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Most directional cluster sets have common values

by

C. L. Belna, M. J. Evans and **P. D. Humke** (Macomb, Ill.)

Abstract. Let f be a measurable function from the open upper half plane into the Riemann sphere, let x be a point on the real line R, and let $\Theta(x)$ denote the set of all directions θ $(0<\theta<\pi)$ in which the essential directional cluster set of f at x contains the total essential cluster set of f at x. It is shown that $\Theta(x)$ is of full measure for almost every and nearly every $x \in R$; furthermore, if f is continuous, then $\Theta(x)$ is residual for almost every and nearly every $x \in R$. Then an application of this result to regular directional cluster sets is given.

§ 1. Introduction. Let H denote the open upper half plane, and let x be a point on the real line R. For each direction θ $(0<\theta<\pi)$ and each r $(0<r\leqslant\infty)$, let

$$L_\theta(x, r) = \{z \in H\colon \arg(z-x) = \theta \text{ and } |z-x|<r\}.$$

Let S be a Lebesgue measurable subset of H, and let θ be a direction for which $S \cap L_\theta(x, \infty)$ is (linear) Lebesgue measurable. The upper density $\bar{D}(S; x)$ of S at x is the limit superior as $r \downarrow 0$ of the ratio

$$(*) \qquad |S \cap \{z \in H\colon |z-x|<r\}| / |\{z \in H\colon |z-x|<r\}|.$$

(We note that here and throughout this paper, the symbol $|G|$ denotes the Lebesgue measure of the set G; as to whether the measure is linear or 2-dimensional will always be clear from the context.) Should the limit as $r \downarrow 0$ of $(*)$ exist, the limit value is denoted $D(S; x)$ and is called the density of S at x. The corresponding directional densities $\bar{D}(S; x, \theta)$ and $D(S; x, \theta)$ are defined analogously with the set $\{z \in H\colon |z-x|<r\}$ in $(*)$ replaced by $L_\theta(x, r)$.

Let f be a measurable function from H into the Riemann sphere W. Then the *essential cluster set* $C_e(f, x)$ *of* f *at* x is defined to be the set of all values $w \in W$ for which $\bar{D}(f^{-1}(U); x)>0$ for each open set U containing w; the definition of the *essential cluster set* $C_e(f, x, \theta)$ *of* f *at* x *in the direction* θ is similar with $\bar{D}(f^{-1}(U); x)$ replaced by $\bar{D}(f^{-1}(U); x, \theta)$. Casper Goffman and W. T. Sledd [5, Theorem 2, p. 299] have established the following relationship between these two cluster sets.

THEOREM GS. *If* $f\colon H\to R$ *is measurable and* θ *is a direction, then*

$$C_e(f, x) \subset C_e(f, x, \theta),$$

except for a set of measure zero; furthermore, if f is continuous, then

$$C_e(f, x) \subset C_e(f, x, \theta),$$

except for a set of the first category.

To supplement this result, we prove (§ 4)

THEOREM 2. *Let $f\colon H \to W$, and for each $x \in R$ let*

$$\Theta(x) = \{\theta\colon C_e(f, x) \subset C_e(f, x, \theta)\}.$$

If f is measurable, then $|\Theta(x)| = \pi$ for almost every and nearly every $x \in R$; furthermore, if f is continuous, then $\Theta(x)$ is residual for almost every and nearly every $x \in R$.

(The expressions "for almost every $x \in R$" and "for nearly every $x \in R$" mean that the exceptional set is respectively of measure zero and of the first category. Also, a residual set is one whose complement is of the first category.)

In § 3 we establish a measure-theoretic result (Theorem 1) from which Theorem 2 follows readily; the technical lemmas needed in the proof of Theorem 1 are established in § 2. In § 5 we present (Theorem 3) a corollary of Theorem 2 concerning ordinary directional cluster sets. We conclude (§ 6) by listing several open questions.

