An analogue of the theorem of Hake-Alexandroff-Looman

by

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Abstract. A characterization for monotone increasing functions is given. Based on it an abstract integral of Perron type and one of Denjoy type are defined and the two are proved to be equivalent. Concrete examples are also briefly given.

1. Monotone functions. To give a characterization for monotone increasing functions, we first fix certain terminologies and notations.

A function defined on a (finite) closed interval I is said to be *upper closed monotone* (or simply uCM) *on* I if the function is monotone increasing on the closed interval $[c, d]$ whenever it is so on the open interval $(c, d)\subseteq I$. Clearly, every Darboux function on I is uCM on I. But the converse does not hold.

A function F on I is said to be *lower absolutely continuous* (or simply lAC) *on a set* $E\subseteq I$ if for each $\varepsilon>0$, there exists $\delta>0$ such that

$$\sum [F(b_i)-F(a_i)]>-\varepsilon$$

for each finite set $\{[a_i, b_i]\}$ of non-overlapping intervals with end points in E and $\sum (b_i-a_i)<\delta$. A function F is said to be *generalized lower absolutely continuous* (or simply [lACG]) *on a set* E if E is a union of countably many *closed* sets on each of which F is lAC.

The notion of lAC dates back to Ridder [10] (cf. also Jeffrey [6]), which is a generalization of the notion of absolute continuity in the wide sense (i.e. AC given in Saks [11]). Note that the Cantor singular function is a monotone function which is lAC but not AC on the unit interval.

For convenience, if P is a well-defined property for functions defined on a certain domain, we will also use P to denote the class of all functions having the property P. Unless otherwise stated, the domain of functions will be a fixed finite closed interval $I\equiv[a, b]$, and will be left unspecified. Thus, uCM is also used to denote the class of all functions which are uCM on I, and similar for [lACG]. Furthermore, we also write

$$\mathrm{lCM} = \{F:\ -F\in \mathrm{uCM}\},$$

$$\mathrm{CM} = \mathrm{lCM}\cap \mathrm{uCM},$$

$$[\mathrm{uACG}] = [F:\ -F\in [\mathrm{lACG}]\},$$

$$[\mathrm{ACG}] = [\mathrm{lACG}]\cap[\mathrm{uACG}].$$

Note that the notion of [ACG] generalizes that of ACG in Saks [11] so that the functions concerned are not necessarily continuous (cf. Ellis [5]).

The following result gives a characterization for monotone increasing functions. The proof is a standard argument using Bair's category theorem and Vitali's covering theorem, and hence is only given in a brief manner.

THEOREM 1. *Let F be a function defined on $I \equiv [a, b]$. For the function F to be monotone increasing on I it is necessary and sufficient that $F \in \mathrm{uCM} \cap [\mathrm{lACG}]$, and $\overline{F}(x) \geqslant 0$ for almost all x in I, where $\overline{F}(x)$ is the upper derivate of F at x.*

As an immediate consequence of Theorem 1, one has the following result.

COROLLARY. *For the function F to be a constant function on I, it is necessary and sufficient that $F \in \mathrm{CM} \cap [\mathrm{ACG}]$, and $F'_{ap}(x) = 0$ for almost all x in I, where $F'_{ap}(x)$ denotes the approximate derivative of F at x.*

Proof of Theorem 1. That the condition is necessary is trivial. We prove that the condition is sufficient. Let E be the set of all points $x \in I$ such that there is *no* open interval I_x containing x with the property that F is monotone increasing on $I_x \cap I$. If one shows that E is empty, then applying the Heine-Borel theorem, one concludes easily that F is monotone increasing on I, and the proof will be done. To see that E is empty, suppose, on the contrary, that E is non-empty. Then one sees easily that E is closed in I, and since F is [lACG] on I, one concludes from Bair's category theorem that there exists an interval $[c, d] \subseteq I$ such that F is lAC on $[c, d] \cap E$ and $(c, d) \cap E$ is non-empty. Using the condition $F \in \mathrm{uCM}$, one sees that F is monotone increasing in the *closure* of each contiguous interval of $[c, d] \cap E$ relative to $[c, d]$. Then it follows easily that F, being lAC on $[c, d] \cap E$, is in fact lAC on the whole interval $[c, d]$. Then since $\overline{F}(x) \geqslant 0$ for almost all x in I, applying Vitali's covering theorem, one proves easily that F is monotone increasing on $[c, d]$. This contradicts the assumption that $(c, d) \cap E$ is non-empty, and hence the proof is completed.

