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Metric criteria for Banach and Euclidean spaces

by

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Abstract. Let M be a complete, convex, externally convex, metric space. It is known that M is a Banach space if and only if M satisfies the Quadrilateral Midpoint Postulate. In this paper Banach spaces with unique metric lines are characterized by somewhat weaker four-point properties. These weaker versions are analogues of the Weak, Feeble, and Queasy Euclidean Four-point Properties used by Blumenthal and Day to characterize Euclidean spaces. In addition localized versions of the Euclidean and Banach four-point properties are used to characterize inner-product spaces.

1. Introduction. Andalafte and Blumenthal [1] characterized the class of real Banach spaces among the class of complete, convex, externally convex metric spaces which have the two-triple property by means of a specified relation between the segment joining the two midpoints of two sides of a triangle and the base of that triangle. More specificially Andalafte and Blumenthal said that a metric space satisfies the *Young Postulate* provided for each three of its points p, q, and r, if q' and r' are the respective midpoints of p and q and p and r, then $q'r' = qr/2$. They showed that a complete, convex, externally convex metric space with the two-triple property is a real Banach space if and only if it satisfies the Young Postulate. Valentine and Wayment [8] said that a metric space satisfies the *Quadrilateral Midpoint Postulate* provided for each four points p, q, r, and s of the space, no three of which are collinear, if m_1, m_2, m_3, and m_4 are the midpoints of p and q, q and r, r and s, and s and p, respectively, then $m_1m_2 = m_3m_4$ and $m_2m_3 = m_1m_4$. They showed a complete, convex, externally convex metric space with the two-triple property satisfies the Quadrilateral Midpoint Postulate if and only if it satisfies the Young Postulate, and thus obtained a characterization of real Banach spaces.

The Quadrilateral Midpoint Postulate is analogous to the characterization of Hilbert space by the parallelogram law [7]. It is also analogous to the *Euclidean Four-point Property* which assumes each quadruple of points of a space is congruent to a quadruple of points of Euclidean space. The Euclidean Four-point Property was introduced by Wilson [9] who showed that a complete, convex, externally convex metric space has the Euclidean Four-point Property if and only if is congruent to a Euclidean space (possibly infinite dimensional). Since the Quadrilateral Midpoint Postulate effects a characterization of Banach spaces (see Theorem 2.1

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and [8]) it is more appropriate to call it the *Banach Four-point Property* as we do here. The condition that no three of $\{p, q, r, s\}$ be collinear is superfluous and will not be considered a part of the definition of the Banach Four-point Property.

Blumenthal [3] characterized Euclidean space with the *Euclidean Weak Four-point Property* (each quadruple which contains a linear triple is congruent to a quadruple in Euclidean space), and he [4] further showed that one need only consider quadruples containing a triple of which one is a midpoint of the other two (the *Euclidean Feeble Four-point Property*). A metric space has the *Euclidean Queasy Four-Point Property* provided for each pair of distinct points q and s, there is a point r between q and s such that for each point p of the space the quadruple $\{p, q, r, s\}$ is congruent to a quadruple of points of Euclidean space. Day [5] showed this property also characterizes Euclidean space among the class of complete, convex, externally convex metric spaces. In Section 2 we define the Banach Weak, Feeble, and Queasy Four-point properties and show they characterize rotund Banach spaces over the reals among the class of complete, convex, externally convex metric spaces.

Since the time of Blumenthal's characterizations of Euclidean space by means of the Euclidean Weak and Feeble Four-point Properties, much work has been done toward reducing the types of quadruples in a space that need be congruently embedable in Euclidean space to obtain the characterization. For example, Valentine and Wayment [9] have shown that the Euclidean Weak, Feeble, or Queasy Four-point Properties need only be vaid in a small neighborhood of each point. In Section 3 we investigate slightly different local versions of these properties, where we require that only those quadruples containing a given fixed point of the space be congruently embeddable in Euclidean space. In this situation we say that a particular four-point property is possessed by the space "at the point p". Theorem 3.3 states that a reasonable space with the Euclidean Feeble Four-point Property at a point p must be an inner-product space. The various Banach Four-point Properties at a point p can be used to characterize inner-product spaces provided there is also a neighborhood of p in which one of the Euclidean Four-point Properties is valid (see Theorems 3.1 and 3.2 for clarification).

Throughout the paper M will denote a complete, convex, externally convex metric space.

