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On the category of commutative connected graded Hopf algebras over a perfect field

by

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Abstract. Let $\mathcal{H}$ be the category of all commutative, cocommutative, connected, graded Hopf algebras over a given perfect field k of finite characteristic p. By [13] $\mathcal{H}$ is a locally noetherian Grothendieck category of global dimension two. Using functor category methods [10], we prove that

(a) $\mathcal{H}$ is semiperfect, i.e. each of its noetherian objects has a projective cover.

(b) The endomorphism ring of any noetherian object in $\mathcal{H}$ is a module of finite length over the ring of infinite Witt p-vectors over k.

(c) Any flat object in $\mathcal{H}$ is a directed union of countably generated pure flat subobjects and has the projective dimension at most 1.

(d) Every primitively generated Hopf algebra from $\mathcal{H}$ is a coproduct of Hopf algebras of the form $k[x]/(x^{p^i})$.

We describe local noetherian objects in $\mathcal{H}$.

Introduction. Let k be a perfect field of finite characteristic p and let $\mathcal{H}$ denote the category of all commutative, cocommutative, connected, graded Hopf k-algebras. In [13] Schoeller showed that $\mathcal{H} = \mathcal{H}^- \times \mathcal{H}^+$, where $\mathcal{H}^-$ is the full subcategory of $\mathcal{H}$ consisting of Hopf algebras generated by elements of odd degrees and $\mathcal{H}^+$ consists of all Hopf algebras which are zero in odd degrees. Furthermore, $\mathrm{gl.dim}\,\mathcal{H}^- = 0$ and $\mathcal{H}^+$ is a product of a countable number of copies of a full subcategory $\mathcal{H}_1$ of $\mathcal{H}^+$ consisting of all Hopf algebras generated by elements of degrees $2p^i$ where $i = 0, 1, \ldots$ Moreover, $\mathcal{H}_1$ has enough noetherian projective objects and therefore $\mathcal{H}_1 = \mathcal{P}^{\mathrm{op}}$-Mod, where $\mathcal{P}$ consists of all indecomposable noetherian projective objects in $\mathcal{H}_1$. Then we can apply to the study of $\mathcal{H}$ functor category methods [10].

Section 1 contains the basic results on semiperfect functor categories needed in the paper. In Section 2 we recall some fundamental facts concerning the category $\mathcal{H}$. In Section 3 we define a useful $W(k)$-category structure on $\mathcal{H}_1$ and on $\mathcal{P}$-Mod, where $W(k)$ is the ring of infinite Witt p-vectors. Using this fact, we show that $\mathrm{Hom}_{\mathcal{H}_1}(N, N')$ is a $W(k)$-module of finite length for any noetherian objects N and N' in $\mathcal{H}_1$. It is also proved that the category $\mathcal{P}$-Mod is locally noetherian of global dimension two and the set of one-sided ideals in $\mathcal{P}$ is countable.

Section 4 contains a generalization of the results in [14] and [16] concerning the projective dimension of flat objects in a functor category. As a consequence, we infer that the projective dimension of any flat object in $\mathscr{H}_1$ as well as of any flat $\mathscr{P}$-module is at most 1. In Section 5 we study primitively generated Hopf algebras from $\mathscr{H}$. Our main result asserts that every such Hopf algebra is a coproduct of Hopf algebras of the form $k[x]/(x^{p^i})$, which is a generalization of well-known Milnor–Moore result in [9]. In the last section a complete description of all local noetherian objects in $\mathscr{H}_1$ is given.

Throughout the paper k is a perfect field of finite characteristic $p\geqslant 2$ and R denotes a commutative ring with an identity element. If $\mathscr{A}$ is a locally finitely presented Grothendieck category, we denote by $\mathrm{fp}(\mathscr{A})$ (resp. $\mathrm{fg}(\mathscr{A})$) its full subcategory consisting of all finitely presented (resp. finitely generated) objects.

§ 1. Semiperfect functor categories.

Let $\mathscr{C}$ be a skeletally small additive category (not necessarily with coproducts). A $\mathscr{C}$-module is a covariant additive functor from $\mathscr{C}$ to the category of abelian groups. The category $\mathscr{C}$-Mod of all $\mathscr{C}$-modules is a Grothendieck category and $\mathscr{C}$-modules of the form

$$h^X = \mathrm{Hom}_{\mathscr{C}}(X, -), \qquad X\in\mathscr{C},$$

form a set of finitely presented projective generators of $\mathscr{C}$-Mod.

A $\mathscr{C}$-module is free if it is isomorphic with a coproduct of modules h^X. A left ideal in $\mathscr{C}$ is a $\mathscr{C}$-submodule of an h^X, $X\in\mathscr{C}$; a right ideal in $\mathscr{C}$ is a submodule of an $\mathscr{C}^{\mathrm{op}}$-module $h_X = \mathrm{Hom}_{\mathscr{C}}(-, X)$, $X\in\mathscr{C}$. A two-sided ideal in $\mathscr{C}$ is a subfunctor of the functor

$$\mathrm{Hom}_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to\mathscr{A}b\,.$$

If I is a two-sided ideal in $\mathscr{C}$, then we define the quotient category $\mathscr{C}/I$, which has the same objects as $\mathscr{C}$ and

$$\mathrm{Hom}_{\mathscr{C}/I}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)/I(X, Y)\,.$$

The *Jacobson radical* of an additive category $\mathscr{C}$ is a two-sided ideal $J(\mathscr{C})$ defined by

$$J(\mathscr{C})(A, B) = \{f\in\mathrm{Hom}_{\mathscr{C}}(A, B),\ 1_A\text{-}gf \text{ has a two-sided inverse for every } g\}$$

(see [10]). It is not difficult to check that $J(\mathscr{C}/J(\mathscr{C})) = 0$ and that $J(\mathscr{C})(X, X)$ is the Jacobson radical of the endomorphism ring

$$\mathrm{End}_{\mathscr{C}}(X) = \mathrm{Hom}_{\mathscr{C}}(X, X)$$

for every object X from $\mathscr{C}$. Moreover, the following simple lemma holds:

LEMMA 1.1. *If X and Y have local endomorphism rings, then $J(\mathscr{C})(X, Y)$ is a group of all non-isomorphisms from X to Y.*

A small additive category $\mathscr{C}$ is *semi-simple* if each $\mathscr{C}$-module h^X, $X\in\mathscr{C}$, is a coproduct of simple left ideals. $\mathscr{C}$ is *regular* in the sense of von Neumann if for each of its morphisms f there exists a g such that $f = fgf$.