2. Perron and Denjoy integrals. A class of functions, say $\mathscr{F}$, is termed an *upper semi-linear space* if $\mathscr{F}$ is closed under linear combinations with non-negative coefficients, i.e. if $\alpha f + \beta g \in \mathscr{F}$ whenever $f, g \in \mathscr{F}$ and α, β are non-negative constants. It is noted that the intersection of finitely many such spaces is also such a space.

The class [lACG] given in the previous section is an example of upper semi-linear spaces. But the class uCM is *not* such a space. Examples for upper semi-linear spaces contained in uCM will be given in the next section. Here, we will consider an abstract upper semi-linear space contained in uCM.

Throughout the rest of the section, let uL denote an upper semi-linear space contained in uCM, and f a function defined on $I \equiv [a, b]$.

A function M is said to be a LPG-*major function for f on I* if

(i) $M(a) = 0$;

(ii) $M \in \mathrm{uL}$ on I;

(iii) $\mathrm{lD_{ap}} M(x) \geqslant f(x)$ for almost all x in I;

(iv) $M \in [\mathrm{lACG}]$ on I.

Here, the notation $\mathrm{lD_{ap}} M(x)$ denotes the lower approximate derivate of M at x. (Later on, we will also use $D_{ap}M(x)$ to denote the approximate derivative of M at x).

A function m is said to be a LPG-*minor function for f on I* if $-m$ is a LPG-major function for $-f$ on I. It then follows from Theorem 1 that $M-m$ is monotone increasing and non-negative on I for any LPG-major function M and minor function m. Then by a standard procedure (cf. [3]), an integral of Perron type is defined, which we denote as LPG-integral, and call it the *generalized Perron integral using* uL. As usual, it can be proved easily that an indefinite LPG-integral is in $\mathrm{L} \equiv \{F\colon F \text{ and } -F \in \mathrm{uL}\}$ and its approximate derivative exists and is equal to its integrand almost everywhere provided that uL is closed under uniform convergences (i.e. $M_n \to M$ uniformly in I with $M_n \in \mathrm{uL} \Rightarrow M \in \mathrm{uL}$). Furthermore, we have the following non-trivial result.

LEMMA. *Every indefinite* LPG-*integral is* [ACG].

Proof. Let F be an indefinite LPG-integral for f on $[a, b]$ and without loss of generality let $F(a) = 0$. Let M, m, with or without indices, denote respectively, the LPG-major and minor functions for f on $[a, b]$. Then it follows from Theorem 1 that one has

(1) $M-m$ is monotone increasing and non-negative on $[a, b]$ for every pair M, m.

Then, routinely one shows that

(2) there exist sequences $\{M_n\}$, $\{m_n\}$ such that $M_n \to F$ and $m_n \to F$ uniformly on $[a, b]$, and

(3) $M-F$ and $F-m$ are both monotone increasing and non-negative on $[a, b]$ for every pair M, m.

Using (2) and (3), one proves easily that F is [ACG] on $[a, b]$ *provided that* the following holds.

(4) There exists a sequence $\{E_i\}$ of closed sets covering $[a, b]$ and over each of which M and $-m$ are lAC for *all* pairs M, m.

To show that (4) holds, let M_0, m_0 be a fixed pair. Then it is clear that there exists a sequence $\{E_i\}$ of closed sets covering $[a, b]$ and on each E_i both M_0 and $-m_0$ are lAC. Then using (1), one proves that both M_0 and m_0 are BV (i.e. bounded variation) on each E_i. Furthermore, since $M = (M-m_0)+m_0$ and $m = M_0-(M_0-m)$, it follows from (1) again that every M and every m are BV on E_i. Using this fact, we show that every M (and similarly every $-m$) is lAC on each E_i as follows: Let $M_*(x) = M(x)$ for x in E_i and on each interval (c, d) contiguous to E_i, M_* be defined to have its graph the linear segment joining the points $(c, M(c))$ and $(d, M(d))$. Then since E_i is closed and since M is a LPG-major function and M is BV on E_i, one shows that M_* is uCM, [lACG] and BV on $[a, b]$. Consider the function G defined by