§ 2. Lemmas concerning α-trapezoids and $\gamma\varkappa$-sectors. By an *α-trapezoid* we mean a trapezoid T in H with one of its bases I (henceforth referred to as "the base of T") being contained in R and having length twice that of the other base and for which the corresponding base angles are both equal to α $(0<\alpha<\frac{1}{2}\pi)$. An α-trapezoid whose base is centered at $x \in R$ is called an *α-trapezoid at x*. By a *$\gamma\varkappa$-sector at x* we mean the angular region in H lying between $L_\gamma(x, \infty)$ and $L_\varkappa(x, \infty)$.

In the interest of expediting both the statements and proofs of our first two lemmas, we introduce some new notations.: Suppose T is an α-trapezoid with base I, suppose $x_1, x_2, \ldots, x_{2N} \in I$, and suppose $\sigma(x_n)$ denotes the $\gamma\varkappa$-sector at x_n $(n = 1, 2, \ldots, 2N)$ for some γ and $\varkappa$ satisfying $\alpha \leqslant \gamma < \varkappa \leqslant \pi - \alpha$. Then with each sector $\sigma(x_n)$ we associate a "capped sector" $\sigma^*(x_n)$ as follows: $\sigma^*(x_1) = \sigma(x_1) \cap T$, and

$$\sigma^*(x_n) = [\sigma(x_n) \cap T] - \overline{\bigcup_{k=1}^{n-1} \sigma^*(x_k)}$$

for $n = 2, 3, \ldots, 2N$ (the bar denotes closure in H). Finally, we define the "radial proportional" of the capped sector $\sigma^*(x_n)$ to be the number

$$\operatorname{rp}[\sigma^*(x_n)] = \sup\Omega / \inf\Omega,$$

where $\Omega = \{|L_\theta(x_n, \infty) \cap \sigma^*(x_n)|\colon \gamma<\theta<\varkappa\}$.

We now present our first lemma which will be used only in the proof of Lemma 2.

LEMMA 1. *Let T be an α-trapezoid with base I and height h. Let $0<\hat{h}<h$, and set*

$$\hat{T} = \{z \in T\colon \operatorname{Im}(z) > \hat{h}\} \text{ and } \Lambda = \{z \in T\colon \operatorname{Im}(z) = \hat{h}\}.$$

Let γ and $\varkappa$ be such that $\alpha \leqslant \gamma < \varkappa \leqslant \pi - \alpha$, and let $\hat{I} = (a, b)$ be the subinterval of I where a and b are such that $L_\varkappa(a, \infty)$ and $L_\gamma(b, \infty)$ contain the left and right endpoint of Λ, respectively. Set

$$N = \operatorname{INT}\{2[2(h/\hat{h})-1]\cot\alpha/(\cot\gamma - \cot\varkappa)\}$$

(INT $\equiv$ greatest integer function) and partition $\hat{I}$ into $2N$ equal subintervals $I_1, I_2, \ldots, I_{2N}$. For each index n, let x_n be any point lying in the middle half of the interval I_n and let $\sigma(x_n)$ denote the $\gamma\varkappa$-sector at x_n. Then

(i) $$\Big|\bigcup_{n=1}^{2N} \sigma^*(x_n)\Big| > |\hat{T}|$$

and

(ii) $$\operatorname{rp}[\sigma^*(x_n)] \leqslant 8(h/\hat{h})\csc\alpha \quad (n = 1, 2, \ldots, 2N).$$

Proof. Since $|\hat{I}| = 2(2h-\hat{h})\cot\alpha - \hat{h}(\cot\gamma - \cot\varkappa)$, it is easy to see that

$$N = 1 + \operatorname{INT}\left\{\frac{|\hat{I}|}{\hat{h}(\cot\gamma - \cot\varkappa)}\right\}.$$

Hence, for each of the intervals I_n $(n = 1, 2, \ldots, 2N)$, we have

(1) $$\frac{\hat{h}(\cot\gamma - \cot\varkappa)}{4} \leqslant |I_n| < \frac{\hat{h}(\cot\gamma - \cot\varkappa)}{2}.$$