A Grothendieck category is *semiperfect* (resp. *F-semiperfect*) if each of its finitely generated (resp. finitely presented) objects has a projective cover (see [1], [8], [16] and [23]).

Recall that an object is called *local* if it has a unique maximal proper subobject [22].

For a functor category we have the following results:

THEOREM 1.2. *Let $\mathscr{C}$ be a skeletally small additive category. The following conditions are equivalent:*

(a) *$\mathscr{C}$-Mod is semiperfect.*

(b) *Every simple $\mathscr{C}$-module has a projective cover.*

(c) *$\mathscr{C}/J(\mathscr{C})$ is semi-simple and idempotents can be lifted modulo $J(\mathscr{C})$.*

(d) *Every finitely generated projective (free) $\mathscr{C}$-module has a semiperfect endomorphism ring.*

(e) *Any $\mathscr{C}$-module h^X, $X\in\mathscr{C}$, is a finite coproduct of local left ideals generated by idempotents.*

(f) *Any projective $\mathscr{C}$-module P is a coproduct of local left ideals generated by idempotents.*

(a′), (b′), (d′), (e′), (f′) *for $\mathscr{C}^{\mathrm{op}}$-Mod and right ideals.*

Proof. The equivalences (a)↔(c)↔(d) are proved in [16], Theorem 5.6 and (a)↔(b)↔(e) may be proved as in Theorem 2.1 in [18]. Since (f)→(e) is trivial and since (c) is left-right symmetric, it remains to show that (e) implies (f).

Assume (e) and let P be a projective $\mathscr{C}$-module. Then there exists a Q such that $P\oplus Q = \bigoplus_{i\in I} L_i$, where L_i are local left ideals generated by idempotents. By (d) the endomorphism ring of any L_i is semiperfect and hence it is local because L_i is indecomposable. Then (f) is a consequence of the following theorem:

THEOREM 1.3. *Suppose that M is an object of a Grothendieck category which is a coproduct of countably generated objects M_i, $i\in I$, each with a local endomorphism ring. Then*

(a) *any two such decompositions are isomorphic.*

(b) *a direct summand of M is again a coproduct of subobjects, each isomorphic to one of the original summands M_i.*

Proof. See [4], Th. 4.2, [12], Th. 1.3, [19], Th. 1 and [20], Th. 7.

COROLLARY 1.4. *If $\mathscr{C}$-Mod is semiperfect, then any projective $\mathscr{C}$-module is a coproduct of local left ideals generated by idempotents and any two such decompositions are isomorphic.*

Generally, we can prove the following:

THEOREM 1.5. *If $\mathscr{C}$-Mod is F-semiperfect, then any projective $\mathscr{C}$-module is a coproduct of left ideals generated by idempotents.*

Proof. In view of Theorem 5.6 in [16] the theorem may be proved as Theorem 3 in [21].

An object M in a Grothendieck category is hollow if the equality $M = X+Y$ implies either $X = M$ or $Y = M$ (see [6]). It is clear that any local noetherian object is hollow. If M is noetherian and hollow, then M is also local. In fact, the family of all proper subobjects of M has a maximal element which is a unique maximal subobject of M.

LEMMA 1.6. *Let P be a projective finitely generated object of a semiperfect category $\mathscr{A}$. Then the following conditions are equivalent:*

(a) *P is indecomposable;*

(b) *P is hollow;*

(c) *P is a projective cover of a hollow object.*

Proof. First we remark that if $f\colon M\to N$ is an essential epimorphism, then M is hollow if and only if N is hollow. So (b)$\leftrightarrow$(c). Since (b)$\to$(a) is trivial, it remains to prove (a)$\to$(b).

Suppose that $P = M_1+M_2$, where M_1, M_2 are proper subobjects of P. Then the natural morphism $\nu\colon P\to P/M_1\oplus P/M_2$ is an epimorphism. The non-zero objects P/M_1, P/M_2 are finitely generated; let $r_1\colon p'\to P/M_1$, $r_2\colon P''\to P/M_2$ be their projective covers. Then $P'\oplus P''\xrightarrow{r_1\oplus r_2} P/M_1\oplus P/M_2$ is also the projective cover and hence there exists an epimorphism $P\to P'\oplus P''$. But this is impossible because P is indecomposable.

§ 2. Preliminary results on graded Hopf algebras. Recall that $\mathscr{H}$ is a locally noetherian Grothendieck category and $H\in\mathscr{H}$ is a noetherian object if H is finitely generated as a k-algebra (see [9] and [13]). Denote by $\mathscr{N}$ the full subcategory of $\mathscr{H}$ consisting of all noetherian objects.

