Semi-confluent and weakly confluent images of tree-like and atriodic continua *

by

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Abstract. Let X be a compact metric continuum and f be a continuous function from X onto Y. The principle results are as follows. (1) If X is arc-like and f is semi-confluent, then Y is atriodic. (2) If X is tree-like and f is semi-confluent, then Y is hereditarily unicoherent. (3) A new proof that Y is atriodic if X is atriodic and Suslinian and f is weakly confluent. (4) An example is given where X is arc-like and non-Suslinian and f is a weakly confluent map onto a simple triod.

Confluent maps were introduced by Charatonik [2] in 1964 and have been studied extensively since. More recently this notion has been generalized by Maćkowiak [5] and Lelek [4] to that of semi-confluent maps and weakly confluent maps, respectively. Maćkowiak has extended to semi-confluent maps [5, p. 262] the theorem due to Charatonik [2, p. 217] that the image of a λ-dendroid under a confluent map is a λ-dendroid and in addition he has shown that the semi-confluent image of an arc is an arc [5, p. 262]. Two theorems of the present paper give the results that the image of an arc-like continuum under a semi-confluent map is atriodic and that the image of a tree-like continuum under a semi-confluent map is hereditarily unicoherent.

As corollaries we obtain the results of Maćkowiak that the semi-confluent image of an arc is an arc and, that the semi-confluent image of a λ-dendroid is a λ-dendroid. Using Bing's theorem that any hereditarily decomposable, hereditarily unicoherent, atriodic continuum is arc-like, we also obtain as a corollary the result that the semi-confluent image of a hereditarily decomposable arc-like continuum is arc-like (and hereditarily decomposable).

Concerning weakly confluent maps, the main result is that the weakly confluent image of an atriodic continuum is atriodic, provided that the domain is Suslinian. An example of a weakly confluent mapping of an arc-like continuum onto a simple triod is given to show that the Suslinian condition cannot be dropped from the hypothesis even with the stronger hypothesis that the continuum is arc-like. A cor-

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ollary of the main result on weakly confluent maps establishes that the weakly confluent image of an arc is an arc or a simple closed curve.

For each of the foregoing results on semi-confluent maps, a corresponding result (having a similar proof) for circle-like continua is also given.

Throughout this paper a continuum is a compact, connected metric space and all maps are continuous. A map f from a continuum X onto a continuum Y is *confluent* if for each continuum $K \subset Y$ and each component C of $f^{-1}(K)$, $f(C) = K$; if for every continuum $K \subset Y$ there exists at least one component C of $f^{-1}(K)$ such that $f(C) = K$, then f is *weakly confluent*; and if for each continuum $K \subset Y$ and each two components C_1, C_2 of $f^{-1}(K)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$ then f is *semi-confluent*. It is obvious that every confluent map is semi-confluent and it is known that every semi-confluent map is weakly confluent [5, p. 254]. A *chain* or *linear chain* (respectively, *circular chain*) is a finite collection $\{C_1, \ldots, C_n\}$ of open sets such that $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$ (respectively, $|i-j| \leqslant 1$ or $|i-j| = n-1$). A continuum X is *arc-like* (respectively, *circle-like*) if for every $\varepsilon > 0$ there exists a cover of X by a linear chain (respectively, circular chain) of mesh less than ε, i.e., a linear chain (respectively, circular chain) whose elements, called *links*, have diameters less than ε. A *tree-chain* is a finite coherent collection of open sets such that no three of the sets have a point in common and no subcollection is a circular chain. A continuum X is *tree-like* if for every $\varepsilon > 0$ there exists an open cover of X by a tree chain of mesh less than ε. A continuum is *Suslinian* if it does not contain an uncountable collection of mutually disjoint, nondegenerate subcontinua. A *triod* is the union of three continua such that the common part of all three of them is a nonempty proper subcontinuum of each of them and is also the common part of each two of them. A continuum is *atriodic* if it does not contain a triod.

