Indecomposable continua with one and two composants *

by

David P. Bellamy (Warszawa)

Abstract. It is proven that there exist compact Hausdorff indecomposable continua with exactly one and exatly two composants, showing that the result that a compact metric indecomposable continuum has uncountably many composants does not generalize to the non-metric case. The construction involves an inverse limit of ω_1 metric indecomposable continua.

In this paper "continuum" means compact connected Hausdorff space M. Smith [7] has recently proved that given any infinite cardinal number m, there exists an indecomposable continuum with 2^m composants. Herein a somewhat similar technique is used to construct indecomposable continua with one and two composants, showing that the classical result that a metric indecomposable continuum has uncountably many composants cannot be generalized to the Hausdorff case. For background on this problem, the reader is referred to [1], [2], [4], and [6].

First some definitions and notation are established. If $f\colon X\to Y$ and $A\subseteq X$, $f\langle A\rangle$ denotes the image of A under f. If X is a continuum, $C(X)$ will denote the hyperspace of subcontinua of X with the exponential topology ([5, p. 45]). If $f\colon X\to Y$ is a map, $\hat{f}\colon C(X)\to C(Y)$ will denote the hyperspace map induced by f, defined by $\hat{f}(W)=f\langle W\rangle$. I is the closed interval $[0,1]$.

If $\{X_\alpha; r[\alpha,\beta]\}_{\alpha,\beta<\gamma}$ is an inverse system of continua indexed by a limit ordinal number γ, where for $\alpha<\beta$, $X_\alpha\subseteq X_\beta$ and $r[\beta,\alpha]\colon X_\beta\to X_\alpha$ is a retraction, and X_γ is the inverse limit of the system, there is a natural embedding $i_\alpha\colon X_\alpha\to X_\gamma$, defined by $i_\alpha(x)=\langle x_\beta\rangle_{\beta<\gamma}$, where $X_x = x$ for all $\beta\geqslant\alpha$. The other coordinates are determined by the bonding maps. Henceforth, in such an inverse system, each X_α will be considered to be identified with its image under i_α, to avoid cumbersome notation. The hyperspace map induced by $r[\alpha,\beta]$ will be denoted $\hat{r}[\alpha,\beta]$, rather than with a hat over the whole term $r[\alpha,\beta]$. In any inverse system $\{X_\alpha; h[\alpha,\beta]\}_{\alpha,\beta<\gamma}$, if X is its

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* This research was done while the author was a participant in an exchange program sponsored by the National Academy of Sciences, U.S.A. and the Polish Academy of Sciences.

inverse limit, $h[\gamma, \alpha]\colon X \to X_\alpha$ will denote the projection of X onto X_α. Herein all inverse systems considered have surjective bonding maps and are indexed by limit ordinal numbers.

The following result will pave the way and motivate the construction.

THEOREM 1. *Let $\{X_\alpha; r[\alpha, \beta]\}_{\alpha,\beta<\omega_1}$ be an inverse system of metric indecomposable continua such that:*

1. *For $\beta<\alpha$, X_β is a proper subcontinuum of X_α, and $r[\alpha, \beta]\colon X_\alpha \to X_\beta$ is a retraction, and*

2. *For each β, there is a composant C_β of X_β such that*

$$\Big(\bigcup_{\gamma<\beta} X_\gamma\Big) \cap C_\beta = \emptyset$$

and such that for each $\alpha>\beta$,

$$r[\alpha, \beta]\langle X_\alpha - X_\beta\rangle \subseteq C_\beta .$$

Let $X = \varprojlim\{X_\alpha; r[\alpha, \beta]\}_{\alpha,\beta<\omega_1}$. Then X is an indecomposable continuum with at most two composants. In particular, if $E = \bigcup_{\alpha<\omega_1} X_\alpha$, and $C = X-E$, then E is a composant of X and either $C = \emptyset$ or C is a composant of X.

