroup, and denote the right simple elements by R. If γ is a right simple homomorphism on S which preserves the set R in a one-to-one manner, then for $N = S\setminus R \neq \square$,*

(1) *N is a disjoint union of right neat subsets.*

(2) *$N\gamma \cong R$ if and only if $S\gamma \cong R$.*

(3) *$R\gamma \subseteq N\gamma$.*

Proof. By Lemma 3.1, $N\gamma = S\gamma$. Thus by [6; Section 2, Theorem], (1) is true. Also, $N\gamma = S\gamma \cong R$ implies (2) is true. Part (3) follows since $R\gamma \subseteq S\gamma = N\gamma$. ■

In considering a possible converse for Theorem 3.7, we show that one can have $R \cong N\gamma$, where γ is not a right simple homomorphism, nor is $S\gamma$ isomorphic to R.

EXAMPLE 3.8. Let $S = G \cup H$, where G and H are disjoint isomorphic finite groups, and S is the Clifford semigroup with multiplication homomorphism $\emptyset: G \to H$ any isomorphism of G onto H. Finally, let γ be the identity map on S. Then $R = G$, $N = H$, and $N\gamma \cong R$, but S is not right simple nor isomorphic to R.

Lastly, one can consider the image of R under a homomorphism of S.

THEOREM 3.9. *Let S be a right simple element semigroup. If γ is a homomorphism on S, then $R\gamma$ is a subset of the right simple elements of $S\gamma$.*

Proof. If $t\gamma \in S\gamma$ and $x \in R\gamma$, then there exists an r in R such that $x = r\gamma$. For r, there exists s in S such that $rs = t$. Thus $x \cdot s\gamma = r\gamma \cdot s\gamma = rs\gamma = t\gamma$, so x is a right simple element of S. ■

Note that the containment in Theorem 3.9 may be proper as demonstrated by Theorem 3.2.

4. Further results and applications. The theorems of the preceding sections are now applied to RSE semigroups where various other conditions exist, such as finiteness, left cancellativity, the existence of idempotents, or the descending chain condition on right ideals of S. In particular, the results of [1; p. 39] are generalized for RSE semigroups.

The first theorem combines some earlier results and the concept of descending chain conditions (DCC), [7].

THEOREM 4.1. *Let S be a right simple element semigroup, and let N denote the right ideal of S of non-right simple elements. If N is non-empty and satisfies the descending chain condition, then N is an ideal of S, and S is a partial right simple semigroup.*

Proof. If the DCC holds in N, then by the contrapositive of (4) of Theorem 2.6, B is empty. Thus by Theorem 2.5, N is an ideal of S, and S is a PRS semigroup. ■

COROLLARY 4.2. *If S is a right simple element semigroup such that $N \neq \square$ and finite, then S is a partial right simple semigroup.*

Proof. By the DCC on N, Theorem 4.3 applies. It can also be shown directly that $B = \square$. ■

Recall that if a semigroup S has idempotents, the set of idempotents will be denoted by E_S. Also, for right simple semigroups, we have the following result.

LEMMA 4.3 [1; Theorem 1.27, p. 38]. *A semigroup S is a right group if and only if S is right simple and contains an idempotent.*

The generalization is evident if R is a right group, i.e., S is then a PRG. However, Example 2.8 shows that $E_R \neq \square$ or $E_C \neq \square$ does not necessarily imply that S is a PRG. The same example shows that C finite is also not strong enough for S to be a PRG. Thus the following are some conditions in order that a RSE semigroup be a PRG.

THEOREM 4.4. *Let S be a right simple element semigroup, and let $N \neq \square$ denote the right ideal of S of non-right simple elements of S. If $R = S\setminus N$ is finite, S is a partial right group. Moreover, N is an ideal of S.*

Proof. By Theorem 2.6 (1), $B = \square$. Thus $R = C$ and therefore by Lemma 4.3, R is a right group and S is a PRG. ■

The previous theorem therefore implies that any finite RSE semigroup is a PRG. Additionally, it says that if S is an infinite RSE semigroup with R finite, then R must necessarily be right simple.