THEOREM 1. *If f is a semi-confluent map defined on an arc-like continuum X, then $Y = f(X)$ is atriodic.*

Proof. Suppose Y contains a triod $W = A \cup B \cup C$ where A, B and C are continua and $A \cap B = A \cap C = B \cap C = Q$, a proper subcontinuum of A, B and C. Let a, b and c be points in $A \setminus Q$, $B \setminus Q$ and $C \setminus Q$, respectively. If there exists a component of $f^{-1}(A \cup B)$ that contains a point of $f^{-1}(a)$ but no point of $f^{-1}(b)$, then every component of $f^{-1}(A \cup B)$ that contains a point of $f^{-1}(b)$ must also contain a point of $f^{-1}(a)$, due to the semi-confluence of f. Hence, either every component of $f^{-1}(A \cup B)$ that intersects $f^{-1}(a)$ also intersects $f^{-1}(b)$ or every component that intersects $f^{-1}(b)$ also intersects $f^{-1}(a)$; without loss of generality we will assume the former. Now let X_{ac} be a component of $f^{-1}(A \cup C)$ containing points a' and c' such that $f(a') = a$, $f(c') = c$. Then let X_{ab} be the component of $f^{-1}(A \cup B)$ that contains a' and hence a point b' such that $f(b') = b$. Let $\varepsilon = \min\{\mathrm{dist}(a, B \cup C), \mathrm{dist}(c, A \cup B), \mathrm{dist}(b, A \cup C)\}$ and let $\mathcal{U}$ be an open cover of Y by sets of diameter less than $\frac{1}{2}\varepsilon$. Denote by $\mathcal{T}$ a linear chain covering X which refines $\{f^{-1}(U) \mid U \in \mathcal{U}\}$. Let $T_{a'}$, $T_{b'}$ and $T_{c'}$ be links in $\mathcal{T}$ containing a', b' and c', respectively, and U_a, U_b and U_c be members of $\mathcal{U}$ such that $f(T_{a'}) \subset U_a$, $f(T_{b'}) \subset U_b$ and $f(T_{c'}) \subset U_c$. Designate by A_0 a subcontinuum of $A \setminus U_a$ that contains Q and intersects $\overline{U}_a$. Designate by B_0 a subcontinuum of $B \setminus U_b$ that contains Q and intersects $\overline{U}_b$ and for which there is a point b'', lying in X_{ab} and in a link of $\mathcal{T}$ between $T_{a'}$ and $T_{b'}$, such that $f(b'') \in B_0 \cap \overline{U}_b$. Define C_0 and c'' analogously to B_0 and b''. Consider the components of $f^{-1}(W_0)$, where $W_0 = A_0 \cup B_0 \cup C_0$, and let $X_{b''}$ and $X_{c''}$ be the components of $f^{-1}(W_0)$ containing b'' and c'', respectively.

If T is a link of $\mathcal{T}$ between $T_{a'}$ and $T_{c'}$ then T intersects X_{ac} and so $f(T)$ is contained in the $\frac{1}{2}\varepsilon$ neighborhood of $A \cup C$. Since $\mathrm{dist}(f(b''), A \cup C) > \frac{1}{2}\varepsilon$, it follows that $f(b'') \notin f(T)$. But $X_{c''} \subset \bigcup\{T \mid T$ is a link of $\mathcal{T}$ between $T_{a'}$ and $T_{c'}\}$, since $X_{c''} \cap (T_{a'} \cup T_{b'} \cup T_{c'}) = \emptyset$, and so $f(b'') \notin f(X_{c''})$. Similarly $f(c'') \notin f(X_{b''})$. This contradicts the semi-confluence of f, since $f(c'') \in f(X_{c''})$ and $f(b'') \in f(X_{b''})$, and completes the proof.

THEOREM 2. *If f is a semi-confluent map defined on a tree-like continuum X, then $Y = f(X)$ is hereditarily unicoherent.*

Proof. Suppose Y is not hereditarily unicoherent, i.e., assume Y contains a subcontinuum S such that $S = A \cup B$ where A and B are subcontinua and $A \cap B = P \cup Q$, a separation. Since f is weakly confluent, there is a component X' of $f^{-1}(S)$ such that $f(X') = S$. It is clear that $f|X'$ is semi-confluent so no generality will be lost if we take X to be X'. Let ε be less than $\frac{1}{4}\,\mathrm{dist}(P, Q)$ and let $\mathcal{U}$ be an open cover of S such that if $U \in \mathcal{U}$ then the diameter of U is less than ε and if both $U \cap A$ and $U \cap B$ are nonvoid then $U \cap (A \cap B) \neq \emptyset$. Let $\mathcal{T}$ be a tree-cover of X that refines $\{f^{-1}(U) \mid U \in \mathcal{U}\}$. Let f_A be a continuous function from A onto $[0, 1]$ such that $f_A^{-1}(0) = P$ and $f_A^{-1}(1) = Q$ and let f_B be a continuous function from B onto $[1, 2]$ such that $f_B^{-1}(1) = Q$ and $f_B^{-1}(2) = P$.