Proof. X is indecomposable, since it is an inverse limit of indecomposable continua with onto bonding maps. Let E and C be as in the statement of the theorem. E is then just the set of eventually constant ω_1-sequences. E is contained in some composant of X, since it is a monotone union of proper subcontinua of X. To prove that E is a composant of X, it suffices to prove that X is irreducible between any point $\langle y_\alpha\rangle_{\alpha<\omega_1}$ of E and any point $\langle x_\alpha\rangle_{\alpha<\omega_1}$ not in E. Let $\beta<\omega_1$. Since $\langle x_\alpha\rangle_{\alpha<\omega_1} \notin E$, there is a $\lambda>\beta$ such that $x_\lambda \neq x_\beta$. Thus, since $r[\lambda, \beta]$ is a retraction and $r[\lambda, \beta](x_\lambda) = x_\beta$, $x_\lambda \notin X_\beta$, so that $r[\lambda, \beta](x_\lambda) \in C_\beta$. Thus, for all β, $x_\beta \in C_\beta$. Since for some $\eta<\omega_1$, $y_\eta = y_\alpha$, and hence $y_\alpha \in X_\eta$, for all $\alpha>\eta$, it follows that for all $\alpha>\eta$, $y_\alpha \notin C_\alpha$. Then if $W\subseteq X$ is a continuum containing $\langle x_\alpha\rangle_{\alpha<\omega_1}$ and $\langle y_\alpha\rangle_{\alpha<\omega_1}$, it follows that for every $\alpha>\eta$, $r[\omega_1, \alpha]\langle W\rangle = X_\alpha$, since each such X_α is irreducible from x_α to y_α. This implies that $W = X$, as required.

To complete the proof it suffices to prove that C is contained in some composant of X. Suppose $\langle x_\alpha\rangle_{\alpha<\omega_1}, \langle y_\alpha\rangle_{\alpha<\omega_1} \in C$. For each α, x_α and y_α both belong to C_α, as noted above. For each α, let

$$\mathfrak{A}_\alpha = \{W \in C(X_\alpha)\colon x_\alpha, y_\alpha \in W\} .$$

Then $\mathfrak{A}_\alpha$ is a closed subset of $C(X_\alpha)$ for each α; furthermore, if $\alpha\geqslant\beta$, $\hat{r}[\alpha, \beta]\langle\mathfrak{A}_\alpha\rangle \subseteq \mathfrak{A}_\beta$, and $\hat{r}[\alpha, \beta]\langle\mathfrak{A}_\alpha\rangle$ is nondegenerate, since $X_\alpha \in \mathfrak{A}_\alpha$ and $r[\alpha, \beta](X_\alpha) = X_\beta$, while there is a $K_\alpha \neq X_\alpha$ such that $K_\alpha \in \mathfrak{A}_\alpha$, because $x_\alpha, y_\alpha \in C_\alpha$, and thus, since $K_\alpha \subseteq C_\alpha \subseteq X_\alpha - X_\beta$, and $r[\alpha, \beta]\langle K_\alpha\rangle \subseteq r[\alpha, \beta]\langle X_\alpha - X_\beta\rangle \subseteq C_\beta$, so that $\hat{r}[\alpha, \beta](K_\alpha) \neq X_\beta$. Now, for a fixed β, the family

$$\{\hat{r}[\alpha, \beta]\langle\mathfrak{A}_\alpha\rangle\colon \omega_1>\alpha\geqslant\beta\}$$