$$G(x) = M_*(x) - H(x), \quad H(x) = \int_a^x M_*'(t)\,dt.$$

Note that since H is continuous and AC on $[a, b]$, the function G is uCM, [lACG] on $[a, b]$, and furthermore $G'(x) = 0$ for almost all x in $[a, b]$. Hence by Theorem 1, G is monotone increasing in $[a, b]$. Then

$$M_*(y) - M_*(x) \geqslant H(y) - H(x)$$

for all $[x, y] \subseteq [a, b]$. It then follows that M_* is lAC on $[a, b]$ since H is AC on $[a, b]$. Hence M, being identical with M_* on E_i, is lAC on E_i, completing the proof.

For an upper semi-linear space uL contained in uCM, note that

$$\mathrm{L} \equiv \{F\colon F \text{ and } -F \in \mathrm{uL}\}$$

is a linear space. Hence it follows from the corollary to Theorem 1 that if $F, G \in \mathrm{L} \cap$ [ACG] and $\mathrm{D_{ap}}F(x) = \mathrm{D_{ap}}G(x)$ for almost all x in I, then the difference function $F-G$ is a constant on I. Therefore the following definition of LDG-integral, which may be called as the *generalized Denjoy integral using* L, is well-defined. A function f is said to be LDG-*integrable on* $I = [a, b]$ if there exists a function $F \in \mathrm{L} \cap$ [ACG] such that $D_{\mathrm{ap}}F(x) = f(x)$ for almost all x in I, and in this case, the LDG-*integral of* f *over* I is defined to be $F(b) - F(a)$, and the function F is called an *indefinite* LDG-*integral of* f *on* I.

Now, we are in a position to give a result which is an analogue of the theorem of Hake–Alexandroff–Looman, which asserts that the Denjoy integral in the restricted sense is equivalent to the classical Perron integral (see Saks [11]).

THEOREM 2. *Let* uL *be an upper semi-linear space contained in the class* uCM, *and be closed under uniform convergences. Then the generalized Denjoy integral using* L (*i.e. the* LDG-*integral*) *is equivalent to the generalized Perron integral using* uL (*i.e. the* LPG-*integral*).

Proof. Clearly, if f is LDG-integrable on $I \equiv [a, b]$ and if F is an indefinite LDG-integral of f on I, then $F - F(a)$ serves as both a LPG-major and minor function for f on I and hence f is LPG-integrable with F as an indefinite integral. Conversely, if f is LPG-integrable on I with F as an indefinite integral, then by the remark preceding the lemma and the lemma itself, one concludes that f is LDG-integrable.

3. Examples and remarks. For functions defined on a fixed finite closed interval I, let us denote

$$\mathrm{C}_0 = \{F\colon F \text{ is continuous (in } I)\},$$
$$\mathrm{AC}_0 = \{F\colon F \text{ is approximately continuous}\},$$

$$\mathrm{C}_n = \{F\colon F \text{ is an exact } n\text{th Peano derivative}\},$$
$$\mathrm{AC}_n = \{F\colon F \text{ is an exact } n\text{th approximate Peano derivative}\},$$

where $n = 1, 2, 3, \ldots$ Then it is trivial that each of these classes is a linear space (and hence an upper semi-linear space). Furthermore, each is contained in the class uCM since it is known that every function in any of the above classes has the Darboux property (cf. [15] for approximately continuous functions, [9] for Peano derivatives and [1] for approximate Peano derivatives). Hence, taking the upper semi-linear space uL considered in the previous section to be C_n, AC_n, respectively, and noting that in this situation, uL = L, one obtains C_nPG-, C_nDG-, AC_nPG- and AC_nDG-integrals for $n = 0, 1, 2, \ldots$

The C_0DG-integral is just the Denjoy integral in the wide sense, or the so-called Denjoy–Kintchine integral (see Saks [11]). Similar to the Hake–Alexandroff–Looman Theorem, Theorem 2 asserts that the C_0DG-integral is equivalent to the C_0PG-integral of Perron type. Note that Ridder has also defined an integral of Perron type which is equivalent to the Denjoy–Kintchine integral. But his definition is not the same as that given here for the C_0PG-integral (cf. Jeffery [6]).