It will be helpful to picture S with A on the right-hand side, B on the left-hand side, P at the top and Q at the bottom. Using $\mathcal{T}$, the function f, and the functions f_A and f_B, we now define a function g from X into the reals that keeps track of the net amount of "clockwise wrapping" of X around S by f.

Let T be a member of $\mathcal{T}$ such that $f(T) \subset A \setminus B$ and $\mathrm{dist}(f(T), Q) < \mathrm{dist}(f(T), P)$. Consider any linear chain $\{T_1, \ldots, T_n\}$ in $\mathcal{T}$ with T as the first link, i.e., $T = T_1$. For each x in T, let $g(x) = f_A(f(x))$. Suppose g has been defined on $\bigcup_{j=1}^{i} T_j$ so that, for $j = 1, 2, \ldots, i$, there exist integers k_j' and k_j'', with $k_1' = k_1'' = 0$, such that

(1) for each x in T_j, $g(x) = f_A(f(x)) + 2k_j'$ if $f(x) \in A$ and $g(x) = f_B(f(x)) + 2k_j''$ fi $f(x) \in B$ and

(2) $k_j' = k_j''$ if $\mathrm{dist}(f(T_j), Q) < \mathrm{dist}(f(T_j), P)$ and $k_j' - 1 = k_j''$ if $\mathrm{dist}(f(T_j), P) \leqslant \mathrm{dist}(f(T_j), Q)$.

If $x \in T_{i+1}$, then let

$$g(x) = \begin{cases} f_A(f(x)) + 2k_i' & \text{if } f(x) \in A, \\ f_B(f(x)) + 2k_i'' & \text{if } f(x) \in B. \end{cases}$$

(1) If $f(T_i) \cup f(T_{i+1}) \subset A \setminus B$, then

a) let $k'_{i+1} = k'_i$ and $k''_{i+1} = k''_i + 1$, if $\operatorname{dist}(f(T_i), P) \leqslant \operatorname{dist}(f(T_i), Q)$ and $\operatorname{dist}(f(T_{i+1}), P) > \operatorname{dist}(f(T_{i+1}), Q)$ and

b) let $k'_{i+1} = k'_i$ and $k''_{i+1} = k''_i - 1$, if $\operatorname{dist}(f(T_i), P) > \operatorname{dist}(f(T_i), Q)$ and $\operatorname{dist}(f(T_{i+1}), P) \leqslant \operatorname{dist}(f(T_{i+1}), Q)$.

2) If $f(T_i) \cup f(T_{i+1}) \subset B \setminus A$, then

a) let $k'_{i+1} = k'_i + 1$ and $k''_{i+1} = k''_i$, if $\operatorname{dist}(f(T_i), Q) < \operatorname{dist}(f(T_i), P)$ and $\operatorname{dist}(f(T_{i+1}), Q) \geqslant \operatorname{dist}(f(T_{i+1}), P)$ and

b) let $k'_{i+1} = k'_i - 1$ and $k''_{i+1} = k''_i$, if $\operatorname{dist}(f(T_i), Q) \geqslant \operatorname{dist}(f(T_i), P)$ and $\operatorname{dist}(f(T_{i+1}), Q) < \operatorname{dist}(f(T_{i+1}), P)$.

3) Otherwise let $k'_{i+1} = k'_i$ and $k''_{i+1} = k''_i$.