is a nonincreasing ω_1-sequence of closed sets in the compact metric space $C(X_\beta)$, and is thus eventually constant. Hence, since $\hat{r}[\alpha, \beta]\langle\mathfrak{A}_\alpha\rangle$ is always nondegenerate, as above, it follows that $\mathscr{K}_\beta = \bigcap_{\omega_1>\alpha>\beta} \hat{r}[\alpha, \beta]\langle\mathfrak{A}_\alpha\rangle$ is also nondegenerate for each β and furthermore, since the $\mathfrak{A}_\alpha$'s are compact and the intersection is monotone, $\hat{r}[\beta, \gamma]\langle\mathscr{K}_\beta\rangle = \mathscr{K}_\gamma$. Thus, $\{\mathscr{K}_\alpha; \hat{r}[\alpha, \beta]|\ \mathscr{K}_\alpha\}_{\alpha,\beta<\omega_1}$ is an inverse system of nondegenerate compact sets and surjective bonding maps, so that its inverse limit, $\mathscr{K}$, has more than one point. Thus, let $\langle W_\alpha\rangle_{\alpha<\omega_1} \in \mathscr{K}$ such that $\langle W_\alpha\rangle_{\alpha<\omega_1} \neq \langle X_\alpha\rangle_{\alpha<\omega_1}$. Then, $\hat{r}[\alpha, \beta](W_\alpha) = W_\beta$; that is $r[\alpha, \beta]\langle W_\alpha\rangle = W_\beta$, whenever $\alpha\geqslant\beta$. Thus, if W is the inverse limit of $\{W_\alpha; r[\alpha, \beta]|\ W_\alpha\}_{\alpha,\beta<\omega_1}$, W is a subcontinuum of X, $W\neq X$ since this would imply $W_\alpha = X_\alpha$ for all α, contradicting the choice of $\langle W_\alpha\rangle_{\alpha<\omega_1}$; and $\langle x_\alpha\rangle_{\alpha<\omega_1}, \langle y_\alpha\rangle_{\alpha<\omega_1} \in W$, since for each α, $W_\alpha \in \mathscr{K}_\alpha \subseteq \mathfrak{A}_\alpha$, so that x_α, $y_\alpha \in W_\alpha$. Thus, $\langle x_\alpha\rangle_{\alpha<\omega_1}$ and $\langle y_\alpha\rangle_{\alpha<\omega_1}$ lie in the same composant of X, and the proof is complete.

The preceeding argument does not prove that X has exactly two composants since it is not obvious that $C\neq\emptyset$. Indeed, one can readily verify that

$$C = \varprojlim\{C_\alpha; r[\alpha, \beta]|\ C_\alpha\}_{\alpha,\beta<\omega_1} ;$$

and since this is an inverse limit of noncompact sets and possibly nonsurjective bonding maps, it is not assured to be nonempty. The author has been unable to determine whether the case $C=\emptyset$ can in fact be realized. The remainder of this article is devoted to constructing an inverse system satisfying the hypotheses of Theorem 1. It will be shown that $C\neq\emptyset$ for this system by choosing recursively a point $\langle p_\alpha\rangle_{\alpha<\omega_1} \in C$ along with the recursive choice of the X_α's.

LEMMA 1. *If $\{X_\alpha; r[\alpha, \beta]\}_{\alpha,\beta<\gamma}$ is an inverse system satisfying all the hypotheses of Theorem 1, except that γ is a limit ordinal which need not equal ω_1, let $E_\gamma = \bigcup_{\alpha<\gamma} X$. Then E_γ is a composant of the inverse limit continuum X_γ.*

Proof. The proof of this can be lifted directly from the first paragraph of the proof of Theorem 1, with E_γ replacing E and γ replacing ω_1.

The next lemma provides the principle tool for the construction.

LEMMA 2. *If X is any nondegenerate metric indecomposable continuum and $K\subseteq X$ is a composant, there exists an indecomposable metric continuum Y, such that X is a proper subcontinuum of Y, and a retraction $r\colon Y\to X$ such that $r\langle Y-X\rangle \subseteq K$. Further, Y can be chosen so that if D is any composant of Y which does not contain X, then $r\langle D\rangle = K$.*

Proof. Let P_0 denote the usual Cantor ternary set in I. Let $P_n = \{x\in P_0\colon x\leqslant 3^{-n}\}$, and let $F_n = P_n - P_{n+1}$. F_n consists of those points of P_0 between $2\cdot 3^{-n-1}$ and 3^{-n}, inclusive, and the midpoint of this interval, $\frac{5}{6}\cdot 3^{-n}$, is the center of symmetry of F_n.

Consider the following homeomorphic copy, N, of Knaster's chainable indecomposable continuum with one endpoint: N is the union of all semicircles with center $(\frac{1}{2}, 1)$, concave downward, and with endpoints in $P_0\times\{1\}$; all semicircles in the lower halfplane with centers $(\frac{5}{6}\cdot 3^{-n}, 0)$ and with endpoints in $F_n\times\{0\}$ for each nonnegative integer n; and $P_0\times I$. Pictures of Knaster's continuum are well-known e.g. [5, p. 205], and the reader can easily sketch this slight modification of it, so no figure is given. Let $Q = \{(x, y)\in N\colon y\geqslant 1\}$ and $R = \{(x, y)\in N\colon y\leqslant 0\}$.