Throughout this paper we assume that k is a perfect field of characteristic $p\geqslant 2$. Let $k[\underline{X}] = k[X_0, X_1, \ldots, X_n, \ldots]$ be the polynomial algebra on variables X_n, $n\in N$, with $\deg X_n = 2p^n$ and the comultiplication given by

$$\Delta(X_0) = X_0\otimes 1+1\otimes X_0,$$

$$\Delta(X_1) = X_1\otimes 1+1\otimes X_1+\frac{1}{p}[X_0^p\otimes 1+1\otimes X_0^p-(\Delta X_0)^p],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta(X_n) = X_n\otimes 1+1\otimes X_n+\frac{1}{p}[X_{n-1}^p\otimes 1+1\otimes X_{n-1}^p-(\Delta X_{n-1})^p]+$$
$$+\frac{1}{p^2}[X_{n-2}^{p^2}\otimes 1+1\otimes X_{n-2}^{p^2}-(\Delta X_{n-2})^{p^2}]+\ldots$$
$$+\frac{1}{p^n}[X_0^{p^n}\otimes 1+1\otimes X_0^{p^n}-(\Delta X_0)^{p^n}]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Next denote by ${}^nP$ the Hopf subalgebra of $k[\underline{X}]$ generated by $X_0, \ldots, X_n$ and consider the Hopf algebra map

$$j_{mn}\colon {}^nP\to {}^mP$$

defined as follows: If $m\geqslant n$, then j_{mn} is the natural injection, and if $m<n$, we put

$$j_{mn}(X_r) = \begin{cases} 0 & \text{for } r<n-m, \\ X_{r-n+m}^{p^{n-m}} & \text{for } r\geqslant n-m. \end{cases}$$

Finally, let ${}^nS = k[X]/(X^p)$, where $\deg X = 2p^n$ and $\Delta(X) = 1\otimes X+X\otimes 1$.

By $W(k)$ we denote the ring of infinite Witt p-vectors over k and by $W_n(k)$ we denote the ring of Witt p-vectors of length n (see [5], [7] and [24]).

We recall that $W(k)$ is a discrete valuation ring with the unique maximal ideal (p), $W(k)/(p)\approx k$ and $W_n(k)\cong W(k)/(p^n)$.

It follows from [13] that

(H1) $\operatorname{gl.dim}\mathscr{H} = \operatorname{gl.dim}\mathscr{H}_1 = 2$.

(H2) ${}^1S, {}^2S, {}^3S, \ldots$ is a complete list of simple objects in $\mathscr{H}_1$, ${}^nP$ is a projective cover of ${}^nS$, and ${}^1P, {}^2P, {}^3P, \ldots$ is a family of generators of $\mathscr{H}_1$.

(H3) $\operatorname{End} k[\underline{X}]\cong W(k)$ and $\operatorname{End}{}^nP\cong W_{n+1}(k)$.

It follows from (H2) that there is an equivalence $\mathscr{H}_1\approx\mathscr{P}^{\mathrm{op}}$-Mod, where $\mathscr{P}$ is the full subcategory of $\mathscr{H}_1$ consisting of all ${}^nP$ (see [13] p. 152). Henceforth we identify $\mathscr{H}_1$ and $\mathscr{P}^{\mathrm{op}}$-Mod.

In view of (H3) the results from Section 1 yield

COROLLARY 2.1. (a) *$\mathscr{H}_1$ is a semiperfect locally noetherian category.*

(b) *Any indecomposable projective object of $\mathscr{H}_1$ is isomorphic with a certain ${}^nP$.*

(c) *Any projective object in $\mathscr{H}_1$ is a coproduct of indecomposable ones and any two such decompositions are isomorphic.*

As an immediate consequence of Lemma 1.1 we get

COROLLARY 2.2.

$$J(\mathscr{P})({}^nP, {}^mP) = \begin{cases} \operatorname{Hom}_{\mathscr{P}}({}^nP, {}^mP) & \textit{for } n\neq m, \\ pW_{n+1}(k) & \textit{for } n = m. \end{cases}$$

Remark. The statement (d) of the theorem on page 140 in [13] is false. In fact, by (H3) $k[\underline{X}]$ is indecomposable, and in view of Corollary 2.1 it is not projective.

§ 3. On $\mathscr{P}$-Mod and $\mathscr{P}^{\mathrm{op}}$-Mod. We now make some observations on the categories $\mathscr{P}$-Mod and $\mathscr{P}^{\mathrm{op}}$-Mod. We start with some general facts.

Let R be a commutative ring. Recall that an additive category $\mathscr{C}$ is an R-category if $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ is an R-module for any $X, Y\in\mathscr{C}$ in such a way that the morphism composition is R-bilinear (see [2] and [10]). A functor $T\colon\mathscr{C}\to\mathscr{C}'$ between R-categories is an R-functor if the natural morphism $\operatorname{Hom}_{\mathscr{C}}(X, Y)\to\operatorname{Hom}_{\mathscr{C}'}(TX, TY)$ given by $f\mapsto T(f)$ is a homomorphism of R-modules for each $X, Y\in\mathscr{C}$. If $\mathscr{C}$ is an R-category and F is a $\mathscr{C}$-module, then $F(X)$ is, in a natural way, an R-module for any $X\in\mathscr{C}$. Moreover, if $f\colon X\to X'$ is a morphism in $\mathscr{C}$, then $F(f)$ is an R-homomorphism. It follows that $\mathscr{C}$-Mod is equivalent to the category of all R-functors from $\mathscr{C}$ to R-Mod (see [2], § 1).

An R-category $\mathscr{C}$ is called *hom-finite* if $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ is an R-module of finite length for any $X, Y \in \mathscr{C}$.

An example of a hom-finite R-category is any finite R-variety in the sense of [2] where R is an artinian ring.

In what follows we need the following result:

PROPOSITION 3.1. *Let R be a commutative ring and let $\mathscr{C}$ be an R-category. Then there is a unique R-category structure on* $\mathscr{C}$-Mod *such that the Yoneda embedding is an R-functor. Furthermore, if $\mathscr{C}$ is hom-finite, then so is* fg($\mathscr{C}$-Mod).

The proof is straightforward and it is left to the reader.

