On shape of hyperspaces

Dedicated to professor Kiiti Morita for his 60th birthday

by

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Abstract. For a compact space X denote by 2^X the hyperspace consisting of all non empty closed subsets of X and by $C(X)$ the hyperspace consisting of all non empty connected closed subsets of X with finite topology. Then it is proved that $\mathrm{Sh}(2^X) = \mathrm{Sh}(2^{\square X})$ and $\mathrm{Sh}(C(X)) = \mathrm{Sh}(\square X)$, where $\square X$ is the decomposition space of X consisting of all components. As a consequence, if X is connected then $\mathrm{Sh}(2^X)$ and $\mathrm{Sh}(C(X))$ are trivial. Also for any compact spaces X and Y such that both $\square X$ and $\square Y$ are countably infinite, we have $\mathrm{Sh}(2^X) = \mathrm{Sh}(2^Y)$. If $X(n)$ denotes the nth symmetric product of X, then it is proved that $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ means $\mathrm{Sh}(X(n)) \geqslant \mathrm{Sh}(Y(n))$. Hence if X is an ASR(ANSP) so is $X(n)$.

§ 1. Introduction. Let X be a compact Hausdorff space. We denote by 2^X the hyperspace with finite topology consisting of all non empty closed subsets of X and by $C(X)$ the hyperspace with finite topology consisting of all non empty closed connected subsets of X. Let $\square X$ be the decomposition space of X consisting of all components. In this paper we shall prove that $\mathrm{Sh}(2^X) = \mathrm{Sh}(2^{\square X})$ and $\mathrm{Sh}(C(X)) = \mathrm{Sh}(\square X)$. Here by $\mathrm{Sh}(X)$ is meant the shape of X (cf. Borsuk [2], Mardešić and Segal [8, 9] and Mardešić [11]). As a consequence the following corollaries are obtained.

(1) If X is connected, then $\mathrm{Sh}(2^X)$ and $\mathrm{Sh}(C(X))$ are trivial.

(2) If $\square X$ and $\square Y$ are metrizable and infinite, then $\mathrm{Sh}(2^X) \equiv \mathrm{Sh}(2^Y)$. (Here we mean by $\mathrm{Sh}(A) \equiv \mathrm{Sh}(B)$ that both the relations $\mathrm{Sh}(A) \leqslant \mathrm{Sh}(B)$ and $\mathrm{Sh}(A) \geqslant \mathrm{Sh}(B)$ hold.)

(3) If both $\square X$ and $\square Y$ are countably infinite, then $\mathrm{Sh}(2^X) = \mathrm{Sh}(2^Y)$.

For a positive integer n, let $X(n)$ be the nth symmetric product of X. We shall show that if $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ then $\mathrm{Sh}(X(n)) = \mathrm{Sh}(Y(n))$.

Throughout this paper all of spaces are Hausdorff and maps are continuous. By an AR-space and an ANR-space we mean always those for metric spaces.

§ 2. Hyperspaces of the inverse limit space. Let X be a space. We denote by 2^X the set of all nonempty closed subsets of X, by $C(X)$ the set of all non empty closed connected subsets of X and by $X(n)$, n a positive integer, the set of all non empty subsets consisting of at most n points. We consider $C(X)$ and $X(n)$ as subsets of 2^X.

Let $\{U_j: j = 1, \ldots, k\}$ be a finite collection of open sets of X. Denote by $\langle U_1, \ldots, U_k\rangle$ the set $\{F \in 2^X: F \subset \bigcup_{i=1}^{k} U_i$ and $F \cap U_j \neq \emptyset$ for each $j\}$. The *finite topology* of 2^X is the one generated by collections of the form $\langle U_1, \ldots, U_k\rangle$ with $U_1, \ldots, U_k$ open sets of X. (See Michael [13, Def. 1.7].) Throughout this paper we assume that 2^X has the finite topology and $C(X)$ and $Y(n)$ are subspaces of 2^X.

Let X and Y be compact spaces and let $f: X \to Y$ be a continuous map. Define $f_*: 2^X \to 2^Y$ by $f_*(F) = f(F)$ for $F \in 2^X$. Then by [13, 5.10] f_* is continuous and $f_*(C(X)) \subset C(Y)$ and $f_*(X(n)) \subset Y(n)$. We say that f_* is a map *induced* by f.

Lemma 1. *Let $f, g: X \to Y$ be maps of a compact space X into a space Y. If $f \simeq g$ then $f_* \simeq g_*$, $f_*|C(X) \simeq g_*|C(X)$ in $C(Y)$ and $f_*|X(n) \simeq g_*|X(n)$ in $Y(n)$.*

Proof. Let $H: X \times I \to Y$ be a homotopy connecting f and g. Define $H': 2^X \times I \to 2^Y$ by $H'(F, t) = H(F \times \{t\})$ for $F \in 2^X$ and $t \in I$. It is easy to show that H' is continuous, $H'(C(X) \times I) \subset C(Y)$ and $H'(X(n) \times I) \subset Y(n)$. Since $H'(F, 0) = f_*(F)$ and $H'(F, 1) = g_*(F)$ for $F \in 2^X$, the lemma is proved.