Let X be the given indecomposable metric continuum and K the specified composant of X. Let $\{K_i\}_{i=1}^{\infty}$ be an increasing sequence of subcontinua of X whose union is K. Let $p\in K_1$. For each K_n, perform the following construction: Let $\{a_i\}_{i=n}^{\infty}$ be a countable dense subset of K_n with $a_n = p$. For $k\geqslant n$, let L_k be a subcontinuum of K_n irreducible from a_k to a_{k+1}, and define the continuum $M_n\subseteq K_n\times I\subseteq X\times I$, irreducible from $(p, 0)$ to $(p, 1)$, by:

$$M_n = (\{p\}\times[1/n, 1])\cup\Big(\bigcup_{k=n}^{\infty}(L_k\times\{1/k\})\Big)\cup\Big(\bigcup_{k=n+1}^{\infty}\{a_k\}\times[1/k, 1/k-1]\Big)\cup(K_n\times\{0\}),$$

and define $M\subseteq P_0\times(X\times I)$ by

$$M = \bigcup_{n=0}^{\infty}(F_n\times M_{n+1})\cup(\{0\}\times(X\times\{0\}))\cup(\{0\}\times(\{p\}\times I)).$$

Equivalently, M is the closure in $P_0\times(X\times I)$ of $\bigcup_{n=0}^{\infty}(F_n\times M_{n+1})$.

Y is now obtained from the disjoint union of Q, R, and M by identifying $(x, 1)\in Q$ with $(x; (p, 1))\in M$ for each $x\in P_0$, and $(x, 0)\in R$ with $(x, (p, 0))\in M$ for each $x\in P_0$. Geometrically, M is sewn into N in place of the vertical segments in $P_0\times I$. Identify X with $\{0\}\times(X\times\{0\})$ and define $r\colon Y\to X$ by $r(x) = p$ if $x\in Q\cup R$; $r(s, (x, t)) = x$ if $(s, (x, t))\in M$. If $q\in M\cap(Q\cup R)$, either definition yields $r(q) = p$, and $r|X$ is the identity. Suppose $x\in Y-X$. If $x\in Q\cup R$, $r(x) = p\in K$. If $x\in M-X$, either $x\in F_n\times M_{n+1}$ for some n, or $x\in\{0\}\times(\{p\}\times I)$. If $x\in F_n\times M_{n+1}$, $x = (y, (q, t))$ for some $y\in P$, $q\in K_{n+1}$, $t\in I$, and in this ease $r(x) = q$ and $q\in K_{n+1}\subseteq K$. If $x\in\{0\}\times(\{p\}\times I)$, $r(x) = p\in K$. Thus $r\langle Y-X\rangle\subseteq K$.

Define $h\colon Y\to N$ by $h(q) = q$ for $q\in Q\cup R$ and $h(x, (y, t)) = (x, t)$ for $(x, (y, t))\in M$. This map is monotone, so Y is connected since it is compact. It is straightforward to verify that no proper subcontinuum of Y is mapped onto N by h, so that Y is indecomposable.

Y is obviously metrizable. If D is any composant of Y which does not contain X, then $r|D\colon D\to K$ is surjective, since if $q\in K$, $q\in K_n$ for some n, and thus q has an inverse image in every component of $F_{n-1}\times M_n$ since $K_n\times\{0\}\subseteq M_n$. But, $F_{n-1}\times M_n$ is a closed, proper subset of Y with nonvoid interior. Hence, D contains some component of $F_{n-1}\times M_n$, completing the proof.

THEOREM 2. *Given any nondegenerate metric indecomposable continuum Y, there exists an inverse system $\{X_\alpha; r[\alpha, \beta]\}_{\alpha,\beta<\omega_1}$ with $X_0 = Y$, satisfying the hypotheses of Theorem 1, with the additional property that $C\neq\emptyset$ for the system, so that the inverse limit, X, of the system has exactly two composants.*

Proof. Let Y be given and set $Y = X_0$ and let C_0 be any composant of X_0, and let p_0 be any point of C_0. $r[0, 0]$ is of course the identity on X_0.