The following useful lemma gives us an important example of a $W(k)$-category.

LEMMA 3.2. *$\mathscr{P}$ is a hom-finite $W(k)$-category such that for any pair n, m $\operatorname{Hom}_{\mathscr{P}}({}^nP, {}^mP)$ is a cyclic $W(k)$-module generated by j_{mn} and isomorphic with $W(k)/(p^{s+1})$ where $s = \min(n, m)$.*

Proof. Since $\operatorname{Hom}_{\mathscr{P}}({}^nP, {}^mP)$ is, in a natural way, a right $\operatorname{End}({}^nP)$-module as well as a left $\operatorname{End}({}^mP)$-module and by (H3) $\operatorname{End}({}^sP) = W_{s+1}(k)$, in virtue of Proposition on page 151 in [13] a $W(k)$-module structure on $\operatorname{Hom}_{\mathscr{P}}({}^nP, {}^mP)$ is given by the formula

$$wf = w_m f = f w_n\,, \tag{3.3}$$

where $f \in \operatorname{Hom}_{\mathscr{P}}({}^nP, {}^mP)$, $w \in W(k)$ and w_s denotes the image of w by the natural projection $W(k) \to W_{s+1}(k)$. Using (3.3), it is easy to check that the morphism composition is $W(k)$-bilinear. Then in view of the proposition on page 151 in [13] the lemma follows.

As an immediate consequence we get

COROLLARY 3.4. *The hom-finite $W(k)$-category structure on $\mathscr{P}$ defined by* (3.3) *may be uniquely extended to a $W(k)$-category structure on $\mathscr{H}_1 = \mathscr{P}^{\mathrm{op}}$*-Mod *and on* $\mathscr{P}$-Mod *such that the corresponding Yoneda embeddings are $W(k)$-functors. Moreover,* $fp(\mathscr{P}$-Mod) *and the full subcategory $\mathscr{N}_1$ of $\mathscr{H}_1$ consisting of all noetherian objects in $\mathscr{H}_1$ are both hom-finite $W(k)$-categories.*

COROLLARY 3.5. (a) *Any indecomposable object in $\mathscr{N}_1$ has an artinian local endomorphism ring.*

(b) *Every object of $\mathscr{N}_1$ is a finite coproduct of indecomposables and any two such decompositions are isomorphic.*

(c) *A pure-projective object in $\mathscr{H}_1$ is a coproduct of indecomposable noetherian objects.*

Proof. Apply the results of Section 1.

We are also able to prove

THEOREM 3.6. *The category* $\mathscr{P}$-Mod *is locally noetherian.*

Proof. Fix n and let M be a $\mathscr{P}$-submodule of h^{nP}. It follows that, for any m, $M({}^mP)$ is a submodule of the cyclic $W(k)$-module $\operatorname{Hom}_{\mathscr{P}}({}^nP, {}^mP)$. Hence $M({}^mP)$ is also cyclic and is generated by $p^r j_{mn}$ for a certain $r \leqslant \min(m, n)$. Let $r(m)$ be a such minimal r. Then we have

1° $M({}^mP) = p^{r(m)} W(k) j_{mn}$,

2° $r(m) \geqslant r(s)$ *provided* $m \leqslant s$.

In fact, if $m \leqslant s$ then the natural injection $j_{sm}\colon {}^mP \to {}^sP$ induces a commutative diagram

$$\begin{array}{ccc} M({}^mP) & \hookrightarrow & \operatorname{Hom}({}^nP, {}^mP) \\ {\scriptstyle M(j_{sm})}\downarrow & & \downarrow{\scriptstyle (j_{sm})_*} \\ M({}^sP) & \hookrightarrow & \operatorname{Hom}({}^nP, {}^sP) \end{array}$$

Hence $\operatorname{Im} M(j_{sm}) = (p^{r(m)}) j_{ns}$ is a $W(k)$-submodule of $M({}^sP) = (p^{r(s)}) j_{ns}$ and therefore $r(m) \geqslant r(s)$.

Choose such a $t \geqslant n$ that $r(m) = r(t)$ for $m \geqslant t$ (this is possible by 2°) and assume $M = \bigcup_{i \in I} M_i$ with I directed. Then $M({}^mP) = \bigcup_{i \in I} M_i({}^mP)$ for every m and each $M_i({}^mP)$ is a submodule of the $W(k)$-module $M({}^mP)$. Since by 1° $M({}^mP)$ is finitely generated, there exists an $i_0 \in I$ such that $M({}^mP) = M_{i_0}({}^mP)$ for $m = 0, \ldots, t$. Now if $m > t$ then the $W(k)$-bilinearity of the morphism composition in $\mathscr{P}$ yields

$$p^{r(m)} j_{mn} = p^{r(t)} j_{mt} j_{tn} = j_{mt}(p^{r(t)} j_{tn}) \in M_{i_0}({}^mP)$$

because M_{i_0} is a left ideal in $\mathscr{P}$ and $p^{r(t)} j_{tn} \in M({}^tP) = M_{i_0}({}^tP)$. It follows by 1° that $M({}^mP) = M_{i_0}({}^mP)$. Consequently $M = M_{i_0}$ and the proof is finished.

In view of Theorem 3.6 and (H1) we obtain

COROLLARY 3.7.

$$\text{gl.dim}\,\mathscr{P}\text{-Mod} = \text{w.gl.dim}\,\mathscr{P}^{\mathrm{op}}\text{-Mod} = \text{gl.dim}\,\mathscr{H}_1 = 2\,.$$

COROLLARY 3.8. *The set of one-sided ideals in $\mathscr{P}$ is countable.*

Proof. By assertions 1° and 2° in the proof of Theorem 3.6 with any left ideal $M \subseteq h^{nP}$ the sequence $r(1) \geqslant r(2) \geqslant \ldots \geqslant r(m) \geqslant \ldots$ of natural numbers is associated; it is in fact finite. This defines an injection of the set of left ideals contained in h^{nP} into the set of finite sequences of natural numbers. The proof for right ideals is similar.