2. The Banach Four-point Properties. In this section we formally define the Banach Weak, Feeble, and Queasy Four-point Properties and show that a complete, convex, externally convex metric space is a rotund Banach space over the reals if and only if it possesses one of the Banach Four-point Properties. We consider a Banach space to be *rotund* provided it has a unique metric line through each pair of its distinct points. Other criteria for a Banach space to be rotund are given in [2]. We first show the Young Postulate implies unique metric lines, and consequently it implies the two-triple property (see [3, Theorem 21.3, p. 57]). This fact was apparently overlooked by Andalafte and Blumenthal [1] and Valentine and Wayment [8], since they postulated unique lines.

THEOREM 2.1. *If M satisfies the Young Postulate, then each pair of distinct points lies on a unique metric line.*

Proof. It is well known that each pair of distinct points of M lie on at least one metric line. If M contains distinct points x and y which lie on at least two lines, then there are at least two metric segments with endpoints x and y; or there is a unique segment with endpoints x and y and this segment admits two prolongations. Suppose x and y endopoints of two segments. Since points x' and y' and distinct subsegments can be found which have only the endopoints x' and y' in common, we assume without loss of generality that the distinct segments with endpoints x and y have only the points x and y in common. Since M is externally convex the segments admit prolongation. Pick p, q, r and m such that m is a midpoint of p and q and m is a midpoint of p and r but $q \neq r$. Then $0 = mm \neq qr/2$ so M does not satisfy the Young Postulate.

If x and y are endpoints of a unique segment which can be prolonged through x, say, in two ways, then it is possible to pick p, q, r and m as above and obtain the same contradiction. The theorem follows.

DEFINITION 2.1. The space M has the *Banach Weak Four-point Property* provided for each quadruple $\{p, q, r, s\}$ of distinct points containing a linear triple $\{q, r, s\}$ with r between q and s, if m_1, m_2, m_3 and m_4 are respective midpoints of p and q, q and r, r and s, and s and p, then $m_1m_2 = m_3m_4$ and $m_1m_4 = m_2m_3$.

DEFINITION 2.2. The space M has the *Banach Feeble Four-point Property* provided for each quadruple $\{p, q, r, s\}$ of distinct points containing a triple $\{q, r, s\}$ with r a midpoint of q and s, if m_1, m_2, m_3 and m_4 are respective midpoints of p and q, q and r, r and s, and s and p, then $m_1m_2 = m_3m_4$ and $m_1m_4 = m_2m_3$.

DEFINITION 2.3. The space M has the *Banach Queasy Four-point Property* provided for each pair $\{q, s\}$ of its distinct points there is a point r between q and s such that for each point p of the space, if m_1, m_2, m_3, and m_4 are the respective midpoints of p and q, q and r, r and s, and s and p, then $m_1m_2 = m_3m_4$ and $m_1m_4 = m_2m_3$.

It is clear that if M has the Banach Weak Four-point Property, then M has the Banach Feeble Four-point Property; and if M has the Banach Feeble Four-point Property, then M has the Banach Queasy Four-point Property. Moreover, every rotund Banach space over the reals possesses all the Banach Four-point Properties. Thus in order to show that each of these Banach Four-point Properties characterizes rotund Banach spaces over the reals it suffices to show that M is a rotund Banach space over the reals if M has the Banach Queasy Four-point Property. We accomplish this by showing that the Banach Queasy Four-point Property implies the Young Postulate in M.

THEOREM 2.2. *If M has the Banach Queasy Four-point Property, then M satisfies the Young Postulate.*

Proof. Let p, q and s be any three points of M and let q' and s' be midpoints of p and q and p and s, respectively. By the Banach Queasy Four-point Property, there is a point r between q and s such that if r' and r'' are the respective midopoints of r and q and r and s, then $q's' = r'r''$. But since $qr' = qr/2$, $sr'' = rs/2$ and $qr+rs = qs$, we have $q's' = qs/2$. Thus M satisfies the Young Postulate.

Theorem 2.3 follows from Theorem 2.2, Theorem 2.1, and [1].

THEOREM 2.3. *A space M is a rotund Banach space over the reals if and only if it has the Banach Weak, Feeble, or Queasy Four-point Property.*

When examining a space for the Banach Four-point Property one may fix a point p rather than checking all possible quadruples, as the following definition and theorem indicate.