Defined in this way, g is continuous on $\bigcup_{i=1}^{n} T_i$. But each member of $\mathscr{T}$ is in a linear chain having T as the first link. Defining g as above on each such linear chain we have g continuous on X. The crucial facts that insure that the definition of g is not ambiguous are that $\mathscr{T}$ has no subcollection that forms a circular subchain and if T is a link such that $f(T) \cap A \neq \emptyset$ and $f(T) \cap B \neq \emptyset$ then $\operatorname{dist}(f(T), P) < \frac{1}{4}\operatorname{dist}(P, Q)$ or $\operatorname{dist}(f(T), Q) < \frac{1}{4}\operatorname{dist}(P, Q)$. Since g is continuous and X is a continuum, $g(X) = [m, M]$ a closed interval of real numbers. Let x_m and x_M be points of $g^{-1}(m)$ and $g^{-1}(M)$, respectively. Note that a continuum in X containing x_M maps onto a set in S "going around" S from $f(x_M)$ in a "counterclockwise" direction. Similarly for x_m, except that the direction is "clockwise."

There are two cases: (1) $f(x_m)$ and $f(x_M)$ do not both belong to the same one of the sets A and B, and (2) $f(x_m)$ and $f(x_M)$ both belong to A or both belong to B.

Case 1. We will prove the theorem for the subcase where $f(x_m) \in B \setminus A$ and $f(x_M) \in A \setminus B$. A similar proof applies to the subcase where $f(x_m) \in A \setminus B$ and $f(x_M) \in B \setminus A$. Let $h \in (f_A(f(x_M)), 1)$. We wish to extend B in the direction of increasing values of f_A to include a point of $f_A^{-1}(h)$ as follows. Let B_m and B_M be the images under f of the x_m-component and x_M-component, respectively, of $f^{-1}(B \cup f_A^{-1}([0, h]))$. The x_m-component of $f^{-1}(B \cup B_m \cup B_M)$ contains a point of $f^{-1}(f_A^{-1}(h))$ but does not contain a point of $f^{-1}(Q)$. The opposite situation holds for the x_M-component of $f^{-1}(B \cup B_m \cup B_M)$. Hence neither of these two components of the preimage of the continuum $B \cup B_m \cup B_M$ maps into a subset of the image of the other. This contradicts the semiconfluence of f.

Case 2. First assume that $f(x_M) \in B$ and $f(x_m) \in B \setminus Q$. Let $h = \frac{1}{2}$ and, using this h, construct B_m as in Case 1. Here the contradiction follows as in Case 1, using $B \cup B_m$ instead of $B \cup B_m \cup B_M$. The subcase where $f(x_m) \in Q$ and $f(x_M) \in B \setminus P$ is handled similarly with the roles of $f(x_m)$ and $f(x_M)$ reversed. In the case where $f(x_m) \in Q$ and $f(x_M) \in P$, the contradiction follows as in Case 1 using A instead of $B \cup B_m \cup B_M$. The subcases where $f(x_M)$ and $f(x_m)$ are in A are essentially like the ones in which they are in B.

As corollaries of Theorems 1 and 2 we obtain the following results referred to in the introductory remarks.

COROLLARY 1 (Maćkowiak). *If f is a semi-confluent map defined on $I = [0, 1]$, then $f(I)$ is an arc or a point.*

Proof. Assume $f(I)$ is nondegenerate. By Theorem 1, $f(I)$ is atriodic and by Theorem 2, $f(I)$ is hereditarily unicoherent. Since $f(I)$ is also locally connected, it follows that $f(I)$ is an arc.

COROLLARY 2 (Maćkowiak). *If f is a semi-confluent map defined on a λ-dendroid X then $f(X)$ is a λ-dendroid.*

Proof. By Theorem 2, $f(X)$ is hereditarily unicoherent and by [5, p. 261], $f(X)$ is hereditarily decomposable.

A hereditarily decomposable continuum is arc-like if and only if it is atriodic and hereditarily unicoherent [1, p. 660]. This result together with Theorems 1 and 2 yields the following.

COROLLARY 3. *If f is a semi-confluent map defined on a hereditarily decomposable arc-like continuum X, then $Y = f(X)$ is arc-like.*

Proof. We know from Theorems 1 and 2 that Y is atriodic and hereditarily unicoherent. From [5, p. 261] we also know that Y is hereditarily decomposable. Therefore it follows that Y is arc-like.