§ 4. On flat $\mathscr{C}$-modules.

We recall that a $\mathscr{C}$-module M is flat if $\operatorname{Tor}_1^{\mathscr{C}}(X, M) = 0$ for every $\mathscr{C}^{\mathrm{op}}$-module X. By fl.d$(\mathscr{C})$ we denote suppd M where M runs through all flat $\mathscr{C}$-modules [16]. Finally, let $\mathfrak{m}$ be an infinite cardinal number. An R-module is $\mathfrak{m}$-noetherian if each of its submodules is generated by at most m elements.

The main result of this section is the following

THEOREM 4.1. *Let R be a commutative $\aleph_n$-noetherian ring, $n \geqslant 0$, and let $\mathscr{C}$ be an R-category such that $\operatorname{Hom}_{\mathscr{C}}(C, C')$ is an $\aleph_n$-generated R-module for any $C, C' \in \mathscr{C}$. If $\mathscr{C}$ has at most $\aleph_n$ finitely generated right ideals, then* fl.d$(\mathscr{C}) \leqslant n+1$.

Observe that the theorem is a generalization of [14] and Corollaries 3.7 and 3.13 in [16].

In view of Lemma 2.15 and Theorem 2.12(c) in [16], Theorem 4.1 is a consequence of the following proposition:

PROPOSITION 4.2. *Let R be a commutative $\mathfrak{m}$-noetherian ring where $\mathfrak{m}$ is an infinite cardinal number. Assume also that $\mathscr{C}$ is an R-category such that $\mathrm{Hom}_{\mathscr{C}}(C, C')$ is $\mathfrak{m}$-generated for all $C, C' \in \mathscr{C}$. If $\mathscr{C}$ has at most $\mathfrak{m}$ finitely generated right ideals, then any flat $\mathscr{C}$-module is an $\mathfrak{m}$-directed union of $\mathfrak{m}$-presented pure submodules.*

To prove the proposition we need some definitions and technical lemmas. Let $F_I = \{F_i, f_{ij}\}_{i,j\in I}$ be a direct system over a directed set I. A set $J\subseteq I$ is called *F_I-closed* if

$$\bigcup_{\substack{k\geq j\\ k\in I}} \ker f_{kj} = \bigcup_{\substack{s\geq j\\ s\in J}} \ker f_{sj} \quad \text{for any } j\in J.$$

LEMMA 4.3. *Let F_I be a direct system in a locally finitely presented Grothendieck category. If $J\subseteq I$ is F_I-closed, then the natural morphism $\operatorname{colim} F_J \to \operatorname{colim} F_I$ is injective.*

Proof. See Lemma 3.5 in [16].

LEMMA 4.4. *Let I be a directed set. If $G_I^\gamma = \{G_i^\gamma, g_{ij}^\gamma\}_{i,j\in I}$, $\gamma\in\Delta$, $|\Delta|\leq\mathfrak{m}$, are direct systems consisting of $\mathfrak{m}$-noetherian R-modules, then every $J\subseteq I$ with $|J|\leq\mathfrak{m}$ is contained in a directed subset J' of I such that J' is G_I^γ-closed for every $\gamma\in\Delta$.*

Proof. By our assumption for every $i\in I$ there exists a subset I_i of I with $|I_i|\leq\mathfrak{m}$ such that

$$\bigcup_{\substack{s\geq i\\ s\in I}} \ker g_{si}^\gamma = \bigcup_{\substack{k\geq i\\ k\in I_i}} \ker g_{ki}^\gamma \quad \text{for all } \gamma\in\Delta.$$

Since the set $L = \bigcup_{j\in J} I_j$ has a cardinality at most $\mathfrak{m}$, by Lemma 1 in [3] there exists a directed set $J_1\subseteq I$ such that $L\subseteq J_1$ and $|J_1|\leq\mathfrak{m}$. Continuing this procedure, we define directed sets $J\subseteq J_1\subseteq J_2\subseteq\ldots$ with $|J_i|\leq\mathfrak{m}$, such that $J' = \bigcup_{i=1}^{\infty} J_i$ satisfies the required conditions.

Proof of Proposition 4.2. Let M be a flat $\mathscr{C}$-module. By Theorem 10.1 in [10] $M = \operatorname{colim} F_I$ for a certain direct system $F_I = \{F_i, f_{ij}\}_{i,j\in I}$ consisting of finitely generated free $\mathscr{C}$-modules. To prove the proposition it is sufficient to show that every subset J of I with $|J|\leq\mathfrak{m}$ is contained in a directed set $J'\subseteq I$ with $|J'|\leq\mathfrak{m}$ such that the natural map $\varphi\colon \operatorname{colim} F_{J'}\to\operatorname{colim} F_I$ is a pure monomorphism.

Since $\otimes_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}}\text{-Mod}\times\mathscr{C}\text{-Mod}\to R\text{-Mod}$ is an R-functor (see [10], p. 52), by our assumptions $h_X/N\otimes_{\mathscr{C}} F_i \cong F_i(X)/N(X)$ is an $\mathfrak{m}$-generated R-module for every finitely generated right ideal $N\subseteq h_X$, $X\in\mathscr{C}$. Let $J\subseteq I$, $|J|\leq\mathfrak{m}$. Since $\mathscr{C}$ has at most $\mathfrak{m}$ finitely generated right ideals $N\subseteq h_X$, it follows from Lemma 4.4 that there exists a directed set $J'\subseteq I$ containing J such that $|J'|\leq\mathfrak{m}$ and J' is $[(h_X/N)\otimes_{\mathscr{C}} F_I]$-closed for each finitely generated right ideal $N\subseteq h_X$. Then the required result follows from Lemma 4.3 because the natural morphism $\varphi\colon \operatorname{colim} F_{J'}\to\operatorname{colim} F_I$ is a pure monomorphism if $(h_X/N)\otimes\varphi$ is a monomorphism for any finitely generated right ideal $N\subset h_X$, $X\in\mathscr{C}$.