Let h_n be the height of the triangle complementary to $\sigma(x_n) \cup \sigma(x_{n+1})$ and having the interval between x_n and x_{n+1} as base, i.e., let

$$h_n = |x_n - x_{n+1}|/(\cot\gamma - \cot\varkappa).$$

It follows from (1) that

(2) $$\tfrac{1}{8}\hat{h} \leqslant h_n < \tfrac{3}{4}\hat{h} \quad (n = 1, 2, \ldots, 2N-1).$$

For each index n, let m_n and M_n denote respectively the infimum and supremum of the values $|L_\theta(x_n, \infty) \cap \sigma^*(x_n)|$ taken over all θ satisfying $\gamma<\theta<\varkappa$. Clearly $m_n \geqslant \min(h_{n-1}, h_n)$ with $h_0 = h_{2N} = \hat{h}$ and $M_n \leqslant h\csc\alpha$; hence, conclusion (ii) follows from the left-hand inequality in (2).

Now, by the right-hand inequality in (2), we have $h_n < \hat{h}$ for $n = 1, 2, \ldots, 2N-1$. Therefore, $\overline{\bigcup_{n=1}^{2N} \sigma^*(x_n)}$ covers all of $\hat{T}$ except for a small triangle Δ_1 in the lower left corner of $\hat{T}$ and a small triangle Δ_2 in the lower right corner of $\hat{T}$. Since the base angles of Δ_1 are α and $\pi - \varkappa$ and since the base of Δ_1 has length $\leqslant \frac{3}{4}|I_1|$, we can utilize the right-hand inequality in (1) to deduce the inequalities

$$\begin{aligned}\operatorname{Area}[\Delta_1] &\leqslant \tfrac{9}{32}|I_1|^2/(\cot\alpha - \cot\varkappa)\\ &< \tfrac{9}{128}(\cot\gamma - \cot\varkappa)\left[\frac{\cot\gamma - \cot\varkappa}{\cot\alpha - \cot\varkappa}\right]\hat{h}^2\\ &\leqslant \tfrac{9}{128}(\cot\gamma - \cot\varkappa)\hat{h}^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\operatorname{Area}[\Delta_2] &\leqslant \tfrac{9}{32}|I_1|^2/(\cot\gamma+\cot\alpha)\\ &< \tfrac{9}{128}(\cot\gamma-\cot\varkappa)\left(\frac{\cot\gamma-\cot\varkappa}{\cot\gamma+\cot\alpha}\right)\hat{h}^2\\ &\leqslant \tfrac{9}{128}(\cot\gamma-\cot\varkappa)\hat{h}^2 .\end{aligned}$$

Consequently,

(3) $$\operatorname{Area}[\Delta_1\cup\Delta_2] < \tfrac{1}{2}(\cot\gamma-\cot\varkappa)\hat{h}^2 .$$

Finally, since for each n

$$\operatorname{Area}[\sigma(x_n)\cap(T-\hat{T})] = \tfrac{1}{2}(\cot\gamma-\cot\varkappa)\hat{h}^2 ,$$

conclusion (i) follows from (3), and the lemma is proved.

In the statement of the next lemma and for the remainder of the paper, we use $|G|^*$ to denote the Lebesgue outer measure of the linear set G.

LEMMA 2. *Let T be an α-trapezoid at x with base I, let γ and $\varkappa$ satisfy $\alpha\leqslant\gamma<\varkappa\leqslant\pi-\alpha$, and let j be a positive integer. Then there exists a positive integer $N = N(\alpha,\gamma,\varkappa,j)$ for which the following is true: Let Q be any subset of I that is either dense in I or satisfies $|Q\cap J|^* > (1-1/4N)|J|$ for each subinterval J of I containing x. Then there exist $2N$ points $x_1, x_2, \ldots, x_{2N}$ in Q such that*

(i) $$\Big|\bigcup_{n=1}^{2N}\sigma^*(x_n)\Big| > \left(1-\frac{1}{2j}\right)|T|$$

and

(ii) $$\operatorname{rp}[\sigma^*(x_n)] \leqslant \frac{8\csc\alpha}{2-\sqrt{4-3/2j}} \qquad (n = 1, 2, \ldots, 2N),$$

where $\sigma(x_n)$ denotes the $\gamma\varkappa$-sector at x_n.