Corollary 3 gives a partial answer to Maćkowiak's question [5, p. 263] (first asked about confluent maps by Lelek [4, p. 102]): Is the image of an arc-like continuum under a semi-confluent map an arc-like continuum?

With slight modifications in the proofs, the preceeding results can be altered to yield the following corresponding results for circle-like continua.

THEOREM 3. *If f is a semi-confluent map defined on a circle-like continuum X, then $Y = f(X)$ is atriodic.*

THEOREM 4. *If f is a semi-confluent map defined on a circle-like continuum X, then every proper subcontinuum of $Y = f(X)$ is unicoherent.*

COROLLARY 4. *If f is a semi-confluent map defined on the simple closed curve J, then $f(J)$ is an arc, a simple closed curve or a point.*

W. T. Ingram [3, p. 198] has proved that if X is an atriodic, hereditarily decomposable continuum that is not unicoherent but for which every proper subcontinuum is unicoherent, then X is circle-like. Using this result and the above theorems we obtain the following.

COROLLARY 5. *If f is a semi-confluent map defined on a hereditarily decomposable circle-like continuum X, then $Y = f(X)$ is arc-like or circle-like.*

Proof. We know from Theorem 3 that Y is atriodic and from [5, p. 261] that Y is hereditarily decomposable. From Theorem 4 every proper subcontinuum of Y is unicoherent. If Y is unicoherent then by [1, p. 660], Y is arc-like. If Y is not unicoherent then by Ingram's result, Y is circle-like.

We now turn our attention to the image of an atriodic continuum under a weakly confluent map.

We first submitted this paper for publication with Theorem 5 (below) proved under the hypothesis that X is arc-like rather than atriodic. Subsequently we learned from Howard Cook and Andrzej Lelek that they had proved this theorem under the more general hypothesis that X is atriodic. With only minor (and, in fact, simplifying) changes, our proof of the arc-like theorem applied to the atriodic case. Cook and Lelek are including their theorem in a joint paper to be submitted to the Canadian Journal of Mathematics. However, at their suggestion, our proof, which is simpler than theirs, is included here.

THEOREM 5 (Cook and Lelek). *If X is an atriodic continuum and f is a weakly confluent map from X onto a continuum Y that contains a triod, then X is not Suslinian.*

Proof. Suppose Y contains a triod T where T is the union of four distinct subcontinua A, B, C, and Q such that $A \cap B = B \cap C = C \cap A = Q$. Let a, b, and c be points of $A \setminus Q$, $B \setminus Q$ and $C \setminus Q$, respectively. For convenience of notation, let the distance function for X be such that $\operatorname{dist}(a, B \cup C) = \operatorname{dist}(b, A \cup C) = \operatorname{dist}(c, A \cup B) = 1$. For each ε in $(0, 1]$, let $N_\varepsilon(Q) = \{y \in Y \mid \operatorname{dist}(y, Q) < \varepsilon\}$, let a_ε, b_ε and c_ε be points of the boundary of $N_\varepsilon(Q)$ in A, B and C, respectively, and let A_ε, B_ε, C_ε and T_ε be subcontinua of A, B, C and T, respectively, in the closure of $N_\varepsilon(Q)$, such that A_ε, B_ε and C_ε contain $\{a_\varepsilon\} \cup Q$, $\{b_\varepsilon\} \cup Q$ and $\{c_\varepsilon\} \cup Q$, respectively, and $T_\varepsilon = A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon$.

For each ε in $(\frac{1}{4}, 1]$, let T'_ε be a component of $f^{-1}(T_\varepsilon)$ that maps onto T_ε let a'_ε, b'_ε and c'_ε be points of T'_ε in $f^{-1}(a_\varepsilon)$, $f^{-1}(b_\varepsilon)$ and $f^{-1}(c_\varepsilon)$ respectively, and let $K_\varepsilon(A)$, $K_\varepsilon(B)$ and $K_\varepsilon(C)$ be the a'_ε-component of $f^{-1}(A_\varepsilon \setminus N_{1/4}(Q))$, the b'_ε-component of $f^{-1}(B_\varepsilon \setminus N_{1/4}(Q))$ and the c'_ε-component of $f^{-1}(C_\varepsilon \setminus N_{1/4}(Q))$, respectively. Note that $T'_\varepsilon \supset K_\varepsilon(A) \cup K_\varepsilon(B) \cup K_\varepsilon(C)$ and that each of $K_\varepsilon(A)$, $K_\varepsilon(B)$ and $K_\varepsilon(C)$ is non-degenerate since $K_\varepsilon(D)$ contains a point of $f^{-1}(\operatorname{Cl}[N_{1/4}(Q)])$, for $D = A$, B and C.