COROLLARY 4.5. *Every flat object in $\mathscr{H}_1$ as well as every flat $\mathscr{P}$-module is an $\aleph_0$-directed union of $\aleph_0$-presented pure flat subobjects.*

Proof. In view of Corollary 3.8 the corollary is a consequence of Proposition 4.2.

COROLLARY 4.6. $\mathrm{fl.d}(\mathscr{P}) = \mathrm{fl.d}(\mathscr{P}^{\mathrm{op}}) = 1$.

Proof. Together with Corollary 3.8, Theorem 4.1 gives $\mathrm{fl.d}(\mathscr{P})\leq 1$ and $\mathrm{fl.d}(\mathscr{P}^{\mathrm{op}})\leq 1$. Then, by Theorem 5.4 in [16], to prove the corollary it is sufficient to show that the Jacobson radical $J(\mathscr{P})$ is neither left nor right T-nilpotent. For this purpose we consider two sequences from $J(\mathscr{P})$ (see Corollary 2.2),

$$ {}^1p \xrightarrow{j_{21}} {}^2p \to \ldots \to {}^np \xrightarrow{j_{n+1,n}} {}^{n+1}p \to \ldots$$

$$ {}^1p \xrightarrow{j_{12}} {}^2p \leftarrow \ldots \leftarrow {}^np \xleftarrow{j_{n,n+1}} {}^{n+1}p \leftarrow \ldots$$

which are non-T-nilpotent because the first is a sequence of monomorphisms and for the second one we have $j_{12}j_{23}\ldots j_{n,n+1}(X_n) = X_0^{p^n}$.

As an immediate consequence of Theorem 3.6 in [16] and Corollary 4.5 we have:

COROLLARY 4.7. *Any direct system as well as any inverse system consisting of finitely generated projective objects in $\mathscr{H}_1$ are $\aleph_0$-factorizable (in the sense of* [16, § 3]).

COROLLARY 4.8. *An inverse limit of flat objects in $\mathscr{H}_1$ is flat.*

Proof. Since $\mathscr{H}_1 \cong P^{\mathrm{op}}\text{-Mod}$ and $\mathscr{P}\text{-Mod}$ are locally noetherian categories of global dimension two, the corollary follows from Corollary 4.5 in [11].

§ 5. Primitively generated Hopf algebras. One of the main theorems of this section is the following generalization of a well-known Milnor–Moore result [9]:

THEOREM 5.1. *Every primitively generated Hopf algebra from $\mathscr{H}$ is a tensor product of Hopf subalgebras of the form $k[X]/(X^{p^r})$, $0\leq r\leq\infty$* [1], *and any two such decompositions are isomorphic.*

It follows from the remarks in Section 2 that it is sufficient to prove the theorem for primitively generated Hopf algebras from $\mathscr{H}_1$.

Let $\mathscr{L}$ be the full subcategory of $\mathscr{H}_1$ consisting of all primitively generated Hopf algebras. It follows from [9] that $\mathscr{L}$ is locally noetherian Grothendieck category and each of its objects of finite type is a coproduct of objects $K_{nr} = k[X]/(X^{p^r})$, where $\deg X = 2p^n$, $1\leq r\leq\infty$, $n = 0, 1, 2, \ldots$ Let us denote by $\mathscr{K}$ the full subcategory of $\mathscr{L}$ consisting of all objects K_{nr}.

We start with the following simple lemma.

(1) $x^\infty = 0$.

LEMMA 5.2. *For any* $n,\ m = 0, 1, 2, \ldots$ *and* $1 \leqslant r,\ l \leqslant \infty$ *we have*

$$\mathrm{Hom}_{\mathscr{K}}(K_{nr}, K_{ml}) = \begin{cases} k & \text{if } 0 \leqslant n-m < l \leqslant r+n-m, \\ 0 & \text{in the opposite case.} \end{cases}$$

Proof. Let $f\colon K_{nr} \to_{ml}$ be an arbitrary morphism in $\mathscr{K}$. It is clear that $f = 0$ whenever $n < m$, and $f(\bar{X}) = a\bar{X}^{p^{n-m}}$, $a \in k$, in the opposite case. Hence $0 = f(\bar{X}^{p^r}) = a\bar{X}^{p^{r+n-m}}$ and the lemma follows because f is uniquely determined by $f(\bar{X})$.

Recall that a Grothendieck category is *pure semi-simple* if each of its objects is a coproduct of finitely presented subobjects (see [15]–[17]).

It follows from the above-mentioned Milnor–Moore result that Theorem 5.1 is a consequence of the following

THEOREM 5.3. $\mathscr{L}$ *is a pure semi-simple category.*

Proof. By Theorem 6.3 in [16] $\mathscr{L}$ is pure semi-simple if and only if the category $\mathrm{fp}(\mathscr{L})^{\mathrm{op}}$-Mod $\cong \mathscr{K}^{\mathrm{op}}$-Mod is perfect, or equivalenty, if the endomorphism ring of any object in $\mathrm{fp}(\mathscr{L})$ is left artinian and the Jacobson radical $J(\mathscr{K})$ is right T-nilpotent. But in virtue of Corollary 3.4 it is sufficient to prove the last statement. For this purpose consider a sequence

$$K_{n_1r_1} \xrightarrow{f_1} K_{n_2r_2} \to \ldots \to K_{n_sr_s} \xrightarrow{f_s} K_{n_{s+1}r_{s+1}} \to \ldots$$

where each f_i belongs to $J(\mathscr{K})$. It follows from Lemmas 1.1 and 5.2 that $J(\mathscr{K})(K_{nr}, K_{ml}) \neq 0$ if and only if either (i) $n = m$ and $r > 1$ or (ii) $0 < n-m < l \leqslant r+n-m$. Assume that each $f_i \neq 0$. Then $n_1 \geqslant n_2 \geqslant n_3 \geqslant \ldots$ and there exists an s such that $n_s = n_{s+1} = \ldots$ It follows that $r_s > r_{s+1} > r_{s+2} > \ldots$ and we get a contradiction. Consequently $f_m = 0$ for a suitable m and the proof is complete.