Proof. Let h be the height of T, set $\hat{h} = (2-\sqrt{4-3/2j})\cdot h$, and let $\hat{T}$ and $\hat{I}$ be as in Lemma 1. (Note that $|\hat{T}| = (1-1/2j)|T|$.) Set

$$N = \operatorname{INT}\{2[2(h/\hat{h})-1]\cot\alpha/(\cot\gamma-\cot\varkappa)\},$$

and partition $\hat{I}$ into the $2N$ equal intervals $I_1, I_2, \ldots, I_{2N}$. Let I_n' denote the middle-half subinterval of I_n. For any choice of points $x_n \in I_n'$, both conclusions (i) and (ii) follow from Lemma 1.

If Q is dense in $\hat{I}$, it is clear that the points x_n can be chosen from Q. Furthermore, since $|\hat{I}| = 4N|I_n'|$, the x_n can also be chosen from Q if $|Q\cap\hat{I}|^* > (1-1/4N)|\hat{I}|$. This proves the lemma.

After applying Lemma 2, we will need the next lemma to determine the relative measure of a certain set in the capped sectors $\sigma^*(x_n)$.

LEMMA 3. *Let S be a measurable subset of H. Let $0<\gamma<\varkappa<\pi$, let $\varrho = \varrho(\theta)$ be a positive continuous function on $[\gamma,\varkappa]$, and set*

$$\Delta_\varrho = \{re^{i\theta}\colon \gamma<\theta<\varkappa \text{ and } 0<r<\varrho(\theta)\}.$$

Let Θ be a subset of $[\gamma,\varkappa]$ with $|\Theta|\geqslant l(\varkappa-\gamma)$ for some l $(0<l<1)$. If for some ε $(0<\varepsilon<1)$,

(1) $$|S\cap L_\theta(0,\varrho(\theta))| \geqslant \varepsilon\varrho(\theta) \quad \textit{for each } \theta\in\Theta ,$$

then

$$|S\cap\Delta_\varrho| \geqslant \varepsilon^2[1-(1-l)\eta^2]\,|\Delta_\varrho| ,$$

where $\eta = M/m$, $M = \max\{\varrho(\theta)\colon \theta\in[\gamma,\varkappa]\}$, and $m = \min\{\varrho(\theta)\colon \theta\in[\gamma,\varkappa]\}$.

Proof. Set $\Delta_\varrho(\Theta) = \{re^{i\theta}\colon \theta\in\Theta \text{ and } 0<r<\varrho(\theta)\}$. Then, letting χ_S denote the characteristic function of S, we have

(2) $$|S\cap\Delta_\varrho(\Theta)| = \int_\Theta\int_0^{\varrho(\theta)}\chi_S(re^{i\theta})\,r\,dr\,d\theta \geqslant \int_\Theta\int_0^{\varepsilon\varrho(\theta)} r\,dr\,d\theta = \varepsilon^2\int_\Theta\int_0^{\varrho(\theta)} r\,dr\,d\theta = \varepsilon^2|\Delta_\varrho(\Theta)| ,$$

where the inequality is easily deduced from hypothesis (1).

Letting $\tilde{\Theta} = [\gamma,\varkappa]-\Theta$, we see that

(3) $$|\Delta_\varrho(\tilde{\Theta})| = \int_{\tilde{\Theta}}\int_0^{\varrho(\theta)} r\,dr\,d\theta = \tfrac{1}{2}\int_{\tilde{\Theta}}[\varrho(\theta)]^2\,d\theta \leqslant \tfrac{1}{2}M^2|\tilde{\Theta}| \leqslant \tfrac{1}{2}M^2(1-l)(\varkappa-\gamma) .$$