For each ε in $(\frac{1}{4}, 1)$ and any α, β and γ in $(\varepsilon, 1]$ either $T'_\varepsilon \cap K_\alpha(A) = \emptyset$ or $T'_\varepsilon \cap K_\beta(B) = \emptyset$ or $T'_\varepsilon \cap K_\gamma(C) = \emptyset$, since otherwise $T'_\varepsilon \cup K_\alpha(A) \cup K_\beta(B) \cup K_\gamma(C)$ is a triod in X. It follows that, for each ε in $(\frac{1}{4}, 1)$, there is $D = A$, B or C such that if δ is in $(\varepsilon, 1]$ then $T'_\varepsilon \cap K_\delta(D) = \emptyset$. For each ε in $(\frac{1}{4}, 1)$ let D_ε be such a D. It follows that there is an uncountable subset E of $(\frac{1}{4}, 1)$ such that if ε and δ are in E then $D_\varepsilon = D_\delta$. Assume without loss of generality that there is an uncountable subset E of $(\frac{1}{4}, 1)$ such that if ε is in E and α is in $(\varepsilon, 1]$ then $T'_\varepsilon \cap K_\alpha(A) = \emptyset$. Then $\{K_\varepsilon(A) \mid \varepsilon \text{ is in } E\}$ is an uncountable disjoint collection of nondegenerate subcontinua of A, since E is uncountable, and if ε and δ are in E and $\varepsilon < \delta$ then $K_\varepsilon(A) \subset T'_\varepsilon$ and $T'_\varepsilon \cap K_\delta(A) = \emptyset$.

Restating Theorem 5 we get the following.

THEOREM 5'. *If f is a weakly confluent map defined on an atriodic, Suslinian continuum X, then $Y = f(X)$ is atriodic.*

COROLLARY 6. *If X is an arc-like or circle-like continuum and f is a weakly confluent map from X onto a continuum Y that contains a triod, then X is not Suslinian.*

COROLLARY 6'. *If f is a weakly confluent map defined on an arc-like or circle-like, Suslinian continuum X, then $Y = f(X)$ is atriodic.*

The following examples show that the Suslinian condition in Theorem 5' is necessary to insure that the image Y is atriodic, even when the domain is arc-like or circle-like.

EXAMPLE 1. A weakly confluent map from an arc-like continuum X onto a simple triod. Let

$A = \{(x, y) \mid -2 \leqslant x \leqslant 2$ and y is in the Cantor set$\}$,

$B = \{(x, y) \mid x = -2$ and y is in some deleted open interval of length $1/3^{2n-1}$ of the Cantor set$\}$,

$C = \{(x, y) \mid x = 2$ and y is in some deleted open interval of length $1/3^{2n}$ of the Cantor set$\}$ and

$X = A \cup B \cup C$.

Let Y be the simple triod consisting of the unit interval on the y-axis and the interval $[-2, 2]$ on the x-axis. Let f be the standard nondecreasing map from the Cantor set onto the unit interval ($f(\frac{1}{3}) = f(\frac{2}{3}) = \frac{1}{2}$, etc.). Let $F\colon X \to Y$ be defined as follows

$$F((x, y)) = \begin{cases} (0, f(y)) & \text{if } x = 0 \text{ and } (x, y) \in A, \\ (0, 0) & \text{if } |x| = y \text{ and } (x, y) \in A, \\ (-2, 0) & \text{if } (x, y) \in B, \\ (2, 0) & \text{if } (x, y) \in C, \\ \left(0, f(y) \cdot \dfrac{y - |x|}{y}\right) & \text{if } 0 < |x| < y \text{ and } (x, y) \in A, \\ \left(\dfrac{2(y - |x|)}{y - 2} \cdot \dfrac{|x|}{x}, 0\right) & \text{if } y < |x| \text{ and } (x, y) \in A. \end{cases}$$

The last two parts of the definition of F merely say that F is linear on intervals between points where the first four parts of the definition apply. It is straightforward to check that F is a weakly confluent map of the arc-like continuum X onto the simple triod Y. Note that F maps each vertical interval in X onto either $(-2, 0)$ or $(2, 0)$. If X' is the decomposition of X gotten by "shrinking" each maximal vertical interval in X to a point, and F' is the map on X' derived from F, then X' is arc-like and F' is a light, weakly confluent map from X' onto Y.