As a consequence of Corollary 2.4 in [17] and Theorem 5.1 we have

COROLLARY 5.4. *Every primitively generated Hopf algebra from* $\mathscr{H}$ *has an F-semi-perfect endomorphism ring.*

We end this section with

COROLLARY 5.5. $\mathrm{fl.d}(\mathscr{K}) = 1$.

Proof. It is not hard to check that $\mathscr{K}$ has at most $\aleph_0$ principal (and so finitely generated) left ideals. Then by Theorem 4.1 $\mathrm{fl.d}(\mathscr{K}) \leqslant 1$ and to prove the equality it is sufficient to show that $J(\mathscr{K})$ is not left T-nilpotent. For this purpose we consider a non-T-nilpotent sequence of natural projections

$$k[X]/(X^p) \leftarrow k[X]/(X^{p^2}) \leftarrow k[X]/(X^{p^3}) \leftarrow \ldots$$

with $\deg X = 2p$.

§ 6. Noetherian local objects in $\mathscr{H}_1$. It follows from Lemma 1.6 that a noetherian object in $\mathscr{H}_1$ is local if and only if it is a quotient object of an object ${}^nP$. Then the next theorem gives a complete description of all noetherian local objects in $\mathscr{H}_1$.

THEOREM 6.1. *Every Hopf subalgebra of* ${}^nP$ *has the form* $(X_0^{p^{r_0}}, \ldots, X_n^{p^{r_n}})$, $0 \leqslant r_0 \leqslant r_1 \leqslant \ldots \leqslant r_n \leqslant \infty$, *i.e. it is generated as a k-algebra by elements* $X_0^{p^{r_0}}, \ldots, X_n^{p^{r_n}}$.

Proof. Let H be a non-zero Hopf subalgebra of ${}^nP$. If $n = 0$, then H is primitively generated because so is ${}^0P$. Since every primitive element of ${}^0P$ has the form $tX_0^{p^r}$, $t \in K$, then $H = (X_0^{p^{r_0}})$, where $r_0 \geqslant 0$ is a minimal number such that $X_0^{p^{r_0}} \in H$.

Let $n \geqslant 1$ and suppose that the theorem is proved for $n-1$. Then $H' = H \cap {}^{n-1}P = (X_0^{p^{r_0}}, \ldots, X_{n-1}^{p^{r_{n-1}}})$ for certain $0 \leqslant r_0 \leqslant r_1 \leqslant \ldots \leqslant r_{n-1} \leqslant \infty$. If $H = H'$, we put $r_n = \infty$. In the opposite case choose a homogeneous element $c \in H \setminus H'$ of smallest degree and put $r_n = m-n$ where $2p^m = \deg c$. We shall show that $H = (X_0^{p^{r_0}}, \ldots, X_n^{p^{r_n}})$.

First we prove that $r_n \geqslant r_{n-1}$. For this purpose we observe that each element d of ${}^nP$ of degree $2p^k$ may be uniquely expressed in the form

$$d = \sum d_{i_0 \ldots i_n} X_0^{i_0} X_1^{i_1} \ldots X_n^{i_n},$$

where the sum is taken over all natural numbers $i_0, \ldots, i_n$ such that $i_0 + pi_1 + \ldots + p^n i_n = p^k$. In particular the element c has the form

$$c = lX_n^{p^{r_n}} + a + b,$$

where

$$l = c_{0 \ldots 0 p^{m-n}},$$

$$a = \sum_{\substack{i_0 + \ldots + i_{n-1} > 0 \\ i_n > 0}} c_{i_0 \ldots i_n} X_0^{i_0} \ldots X_n^{i_n},$$

$$b = \sum_{\substack{i_0 + \ldots + i_{n-1} > 0 \\ i_n = 0}} c_{i_0 \ldots i_n} X_0^{i_0} \ldots X_{n-1}^{i_{n-1}}.$$

We now prove that $a = 0$. Note that

$$(H \otimes H)_{2p^m} = H_{2p^m} \otimes k \oplus k \otimes H_{2p^m} \oplus \bigoplus_{i=1}^{2p^m - 1} H_i' \otimes H_{2p^m - i}'$$

because $H_i = H_i'$ for $i < 2p^m$ by the minimality of $\deg c$. Recall also that $\Delta(X_k) = X_k + Y_k + \varphi_k$, where on the right side X_k and Y_k denote elements $X_k \otimes 1$ and $1 \otimes X_k$ in ${}^nP \otimes {}^nP$, and φ_k is a polynomial of variables $X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}$ (see § 2).