Suppose that for some ordinal $\gamma<\omega_1$, X_α, C_α, p_α, and, for $\beta\leqslant\alpha$, $r[\alpha, \beta]$ have been defined such that:

1. For $\beta<\alpha<\gamma$, X_β is a subcontinuum of X_α and $C_\alpha\cap X_\beta = \emptyset$.
2. For $\alpha<\gamma$, $p_\alpha\in C_\alpha$.
3. $r[\alpha, \beta](p_\alpha) = p_\beta$.
4. $r[\alpha, \beta]\colon X_\alpha\to X_\beta$ is a retraction and $r[\alpha, \beta]\langle X_\alpha-X_\beta\rangle\subseteq C_\beta$.

If γ is not a limit ordinal, apply Lemma 2 to $X_{\gamma-1}$ to obtain an indecomposable continuum X_γ and a retraction $r[\gamma, \gamma-1]\colon X_\gamma\to X_{\gamma-1}$ such that

$$r[\gamma, \gamma-1]\langle X_\gamma-X_{\gamma-1}\rangle\subseteq C_{\gamma-1},$$

and for every composant D of X_γ missing $X_{\gamma-1}$, $r[\gamma, \gamma-1]\langle D\rangle = C_{\gamma-1}$.

Let C_γ be any composant of X_γ not containing $X_{\gamma-1}$, and choose $p_\gamma\in C_\gamma$ such that $r[\gamma, \gamma-1](p_\gamma) = p_{\gamma-1}$. For $\alpha<\gamma-1$, $r[\gamma, \alpha] = r[\gamma-1, \alpha]\circ r[\gamma, \gamma-1]$, and since $C_{\gamma-1}\subseteq X_{\gamma-1}-X_\alpha$,

$$r[\gamma, \alpha]\langle X_\gamma-X_\alpha\rangle\subseteq r[\gamma-1, \alpha]\langle C_{\gamma-1}\rangle\subseteq C_\alpha.$$

If γ is a limit ordinal, let X_γ be the inverse limit of $\{X_\alpha; r[\alpha, \beta]\}_{\alpha,\beta<\gamma}$, with our customary identifications. Using the notation of Lemma 1, $\langle p_\alpha\rangle_{\alpha<\gamma}\in X_\gamma-E_\gamma$, since $\langle p_\alpha\rangle_{\alpha<\gamma}$ is not an eventually constant γ-sequence. Let C_γ be the composant of X_γ containing $\langle p_\alpha\rangle_{\alpha<\gamma}$ and let $p_\gamma = \langle p_\alpha\rangle_{\alpha<\gamma}$. By Lemma 1, $C_\gamma\cap X_\alpha = \emptyset$ whenever $\alpha<\gamma$. The $r[\gamma, \alpha]$'s in this case are the projections. Suppose $\langle x_\beta\rangle_{\beta<\gamma}$ is a point in $X_\gamma-X_\alpha$. Then for some $\lambda>\alpha$, $x_\alpha\neq x_\gamma$, so that $x_\alpha\in C_\alpha$; that is, $r[\gamma, \alpha](\langle x_\beta\rangle_{\beta<\gamma})\in C_\alpha$, so that $r[\gamma, \alpha]\langle X_\gamma-X_\alpha\rangle\subseteq C_\alpha$, as required. The inverse system $\{X_\alpha; r[\alpha, \beta]\}_{\alpha,\beta<\omega_1}$, obtained by transfinite recursion, then satisfies the hypotheses of Theorem 1, and $C\neq\emptyset$ for the system, since the point $\langle p_\alpha\rangle_{\alpha<\omega_1}$, as chosen, must belong to C, completing the proof.