EXAMPLE 2. A weakly confluent map from a circle-like continuum onto a simple triod.

Let X'' be the continuum obtained from X of Example 1 by identifying $(2, 0)$ and $(2, 1)$ and let F'' be the map on X'' derived from F. Then F'' is a weakly confluent map from the circle-like continuum X'' onto a simple triod.

COROLLARY 7. *If f is a weakly confluent map defined on $I = [0, 1]$, then $f(I)$ is an arc, a simple closed curve or a point.*

Proof. Assume $f(I)$ is nondegenerate. By Theorem 5′ $f(I)$ is atriodic. Since $f(I)$ is also locally connected, $f(I)$ is an arc or a simple closed curve.

To show that a simple closed curve can be obtained as the weakly confluent image of I, and also to show that the weakly confluent image of an arc-like continuum need not be unicoherent, consider the following example.

EXAMPLE 3. A weakly confluent map from $I = [0, 1]$ onto the unit circle, J, in the plane.

If $\theta \in I$, let $f(\theta) = e^{4\pi i\theta}$. Clearly f is a weakly confluent map from I onto J.

COROLLARY 8. *If f is a weakly confluent map defined on a simple closed curve J, then $f(J)$ is an arc, a simple closed curve or a point.*

EXAMPLE 4. A confluent map from the unit circle, J, in the plane onto $[-1, 1]$.

If (x, y) is in J, let $f((x, y)) = x$. Clearly f is a confluent map from J onto $[-1, 1]$.

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A generalization of right simple semigroups

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Abstract. An element s in a semigroup S is called a *right simple* element if $sS = S$. This paper developes the notion of right simple elements, and uses it to generalize right simple semigroups.

A non-right simple semigroup with right simple elements is called a *right simple element* semigroup and denoted as RSE. The subset of S of right simple elements is denoted by R, and the non-right simple elements by N. If R is a right simple subsemigroup (right group, subgroup) of S, then S is called a *partial* right simple semigroup (*partial* right group, *partial* group) and denoted by PRS (PRG, PG). While a PRS semigroup is by definition an RSE semigroup, the converse is shown to be false.

The structure of RSE semigroups is determined, and a decomposition found for R. The existence of a maximum right ideal is found to be a necessary, but not sufficient, condition for right simple elements to exist. A partial converse is given.

The structure theorems are then applied to RSE semigroups possessing other properties, such as the descending chain condition on right ideals of N, finiteness, or left (right) cancellativity. It is shown that, if S is a RSE and left simple (left cancellative), then S is a PG (PRG). For right cancellativity, the development parallels that of the Baer–Levi Theory.

1. Introduction. Recall that a semigroup S is called right simple if for all s in S, $sS = S$. An element x in a semigroup S will be called a *right simple element* if $xS = S$. Note that a semigroup is right simple if and only if each of its elements is a right simple element.

This paper uses the concept of right simple elements to generalize right simple semigroups. In particular, semigroups containing right simple elements are investigated, with some of the results obtained analogous to those obtained for right simple semigroups. Throughout the paper, a semigroup containing right simple elements will be called a *right simple element* semigroup, and it will be denoted by RSE. The class of RSE semigroups therefore contains the class of right simple semigroups.

In Section 2, various structure theorems are presented for RSE semigroups. The results of Section 2 are then used in Section 3 to discuss homomorphisms on RSE semigroups, and to extend group homomorphism results. In Section 4, applications of Sections 2 and 3 are developed for RSE semigroups where other conditions such as right (left) cancellativity, left simplicity, finiteness, are also present. Examples appear in various parts of the paper.