Assume, on the contrary, that $a \neq 0$, and denote by N_a^{n+1} the set of all tuples $\langle i_n, \ldots, i_0 \rangle$ such that the coefficient $c_{i_0 \ldots i_n}$ is non-zero and occurs in the expression of the element a above. Clearly N_a^{n+1} is non-empty. If $\langle j_n, \ldots, j_0 \rangle$ is its maximal element in the lexicographical order, then it is easy to check that the summand c' of $\Delta(c)$ which belongs to $\bigoplus_{i=1}^{2p^m - 1} H_i \otimes H_{2p^m - i}$ has the form $c' = c_{j_0 \ldots j_n} X_n^{j_n} Y_{n-1}^{j_{n-1}} \ldots Y_0^{j_0} + f$, where f contains no monomials of the form $eX_n^{j_n} Y_{n-1}^{j_{n-1}} \ldots Y_0^{j_0}$, $e \in K$. On the other hand, c' is a polynomial of $X_{n-1}^{p^{r_{n-1}}}, \ldots, X_0^{p^{r_0}}, Y_{n-1}^{p^{r_{n-1}}}, \ldots, Y_0^{p^{r_0}}$ because it belongs to $H' \otimes H'$. It follows that $c_{j_0 \ldots j_n} = 0$ and we get a contradiction. Consequently $a = 0$ and hence $l \neq 0$. Thus without loss of generality one can suppose $l = 1$.

To prove $r_n \geqslant r_{n-1}$ we consider the summand c'' of $\Delta(c)$ which belongs to the direct summand $H'_{2p^{m-1}} \otimes H'_{2p^{m-1}(p-1)}$ of $(H \otimes H)_{2p^m}$. In view of the equality

$$\Delta(X_n^{p^{r_n}}) = X_n^{p^{r_n}} + Y_n^{p^{r_n}} + (-X_{n-1} Y_{n-1}^{p-1})^{p^{r_n}} + \ldots$$

it is easy to check that

$$c'' = (-1)^{p^{r_n}} X_{n-1}^{p^{r_n}} Y_{n-1}^{p^{r_n}(p-1)} + g\,,$$

where g contains no monomials of the form $eX_{n-1}^{p^{r_n}} Y_{n-1}^{p^{r_n}(p-1)}$, $e \in K$. Since $c'' \in H' \otimes H'$, it is a polynomial of $X_{n-1}^{p^{r_{n-1}}}, \ldots, X_0^{p^{r_0}}$, $Y_{n-1}^{p^{r_{n-1}}}, \ldots, Y_0^{p^{r_0}}$. Hence we conclude that $r_n \geqslant r_{n-1}$.

We now prove that $X_n^{p^{r_n}} \in H$. Consider the element $h = X_n^{p^{r_n}} + b'$, where b' is the sum of all monomials $c_{i_0 \ldots, i_{n-1}, 0} X_0^{i_0} \ldots X_{n-1}^{i_{n-1}}$ in the expression of the element b such that some exponent i_t is not divisible by p^{r_t}. If is sufficient to show that $b' = 0$. Assume the contrary, i.e. that $b' \neq 0$, and consider a non-empty set $N_{b'}^n$ of all tuples $\langle i_{n-1}, \ldots, i_0 \rangle$ such that $c_{i_0 \ldots i_{n-1}, 0} \neq 0$ and occurs in the expression of b'. Let $\langle j_{n-1}, \ldots, j_0 \rangle$ be the maximal element of $N_{b'}^n$ in the lexicographical order and fix j_t which is not divisible by p^{r_t}. By our assumptions $\varphi_n^{p^{r_n}} \in H' \otimes H'$ and therefore $\Delta(h) - \varphi_n^{p^{r_n}} \in (H \otimes H)_{2p^{r_n}}$. Furthermore, an easy computation shows that the summand h' of $\Delta(h) - \varphi_n^{p^{r_n}}$ which belongs to $\bigoplus_{i=1}^{2p^{r_n}-1} H_i \otimes H_{2p^{r_n}-i}$ has the form

$$h' = c_{j_0 \ldots j_{n-1} 0} X_t^{j_t} Y_0^{j_0} \ldots Y_{t-1}^{j_{t-1}} Y_{t+1}^{j_{t+1}} \ldots Y_{n-1}^{j_{n-1}} + h''\,,$$

where h'' contains no monomials of the form $eX_t^{j_t} Y_0^{j_0} \ldots Y_{t-1}^{j_{t-1}} Y_{t+1}^{j_{t+1}} \ldots Y_{n-1}^{j_{n-1}}$, $e \in K$. On the other hand, h' belongs to $G \otimes G$, where $G = (X_0^{p^{r_0}}, \ldots, X_n^{p^{r_n}})$. This is a contradiction because $c_{j_0 \ldots j_{n-1} 0} \neq 0$ and j_t is not divisible by p^{r_t}. Consequently $X_n^{p^{r_n}} \in H$ and therefore $G \subset H$.

In order to prove the required equality $G = H$, suppose, on the contrary, that $G \neq H$ and choose a homogeneous element $d \in H \setminus G$ of minimal degree. If $\deg d = 2p^s$, then $s > m > n$, $H_i = G_i$ for $i < 2p^s$ and d has the form

$$d = \sum d_{i_0 \ldots i_n} X_0^{i_0} \ldots X_n^{i_n}\,,$$

where the sum is taken over all $i_0, \ldots, i_n$ such that $i_0 + pi_1 + \ldots + p^n i_n = p^s$. Let us denote by d' the sum of all monomials $d_{i_0 \ldots i_n} X_0^{i_0} \ldots X_n^{i_n}$ in the expression of d such that a certain i_t is not divisible by p^{r_t}. Since $G \subset H$ and $d \notin G$, then $d' \notin G$. On the other hand, using the same type of arguments as in the previous part of the proof, one can show that $d' = 0$. We then get a contradiction. Consequently $G = H$ and the theorem is proved.

As an immediate consequence of Theorem 6.1 we have:

Corollary 6.2. *Every Hopf subalgebra of* $k[X]$ *is of the form* $(X_0^{p^{r_0}}, \ldots, X_n^{p^{r_n}}, \ldots)$ *with* $0 \leqslant r_0 \leqslant \ldots \leqslant r_n \leqslant \ldots \leqslant \infty$.

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