COROLLARY. *There exist indecomposable continua with one and two composants. In particular, any metric continuum is a retract of an indecomposable continuum with exactly one composant and also a retract of an indecomposable continuum with exactly two composants.*

Proof. By Corollary 4 of [3, p. 180], any metric continuum Y is a retract of a metric indecomposable continuum, so it suffices to consider the case that Y is indecomposable. Applying Theorem 2 with $Y = X_0$, there is an indecomposable continuum X with two composants of which Y is a retract, since the projection is a retraction. Let p belong to the composant of X not containing Y, and obtain Z from X by identifying p with $r(p)$, where r is the retraction from X onto Y. Then Z is an indecomposable continuum with only one composant, and r induces a retraction $R\colon Z\to Y$.

QUESTION. For which cardinal numbers m do there exist indecomposable continua with exactly m composants? Since $1 = 2^0$ and $2 = 2^1$, all indecomposable continua for which the number of composants is known, with or without additional set-theoretic assumption, have 2^m composants for some cardinal number m. Is this true for all indecomposable continua? A complete affirmative answer would imply that one and two are the only possible finite cardinals here, since the number of composants in an indecomposable continuum with only finitely many composants could always be decreased by one by identifying two points. If the continuum is initially assumed to have infinitely many composants, this is Problem number 926 in The New Scottish Book, posed by the author.

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Accepté par la Rédaction le 27. 4. 1976

Eine Bemerkung über lokalkompakte abelsche Gruppen

von

Peter Flor (Graz)

Sei Q die additive Gruppe der rationalen Zahlen. Für eine beliebige Kardinalzahl n bezeichne Q^{n*} die *schwache* n-te direkte Potenz von Q mit der diskreten Topologie.

Satz 1.2 der Arbeit [1] lautet so:

Sei G eine lokalkompakte abelsche Gruppe. Dann sind die folgenden beiden Eigenschaften äquivalent:

(a) *Jeder Charakter auf G mit Ausnahme des Hauptcharakters nimmt genau abzählbar unendlich viele Werte an.*

(b) $G \cong Q^{n*} \times G_0$, *wobei* $0 \leqslant n \leqslant \aleph_0$ *und* G_0 *eine topologische Torsionsgruppe ist.*

Daß diese Behauptung falsch ist, sieht man leicht ein: wenn man G_0 endlich und nichttrivial wählt, besitzt G nichttriviale Charaktere, deren Wertmengen endlich sind. Der Fehler ist aber leicht zu korrigieren:

SATZ. *Sei G eine lokalkompakte abelsche Gruppe. Dann ist folgende Bedingung zu* (a) *äquivalent:*

(c) $G \cong Q^{n*} \times G_0$, *wobei* $0 \leqslant n \leqslant \aleph_0$ *und* G_0 *eine topologische Torsionsgruppe ist, die eine dichte teilbare Untergruppe besitzt.*

Beweis. Zunächst gelte (a). Daß daraus (b) folgt, ist in [1] bewiesen, ebenso, daß die Dualgruppe $\hat{G}$ torsionsfrei ist. Aus $G \cong Q^{n*} \times G_0$ folgt $\hat{G} \cong \hat{Q}^n \times \hat{G}_0$, und als Untergruppe von $\hat{G}$ ist $\hat{G}_0$ ebenfalls torsionsfrei. Nach Theorem 5.2 von [2] besitzt G_0 daher eine dichte teilbare Untergruppe. Damit ist (c) bewiesen.

Nun gelte (c). Zur Abkürzung werde $Q^{n*} =: D$ gesetzt. Sei γ ein Charakter auf G. Es gilt $\gamma(G) = \gamma(G_0)\gamma(D)$. Da G_0 eine topologische Torsionsgruppe ist, muß die Menge $\gamma(G_0)$ nach Theorem 3.15 von [2] abzählbar sein. Dasselbe gilt für $\gamma(D)$, da D abzählbar ist. $\gamma(D)$ ist teilbar, und $\gamma(G_0)$ enthält eine dichte teilbare Untergruppe; daher ist jede dieser beiden Gruppen trivial oder unendlich. Wenn $\gamma \neq 1$ ist, ergibt sich daraus, daß in $\gamma(G) = \gamma(G_0)\gamma(D)$ mindestens ein Faktor abzählbar unendlich, der andere höchstens abzählbar ist; daher gilt (a). ■