A result related to the preceding theorem is the following:

THEOREM 4.5. *Let S be a right simple element semigroup. If $Sx = Syx$ for all x, y in S, then S is a right group.*

Proof. Suppose $B \neq \square$, $C \neq \square$, and let $b \in B$, $c \in C$. Then $Sc = Sbc$, so that $c^2 = xbc$ for some $x \in S$. Clearly $x \in R$, since x in N would imply that $xbc = c^2 \in N$, contradicting that C is a subsemigroup of S. Next, $c^2N = xbcN$, i.e., $N = xbN = xS = S$, a contradiction. Thus one of B or C is empty.

Let $r \in R$. By hypothesis, $Sr = Sr^2$ so that $r^2 = sr^2$ for some $s \in S$. Also, since $r^2S = S$, it follows that s is a left identity for S, i.e., $E_S \neq \square$. Again, $s \in R$, and, since $sN = N$, we have that $s \in C$. Thus $B = \square$ and therefore by Theorem 2.5, N is an ideal of S.

Suppose $N \neq \square$ and let $n \in N$. Then $Sr = Snr$ for $r \in S$ implies that $r^2 = ynr$ for some y in S. But since N is an ideal of S, then $r^2 \in N$, a contradiction. Thus N is in fact empty, and $S = C$, i.e., S is a right simple semigroup. By Lemma 4.3, S is a right group. ■

The preceding result also generalizes [2; Exercise 4, p. 85], viz., if S is right simple and satisfies $Sx = Syx$ for all x, y in S, then S is a right group.

We now consider RSE semigroups possessing a cancellative property. Recall from [1; p. 39] that if a semigroup is right simple and left cancellative, then it is a right group. A similar result holds for RSE semigroups:

THEOREM 4.6. *If S is a left cancellative, right simple element semigroup such that the set N of non-right simple elements is non-empty, then*:

(1) *N is an ideal of S.*

(2) *S is a partial right group.*

(3) *S/N is a right group with zero.*

Proof. If $B \neq \square$, then for $b \in B$, $bN = S$ implies $bn = b^2$ for some n in N. But by left cancellation, $n = b$, a contradiction. Thus, $B = \square$ and therefore by Theorem 2.5, N is an ideal and R is a right simple subsemigroup of S. Since left cancellativity also holds in R, then by [1; p. 39], R is a right group, and S is a PRG. By Theorem 3.5 (1), part (3) is true. ■

While the preceding theorem generalizes right simplicity with left cancellation, the following development considers right cancellativity and thus extends some of the Baer–Levi theory.

LEMMA 4.7. *Let S be a right simple element semigroup without idempotents. Then:*

(1) *The equation $xy = y$ cannot hold for x in S and y in the set of right simple elements of S.*

(2) *If S is also right cancellative, then the equation $xy = y$ cannot hold between any two elements x, y in S.*

Proof. (1) If $xy = y$ for some y in R, then there exists s in S such that $ys = x$. Thus $x^2 = x$, a contradiction.

(2) Suppose that $xy = y$ for some x, y in S. For r in R, there exists s in S such that $rs = y$. Thus $xrs = rs$, and cancelling on the right implies that $xr = r$, contradicting part (1). ■

Note that Lemma 4.7 is not quite as strong as [2; Lemma 8.3, p. 83]. However, since right cancellativity is a precondition for the Baer–Levi theory, Lemma 4.7 (2) suffices.

LEMMA 4.8. *Let S be an idempotent free, right cancellative, right simple element semigroup such that $N \neq \square$. Then, for any s in S, $|S \setminus Ss| + |N| = |S|$, where N denotes the set of non-right simple elements of S. Moreover, if $|N| \leqslant |S \setminus N|$, then $|S \setminus Ss| = |S|$.*

Proof. Let $s \in S$ and define ϕ on S by:

$$x\phi = \begin{cases} x', & \text{such that } xx' = s \text{ if } x \in R, \\ x, & \text{if } x \in N. \end{cases}$$

When $x \in R$, x' exists since x is then a right simple element of S and we choose one of these to be $x\phi$. If $x, y \in R$ and $x\phi = y\phi$, then $x(x\phi) = s = y(y\phi) = y(x\phi)$, and thus $x = y$ since S is right cancellative. Hence ϕ is one-to-one on R.