Ellis in [5] has defined inductively a GM_n-integral of Denjoy type for $n = 1, 2, 3, \ldots$ His definition for the GM_n-integral is based on the concept of M_n-continuity defined by using GM_{n-1}-integral. Note that it can be proved (cf. Sargent [11]) that a function is M_n-continuous in a *whole* closed interval if and only if it is C_n-continuous (in Burkill's sense [4]) in the interval. Hence a function M_n-continuous in a whole closed interval is just an exact nth Peano derivative (cf. [8], [2]). It then follows that Ellis' GM_n-integral is equivalent to the C_nDG-integral defined here, and hence is equivalent to the C_nPG-integral by Theorem 2, noting that the class C_n can be proved to be closed under uniform convergences.

Kubota [7] has defined an integral of Perron type and proved that his integral is equivalent to the GM_1-integral defined by Ellis. Note, however, that the nontrivial part, similar to the establishment of (4) in the proof of the lemma in the previous section, has been neglected in his proof. A similar neglection has appeared in the interesting paper [13] for the equivalence of the C_nP-integral of Burkill and the C_nD-integral of Sargent (see [14] for a complete proof). The proof of the lemma in the previous section is a slight modification of that given in [6], where the equivalence of Denjoy–Kintchine integral and Ridder's generalized Perron integral was given in detail.

It is clear that the AC_nPG-integral is an "approximate" extension of the C_nPG- (or GM_n- or C_nDG-) integral, and hence is also such an extension of Burkill's C_nP-integral. However, the AC_nPG- and AC_{n+1}PG-integral are incompatible. Examples for this can be obtained by a slight modification of examples given in [8], where another "approximate" extension of the C_nP-integral was considered.

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Metric criteria for Banach and Euclidean spaces

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Abstract. Let M be a complete, convex, externally convex, metric space. It is known that M is a Banach space if and only if M satisfies the Quadrilateral Midpoint Postulate. In this paper Banach spaces with unique metric lines are characterized by somewhat weaker four-point properties. These weaker versions are analogues of the Weak, Feeble, and Queasy Euclidean Four-point Properties used by Blumenthal and Day to characterize Euclidean spaces. In addition localized versions of the Euclidean and Banach four-point properties are used to characterize inner-product spaces.

1. Introduction. Andalafte and Blumenthal [1] characterized the class of real Banach spaces among the class of complete, convex, externally convex metric spaces which have the two-triple property by means of a specified relation between the segment joining the two midpoints of two sides of a triangle and the base of that triangle. More specificially Andalafte and Blumenthal said that a metric space satisfies the *Young Postulate* provided for each three of its points p, q, and r, if q' and r' are the respective midpoints of p and q and p and r, then $q'r' = qr/2$. They showed that a complete, convex, externally convex metric space with the two-triple property is a real Banach space if and only if it satisfies the Young Postulate. Valentine and Wayment [8] said that a metric space satisfies the *Quadrilateral Midpoint Postulate* provided for each four points p, q, r, and s of the space, no three of which are collinear, if m_1, m_2, m_3, and m_4 are the midpoints of p and q, q and r, r and s, and s and p, respectively, then $m_1m_2 = m_3m_4$ and $m_2m_3 = m_1m_4$. They showed a complete, convex, externally convex metric space with the two-triple property satisfies the Quadrilateral Midpoint Postulate if and only if it satisfies the Young Postulate, and thus obtained a characterization of real Banach spaces.

The Quadrilateral Midpoint Postulate is analogous to the characterization of Hilbert space by the parallelogram law [7]. It is also analogous to the *Euclidean Four-point Property* which assumes each quadruple of points of a space is congruent to a quadruple of points of Euclidean space. The Euclidean Four-point Property was introduced by Wilson [9] who showed that a complete, convex, externally convex metric space has the Euclidean Four-point Property if and only if is congruent to a Euclidean space (possibly infinite dimensional). Since the Quadrilateral Midpoint Postulate effects a characterization of Banach spaces (see Theorem 2.1

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