DEFINITION 2.4. The space M has the *Banach Four-point Property at p* provided for each three distinct points q, r and s of $M-\{p\}$ if m_1, m_2, m_3 and m_4 are respective midpoints of p and q, q and r, r and s, and s and p, then $m_1m_2 = m_3m_4$ and $m_1m_4 = m_2m_3$.

THEOREM 2.4. *A space M is a rotund Banach space over the reals if and only if M contains a point p such that M has Banach Four-point Property at p.*

Proof. As in the previous proof, we show this property implies the Young Postulate. Let q, r and s be noncollinear points of M, and, as in Definition 2.4. If $p \notin \{q, r, s\}$, let m_1, m_2, m_3 and m_4 be the respective midpoints of p and q, q and r, r and s, and s and p. We must show $m_2m_3 = qs/2$. By the Banach Four-point Property at p,

(1) $$m_1m_4 = m_2m_3 .$$

Let t be a midpoint of q and s, and let m_2' and m_3' be the respective midpoints of q and t and t and s. The Banach Four-point Property at p applied to the quadruple $\{p, q, t, s\}$ yields $m_1m_4 = m_2'm_3' = qs/2$. This together with (1) shows $m_2m_3 = qs/2$.

If $p \in \{q, r, s\}$, then it may be necessary to re-label the points but the previous paragraph still applies to complete the proof.

Analogous to the Banach Four-point Property at p are the Banach Weak, Queasy, and Feeble Four-point Properties at p. We formally state only the Banach Weak Four-point Property at a point, leaving the other two definitions to the reader.

DEFINITION 2.5. The space M has the *Banach Weak Four-point Property* at p provided for each quadruple $\{q, r, s, t\}$ of distinct points of M containing p and containing a linear triple $\{r, s, t\}$ with s between r and t, if m_1, m_2, m_3 and m_4 are respective midpoints of q and r, r and s, s and t, and t and q, then $m_1m_2 = m_3m_4$ and $m_1m_4 = m_2m_3$.

Whether or not Theorem 2.4 remains valid when the Banach Four-point Property at p is replaced by the Banach Weak, Feeble, or Queasy Four-point Property at p is an open question.

3. Characterizations of Euclidean spaces. None of the neighborhood versions of the Euclidean Weak, Feeble, or Queasy Four-point Properties at a single point imply that M is an inner-product space — Example 3.2 presented below illustrates this. However when one of these local properties is combined with one of the Banach Weak, Feeble, or Queasy Four-point Properties at the same point an inner-product space results. To demonstrate this fact we first need some definitions.

DEFINITION 3.1. The space M satisfies the *Young Postulate at a point p* provided, for each pair of points q and r of M, if q' and r' are the midpoints of p and q and p and r, respectively, then $q'r' = qr/2$.

THEOREM 3.1. *If M has the Banach Weak, Feeble, or Queasy Four-point Property at p, then M satisfies the Young Postulate at p.*

The proof of Theorem 3.1 is an easy exercise. Example 3.1, due to Blumenthal, shows the converse is false. This example also shows that the Young Postulate at a point does not characterize real Banach spaces among the class of complete, convex, externally convex metric spaces.

EXAMPLE 3.1. Let T be the convex, externally convex tripod; that is, T is the union of three metric rays $R(p, q)$, $R(p, r)$ and $R(p, s)$ whose pairwise intersections are each $\{p\}$ and whose pairwise unions are metric lines. With the preceeding notation, it is easily seen that the tripod T satisfies the Young Postulate at p. However if we choose a triple $\{q, r, s\}$ in T such that $pq = pr = ps$ and if we let m_1, m_2, m_3, and m_4 be the respective midpoints of p and q, q and r, r and s, and s and p, then we see that $m_2 = m_3 = p$ and $0 = m_2m_3 \neq m_1m_4$. Consequently T does not have the Banach Weak, Feeble, or Queasy Four-point Properties at p.

DEFINITION 3.2. The space M has the *Local Euclidean Weak Four-point Property at a point p* if there is a neighborhood of p in which the Euclidean Weak Four-point Property is valid (note that p need not be one of the four points under consideration). The other four-point properties are similarly localized.