Then, since

$$|\Delta_\varrho| = \int_\gamma^\varkappa\int_0^{\varrho(\theta)} r\,dr\,d\theta = \tfrac{1}{2}\int_\gamma^\varkappa[\varrho(\theta)]^2\,d\theta \geqslant \tfrac{1}{2}m^2(\varkappa-\gamma) ,$$

it follows from (3) that

(4) $$|\Delta_\varrho(\tilde{\Theta})| \leqslant (M^2/m^2)(1-l)\,|\Delta_\varrho| .$$

Now, in view of (2), we see that

$$|S\cap\Delta_\varrho| \geqslant |S\cap\Delta_\varrho(\Theta)| \geqslant \varepsilon^2|\Delta_\varrho(\Theta)| = \varepsilon^2\{|\Delta_\varrho|-|\Delta_\varrho(\tilde{\Theta})|\} .$$

The conclusion of the lemma now follows from (4).

§ 3. **A preliminary theorem concerning density.** For $S\subset H$ and $x\in R$, we define the set

$$\Theta(S;x) = \{\theta\colon D(S;x,\theta) = 1\};$$

and for certain sets S we will determine the nature of the set of points x at which $D(S;x)\neq 1$ while the set $\Theta(S;x)$ is either of positive measure or of the second category. In so doing, we will make implicit use of both the decomposition

$$\Theta(S;x) = \bigcap_{j=1}^{\infty}\bigcup_{k=1}^{\infty}\Theta_{jk}(S;x)$$

where

$$\Theta_{jk}(S;x) = \{\theta:\ |S \cap L_\theta(x,r)| \geq (1-1/3j)^{1/2}\cdot r \text{ for } 0<r<1/k\},$$

and the fact that each $\Theta_{jk}(S;x)$ is measurable provided S is measurable. (The measurability of the sets $\Theta_{jk}(S;x)$, and hence that of the set $\Theta(S;x)$, can readily be verified utilizing Fubini's theorem.)

THEOREM 1. *If S is a measurable subset of H, then $D(S;x) = 1$ at almost every and nearly every point x of the set*

$$A = \{x:\ |\Theta(S;x)|>0\};$$

furthermore, if S is closed, then $D(S;x) = 1$ at almost every and nearly every point x of the set

$$B = \{x:\ \Theta(S;x) \text{ is second category}\}.$$

Proof. For $0<\alpha<\frac{1}{2}\pi$, set

$$A_\alpha = \{x:\ |\Theta(S;x) \cap (\alpha, \pi-\alpha)|>0\}.$$

For the positive integer j, let $A_\alpha(j)$ be the set of all points $x \in A_\alpha$ such that there exists a sequence of α-trapezoids $T_n(x)$ $(n = 1,2,\ldots)$ at x with $T_n(x)\to x$ and

$$\frac{|\tilde{S} \cap T_n(x)|}{|T_n(x)|} > \frac{1}{j} \quad \text{for each index } n,$$

where $\tilde{S} = H-S$. For the positive integer k, set

$$A_\alpha(j,k) = \{x \in A_\alpha(j):\ |\Theta_{jk}(S;x) \cap (\alpha,\pi-\alpha)|>0\}.$$

Set $\tau = 8\csc\alpha/(2-\sqrt{4-3/2j})$ and choose a value $l \in (0,1)$ for which

$$(1-1/3j)[1-(1-l)\tau^2]>1-1/2j. \tag{1}$$

Then for γ and $\varkappa$ satisfying $\alpha \leq \gamma < \varkappa \leq \pi-\alpha$, set

$$A_\alpha(j,k,l,\gamma,\varkappa) = \{x \in A_\alpha(j,k):\ |\Theta_{jk}(S;x) \cap (\gamma,\varkappa)| \geq l(\varkappa-\gamma)\}.$$

To prove the statement in the theorem concerning the set A, it suffices to show that the set

$$Q = A_\alpha(j,k,l,\gamma,\varkappa)$$

is a nowhere dense set of measure zero on R provided S is measurable. (This follows from the fact that the set of points $x \in A$ for which $D(S;x) \neq 1$ is contained in a countable union of sets of the type Q.)