Suppose next that $R\phi \cap Ss \neq \square$. Then there exists z such that $z = r\phi = ys$ for some r in R and s in S. But $z = r\phi$ implies that $rz = r(r\phi) = s$, and therefore $rys = rz = s$, contradicting Lemma 4.7(2). Hence $R\phi \cap Ss = \square$, and therefore $R\phi \subseteq S \setminus Ss$. Since neither R nor N can be finite, i.e., $E_s = \square$, then we have

$$|S| = |R| + |N| = |R\phi| + |N| \leqslant |S \setminus Ss| + |N| \leqslant |S|.$$

Thus $|S \setminus Ss| + |N| = |S|$, and if $|N| \leqslant |R|$, it follows that $|S \setminus Ss| = |S|$. ■

THEOREM 4.9. *Let S be as in Lemma 4.8. If $|N| \leqslant |R|$, where R is the set of right simple elements, then S can be embedded in a Baer–Levi semigroup of type (p, p), where $p = |S|$.*

Proof. The proof follows that of [2; Theorem 8.5, p. 83] since $|N| \leqslant |R|$ implies by Lemma 4.8 that $|S| = |S \setminus Ss|$. ■

Remark 4.10. Note that in Lemma 4.8 and Theorem 4.9 above, the case of $|N| < |R|$ implies that $rN \neq S$ for any r in R. Thus $B = \square$; R is a right simple subsemigroup, i.e., S is a right cancellative, partial right simple semigroup. If $|R| < |N|$, it is an open question whether or not Theorem 4.9 holds.

EXAMPLE 4.11. Let A be an infinite set, and let H be a permutation group on A. Denote the Baer–Levi semigroup $\mathrm{BL}(|A|, |A|)$ by M and define the semigroup S by $S = M \times U$. One can now show, by $U = H \cup M$. Lastly, define the semigroup S by $S = M \times U$. One can now show, by using Exercise 10, page 86 of [2], that $R = M \times H$ and $N = M \times M$. Moreover, both R and N are right cancellative, right simple, and idempotent free. Thus S is right cancellative and idempotent free, but S is not right simple.

The next property considered is that of left simplicity.

THEOREM 4.12. *If S is a left simple, right simple element semigroup, then S is a group.*

Proof. Let $r \in R$. Since $Sr = S$, there exists $e \in S$ such that $er = r$. For any $s \in S$, there exists $t \in S$ such that $rt = s$. Thus, $es = ert = rt = s$, i.e., e is a left identity for S. Also, $Se = S$ and therefore for any $s \in S$, $s = ue$ for some $u \in S$. It follows that e is an identity for S. In particular, e is an idempotent. By the dual of Theorem 1.27 of [1], S is a left group and must be the direct product $G \times E$ of a group and a left zero semigroup. Evidently, $E = \{e\}$ and so $S \cong G$. ■

Theorem 4.12 also says that $S = R = C$, i.e., $N = \square$. By the contrapositive of Theorem 4.12, we have the following result.

COROLLARY 4.13. *If S is an idempotent free, right simple element semigroup, then S contains left ideals.*

To conclude this section, we compare some of the results obtained here, to those listed in [1; p. 39]:

I. (RS) and (LS): S is a group, [1; p. 39].

(RSE) and (LS): S is a group, Theorem 4.12, above.

II. (RS) and (LC): S is a right group, [1; p. 39].
(RSE) and (LC): S is a partial right group, Theorem 4.6, above.

III. (RS) and (RC): The Baer–Levi theory, [2; Chp. 8].
(RSE) and (RC): Theorem 4.9 and its lemmas, above.

Lastly, there now exist obvious generalizations utilizing the left and right duals of the results of this section, e.g., one should now consider semigroups having right simple elements *and* left simple elements, and so on.

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