THEOREM 3.2. *If M has the Local Euclidean Weak, Feeble, or Queasy Four-point Property at a point p of M, and if M satisfies the Young Postulate at the same point p, then M is an inner-product space.*

Proof. It is known [4] that M is an inner-product space if M satisfies the Euclidean Feeble Four-point Property; thus we proceed to show that M has this latter property. We illustrate the proof with the hypothesis that M has the Local Euclidean Feeble Four-point Property at p, leaving the similar "Queasy" case to the reader. Suppose the Euclidean Feeble Four-point Property is valid in the sphere $S(p; \varepsilon)$, and let $\{q, r, s, t\}$ be any quadruple of points of M with s a midpoint of r and t. If $k = \max\{pq, pr, ps, pt\}$, choose n such that $2^n > k/\varepsilon$. Let q', r', s', t' be points of M such that $px' = px/2^n$ and x' is between p and x, where $x \in \{q, r, s, t\}$. Then $\{q', r', s', t'\} \subset S(p; \varepsilon)$ and from n applications of the Young Postulate at p we have $x'y' = xy/2^n$ where $\{x, y\} \subset \{q, r, s, t\}$. Now s' is a midpoint of r' and t', so the Euclidean plane E^2 contains a quadruple of points which is congruent to $\{q', r', s', t'\}$. Multiplying all the distances of this Euclidean quadruple by 2^n, we obtain a quadruple in E^2 which is congruent to the quadruple $\{q, r, s, t\}$. This completes the proof.

The question arises as to whether the Local Euclidean Weak, Feeble, or Queasy Four-point Property at a single point is strong enough by itself to effect a characterization. The following example due to Freese [6] shows the answer is negative.

EXAMPLE 3.2. The point set involved in the example by Freese is the union of the closed Euclidean half-plane P and the interior C of a Euclidean half-circle (as a model of the hyperbolic half-plane) including the open diameter of C, with the diameter of C and the boundary of P identified under an isometry Γ. The distance between pairs of points is Euclidean or hyperbolic if both points lie in the Euclidean or hyperbolic portion, respectively. The distance between a pair of points, one of which is in the Euclidean portion and one of which is in the hyperbolic portion is defined as follows. If p is in the hyperbolic part of the space and q is in the Euclidean part, the segment joining p and q is the union of the hyperbolic segment $[p, x]$ and the Euclidean segment $[\Gamma(x), q]$ where x is the unique point of the diameter of C such that the angles that $[p, x]$ and $[\Gamma(x), q]$ make with the diameter of C and the boundary of P, respectively, are supplementary. The distance pq is now defined as $px+\Gamma(x)q$. Clearly each point in the interior of the Euclidean half-plane has a neighborhood which has the Local Euclidean Feeble Four-point Property, yet the space is not an inner-product space.

In spite of the above example, the Euclidean Feeble (and "Weak" and "Queasy") Four-point Property at a point together with the local version of this property at the same point do characterize inner-product spaces among the class of complete, convex, externally convex metric spaces.

THEOREM 3.3. *If M has the Euclidean Feeble Four-point Property at p and if M has the Local Euclidean Feeble Four-point Property at p, then M is an inner-product space.*

Proof. By hypothesis M has the Local Euclidean Feeble Four-point Property at p, so by Theorem 3.2 it suffices to show M satisfies the Young Postulate at p. Let q and r be points of M, and let q' and r' be the respective midpoints of p and q and p and r. By the Euclidean Feeble Four-point Property at p, the quadruples $\{p, q', q, r\}$ and $\{p, q', r, r'\}$ are congruent with quadruples of the Euclidean plane. Thus the Euclidean cosine law for the cosine of the angle with vertex p and sides the segments $S(p, q)$ and $S(p, r)$ yields the same result when evaluated with the triples $\{p, q', r'\}$ and $\{p, q, r'\}$. Thus,

$$[(pq^2/4)+(pr^2/4)-q'r'^2]/[pq\cdot pr/2)] = [pq^2+(pr^2/4)-qr'^2]/(pq\cdot pr)\,.$$

Solving the above equation for $q'r'^2$, we obtain

$$q'r'^2 = (-qp^2/4)+(pr^2/8)+(qr^2/2)\,. \tag{2}$$

Similarly using the quadruple $\{p, q, r, r'\}$, applying the law of cosines as above for the angle with vertex r and sides $S(q, r\}$ and $S(p, r)$, and solving the resulting equation for $qr^2/4$, we have

$$qr^2/4 = (-qp^2/4)+(pr^2/8)+qr'^2/2)\,. \tag{3}$$

It follows from (2) and (3) that $q'r' = qr/2$, and the result follows.

It would be interesting to know if Theorem 3.3 remains true when the hypothesis that M has the Local Euclidean Feeble Four-point Property at p is removed.

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