Let $N = N(\alpha,\gamma,\varkappa,j)$ be the integer guaranteed by Lemma 2, and suppose there exists a point $x_0 \in R$ and a $\delta_0>0$ such that either Q is dense on the interval $I_0 = (x_0-\delta_0, x_0+\delta_0)$ or

$$|Q \cap I|^* > (1-1/4N)\cdot|I|$$

for each subinterval I of I_0 containing x_0. Then let T be an α-trapezoid at x_0 with

base $I \subset I_0$ and height $<(1/k)\sin\alpha$. According to Lemma 2, there exist $2N$ points $x_1, x_2, \ldots, x_{2N}$ in $Q \cap I$ such that if $\sigma(x_n)$ denotes the $\gamma\varkappa$-sector at x_n, then

(i) $$\Big|\bigcup_{n=1}^{2N} \sigma^*(x_n)\Big| > (1-1/2j)\cdot|T|$$

and

(ii) $$\operatorname{rp}[\sigma^*(x_n)] \leq \tau \quad (n = 1,2,\ldots,2N).$$

Since

$$|\tilde{S} \cap T| \leq \Big|\tilde{S} \cap \bigcup_{n=1}^{2N} \sigma^*(x_n)\Big| + \Big|T - \bigcup_{n=1}^{2N} \sigma^*(x_n)\Big|,$$

it follows from (i) that

$$|\tilde{S} \cap T| < \Big|\tilde{S} \cap \bigcup_{n=1}^{2N} \sigma^*(x_n)\Big| + (1/2j)|T|. \tag{2}$$

Furthermore, in view of (ii), it follows from Lemma 3 that for each n

$$|S \cap \sigma^*(x_n)| \geq (1-1/3j)[1-(1-l)\tau^2]\cdot|\sigma^*(x_n)|;$$

and by (1) we have

$$|S \cap \sigma^*(x_n)| > (1-1/2j)\cdot|\sigma^*(x_n)| \quad (n = 1,2,\ldots,2N).$$

Hence,

$$\Big|S \cap \bigcup_{n=1}^{2N} \sigma^*(x_n)\Big| = \sum_{n=1}^{2N} |S \cap \sigma^*(x_n)| > (1-1/2j)\sum_{n=1}^{2N} |\sigma^*(x_n)| = (1-1/2j)\cdot\Big|\bigcup_{n=1}^{2N} \sigma^*(x_n)\Big|.$$

That is,

$$\Big|\tilde{S} \cap \bigcup_{n=1}^{2N} \sigma^*(x_n)\Big| < (1/2j)\cdot\Big|\bigcup_{n=1}^{2N} \sigma^*(x_n)\Big| < (1/2j)\cdot|T|.$$

This combined with (2) yields the inequality

$$\frac{|\tilde{S} \cap T|}{|T|} < \frac{1}{j}.$$

Hence $x_0 \notin A_\alpha(j)$. This precludes $x_0 \in Q$; and, in view of the Lebesgue Density Theorem for arbitrary sets, the statement in the theorem concerning the set A is proved.

Now, for $0<\alpha<\frac{1}{2}\pi$, set

$$B_\alpha = \{x:\ \Theta(S;x) \text{ is second category on } (\alpha,\pi-\alpha)\}.$$

For the positive integer j, let $B_\alpha(j)$ be the set of all points $x \in B_\alpha$ such that there exists a sequence of α-trapezoids $T_n(x)$ $(n = 1,2,\ldots)$ at x with $T_n(x)\to x$ and

$$\frac{|\tilde{S} \cap T_n(x)|}{|T_n(x)|} > \frac{1}{j} \quad (n = 1,2,\ldots).$$

For the positive integer k, set

$$B_\alpha(j,k) = \{x \in B_\alpha(j)\colon \Theta_{jk}(S;x) \text{ is second category on } (\alpha, \pi-\alpha)\}.$$

To prove the statement in the theorem concerning the set B, we need only show that the set $B_\alpha(j,k)$ is of the first category and measure zero on R whenever S is closed.

If S is closed in H, then the set $\Theta_{jk}(S;x)$ is closed in $(0,\pi)$. Hence, the set $B_\alpha(j,k)$ is a subset of the set $A_\alpha(j,k)$ which we now know to be of the first category and measure zero on R. This completes the proof of the theorem.

§ 4. Proof of Theorem 2. Let $\mathscr{B}$ be a countable basis for the topology on W, and let $\mathscr{G}$ be the collection of all sets expressible as the closure of a finite union of sets in $\mathscr{B}$.

Using the notation introduced prior to the statement of Theorem 1, we associate with each $G \in \mathscr{G}$ the set

$$E(G) = \{x\colon |\Theta(f^{-1}(G);x)|>0 \text{ and } D(f^{-1}(G),x) \neq 1\}$$

and the set

$$F(G) = \{x\colon \Theta(f^{-1}(G);x) \text{ is second category and } D(f^{-1}(G);x) \neq 1\}.$$

If f is measurable (continuous), it follows from Theorem 1 that each of the sets $E(G)$ $(F(G))$ is of the first category and measure zero on R. Now let

$$E = \{x\colon |\Theta(x)| \neq \pi\} \text{ and } F = \{x\colon \Theta(x) \text{ is not residual}\}.$$

Then, since $E \subset \bigcup_{G\in\mathscr{G}} E(G)$ and $F \subset \bigcup_{G\in\mathscr{G}} F(G)$, the theorem is proved.

§ 5. Applications to ordinary directional cluster sets. Throughout this section, we let $C(f,x,\theta)$ denote the ordinary cluster set of the function f at $x \in R$ in the direction θ. By consolidating results of F. Bagemihl, G. Piranian, and G. S. Young [2, Theorem 6] and Bagemihl [1, Theorem 11], we obtain

THEOREM BPY. *Let $f\colon H \to W$ be holomorphic. Then to almost every and nearly every $x \in R$, there corresponds a set $\Theta^* = \Theta^*(x)$ of directions whose complement contains at most one direction and for which* $\bigcap_{\theta\in\Theta^*} C(f,x,\theta) \neq \emptyset$.

By combining two results of E. F. Collingwood [3, Corollary 2 and Theorem 3] with a result of P. Lappan [6, Theorem 1], we arrive at the following analogue of Theorem BPY for continuous functions.

THEOREM CL. *Let $f\colon H \to W$ be continuous. Then for almost every and nearly every $x \in R$, there corresponds a set $\Theta^* = \Theta^*(x)$ of directions whose complement is of the first category and for which* $\bigcap_{\theta\in\Theta^*} C(f,x,\theta) \neq \emptyset$.

We note that this theorem is also a direct consequence of Theorem 2; furthermore, Theorem 2 yields the following result which supplements both of the theorems cited above.

THEOREM 3. *Let $f\colon H \to W$ be measurable. Then to almost every and nearly every $x \in R$, there corresponds a set $\Theta^* = \Theta^*(x)$ of directions whose complement is of measure zero and for which* $\bigcap_{\theta\in\Theta^*} C(f,x,\theta) \neq \emptyset$.

§ 6. Open questions.

QUESTION 1. *Does Theorem 2 remain true when either* (a) *"continuous" is replaced by "measurable", or* (b) *"measurable" is replaced by "arbitrary"?*

(For arbitrary $f\colon H \to W$, Lebesgue outer measure is used to define the essential cluster sets.)

REMARK. Theorem 2 does not remain true when "continuous" is replaced by "arbitrary". For if so, it would follow that Theorem CL is true for arbitrary functions; however, P. Erdös and G. Piranian [4, Theorem 3] have shown this not to be the case by proving

THEOREM EP. *There exists a function $f\colon H \to W$ with the property that to each $x \in R$ there corresponds a second category set of directions $\Theta^*(x)$ such that, for any three distinct directions θ_1, θ_2, and θ_3 in $\Theta^*(x)$,*

$$\bigcap_{j=1}^{3} C(f,x,\theta_j) = \emptyset.$$

QUESTION 2. *Does Theorem 2 remain valid when the definition of the set $\Theta(x)$ is changed to* $\Theta(x) = \{\theta\colon C_e(f,x) = C_e(f,x,\theta)\}$?

Added in proof. Since submission of this paper, we observed that the statements of our theorems could be strengthened with very little change in the proofs. More specifically, in each of our theorems, the phrase *almost every and nearly every* can be replaced by the phrase *virtually every*, the meaning of which is given below.

Following E. P. Dolženko [Math. USSR — Izvestija 1 (1967), pp. 1–12], we say that the subset P of R is *porous at the point* $x \in R$ provided

$$\limsup_{\varepsilon\to 0} \frac{r(x,\varepsilon,P)}{\varepsilon} > 0,$$

where $r(x,\varepsilon,P)$ is the length of the largest open interval in the complement of P which is entirely contained in the interval $(x-\varepsilon, x+\varepsilon)$. Then P is a *porous* set if it is porous at each of its points, and it is a σ-porous set if it is the countable union of porous sets. Clearly, a σ-porous set is both of the first category and of measure zero; on the other hand, L. Zajíček [Časopis pro pěstování matematiky, roč. 101 (1976), Praha, pp. 350–359] has exhibited a perfect set of measure 0 that is not σ-porous.

Now the expression "for virtually every $x \in R$" means that the exceptional set is a σ-porous set; and, in view of the last sentence in the previous paragraph, we see that this expression is stronger than the expression "for almost every and nearly every $x \in R$".

The essential changes needed to improve our results as indicated above are the following: (1) Change the hypothesis on the set Q in Lemma 2 to read: Let Q be any subset of I such that $r(x,\varepsilon,Q) < \varepsilon/4N$ for $0 < \varepsilon < \frac{1}{2}|I|$. Then the lemma remains valid, as a close examination of its proof will reveal; (2) In the proof of Theorem 1, suppose there exists a point x_0 and a $\delta_0 > 0$ such that $r(x_0,\varepsilon,Q) < \varepsilon/4N$ for $0 < \varepsilon < \delta_0$. Then proceed as before. The contradiction arrived at will now imply that Q is porous, as desired.

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Ambiguity and stratification

by

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Abstract. E. Specker has proved that simple type theory with additional axioms expressing typical ambiguity is consistent iff Quine's "New-Foundations" is. His proof is essentially model-theoretic. In this paper, the same result is established using proof theory. It is also shown that there is a recursive procedure that transforms a proof of a stratified formula in a proof in which all formulas are stratified.

1. Let ST denote simple type theory with, as additional axioms, all sentences of the form:

$$A \leftrightarrow A^1, \tag{1.1}$$

where A^1 is obtained from A by raising all types by 1. Specker [2] has proved that ST is consistent iff Quine's NF is. Specker's proof is model-theoretic. The same result will be obtained, here, using proof theory.

Moreover, it is provable that:

(r. p.) there is a recursive procedure for transforming a cut-free derivation $\mathscr{A}$ of a stratified Theorem A of NF (or of a theory all of whose axioms are stratified) into a derivation $\mathscr{B}$, such that

1. $\mathscr{B}$ is a derivation of A, all of whose formulas are stratified;

2. $\mathscr{A}$ and $\mathscr{B}$ are equivalent in the sense that, removing the cuts from $\mathscr{A}$ and $\mathscr{B}$ in the usual way ([1]), one obtains essentially the same derivation.

In fact, the proof of Theorem 2 (below) gives rise to a recursive procedure for obtaining from a cut-free derivation $\mathscr{A}$ of a stratified theorem of the predicate calculus, a derivation $\mathscr{B}$ in type theory with the additional rule:

$$\frac{A}{A^*}. \tag{$*$}$$

In $(*)$ it is understood that A is a theorem and that A^* is as A except, loosely speaking, for the type indices. The details of the proof of (r. p.), although they are clumsy, do not, however, involve any significant difficulties. For this reason, the proof will not be given.