Let $\{X_\alpha, \pi_\alpha^\beta, \Omega\}$ be an inverse system consisting of compact spaces X_α and projections $\pi_\alpha^\beta: X_\beta \to X_\alpha$, $\alpha < \beta$, $\alpha, \beta \in \Omega$, where Ω is a directed set. Then $\{2^{X_\alpha}, \pi_{\alpha*}^\beta\}$, $\{C(X_\alpha), \pi_{\alpha*}^\beta|C(X_\beta)\}$ and $\{X_\alpha(n), \pi_{\alpha*}^\beta|X_\beta(n)\}$ form inverse systems over Ω, where $\pi_{\alpha*}^\beta: 2^{X_\beta} \to 2^{X_\alpha}$ is induced by π_α^β. The following lemma was essentially proved by Segal [19].

Lemma 2. *Let $X = \varprojlim X_\alpha$. Then $2^X = \varprojlim 2^{X_\alpha}$, $C(X) = \varprojlim C(X_\alpha)$ and $X(n) = \varprojlim X_\alpha(n)$.*

Proof. Let $\pi_\alpha: X \to X_\alpha$, $\alpha \in \Omega$, be the projection. Consider the maps $\pi_{\alpha*}: 2^X \to 2^{X_\alpha}$, $\alpha \in \Omega$. Since $\pi_\alpha^\beta \pi_\beta = \pi_\alpha$ for $\alpha < \beta$, $\pi_{\alpha*}^\beta \pi_{\beta*} = \pi_{\alpha*}$ and hence the collection of maps $\{\pi_{\alpha*}, \alpha \in \Omega\}$ defines uniquely a continuous map $\pi_*: 2^X \to \varprojlim 2^{X_\alpha}$. Obviously $\pi_*(C(X)) \subset \varprojlim C(X_\alpha)$ and $\pi_*(X(n)) \subset \varprojlim X_\alpha(n)$. Let $x = \{F_\alpha: F_\alpha \subset X_\alpha, \alpha \in \Omega\}$ be a point of $\varprojlim 2^{X_\alpha}$. Then $\pi_{\alpha*}^\beta(F_\beta) = F_\alpha$ so that $\pi_\alpha^\beta(F_\beta) = F_\alpha$ for each $\beta > \alpha$. Since $\{F_\alpha, \pi_\alpha^\beta|F_\beta\}$ forms an inverse system of compact sets with onto bonding maps, $F_x = \varprojlim F_\alpha \in 2^X$ and $\pi_\alpha(F_x) = F_\alpha$ for each $\alpha \in \Omega$. If $x \in \varprojlim C(X_\alpha)$ (resp. $x \in \varprojlim X_\alpha(n)$) then $F_x \in C(X)$ (resp. $F_x \in X(n)$). Obviously $\pi_*(F_x) = x$. Thus π_* is onto. Similary it is proved π_* is one-to-one. The lemma is obtained by the compactness of 2^X, $C(X)$ and $X(n)$.

Lemma 3. *Let $f: X \to Y$ be a map from a compact space X into a space Y. Let $A = \{y_j; j = 1, \ldots, k\}$ be a finite set of Y. Then $f_*^{-1}(A) = \prod_{j=1}^{k} 2^{f^{-1}(y_j)}$.*

The lemma follows from the definition of the map f_* induced by f.

The following lemmas were obtained by Wojdysławski [20, Théorème II and II_m] and Ganea [3, Korollar].

Lemma 4 (Wojdysławski). *If X is a compact connected ANR-space then 2^X and $C(X)$ are compact AR-spaces.*

Lemma 5 (Ganea). *If X is a finite dimensional compact ANR, then $X(n)$ is a compact ANR-space.*

§ 3. **Shape of hyperspaces.** For a space X, we mean by $\mathrm{Sh}(X)$ the shape of X defined by Mardešić [11]. By Mardešić [11, Th. 6.8 and § 7] this shape is equal to one defined by Borsuk [2] if spaces are compact metric and one defined by Mardešić and Segal [8] if spaces are compact. For a space X, $\Box X$ denotes the decomposition space defined by the decomposition consisting of all components of X. If $f: X \to Y$ is a map, then a map $\Box f: \Box X \to \Box Y$ satisfying $\pi_Y \cdot f = \Box f \cdot \pi_X$ is uniquely defined, where π_X is the decomposition (quotient) map from X onto $\Box X$.

Theorem 1. *Let X be compact and let $\pi_X: X \to \Box X$ be the decomposition map. Then each of maps $\pi_{X*}: 2^X \to 2^{\Box X}$ and $\pi_{X*}: C(X) \to C(\Box X) = \Box X$ induces a shape equivalence. In particular, $\mathrm{Sh}(2^X) = \mathrm{Sh}(2^{\Box X})$ and $\mathrm{Sh}(C(X)) = \mathrm{Sh}(\Box X)$.*

We need the following lemma.

Lemma 6. *If X is a compact metric ANR, then the map $(\pi_X)_*: 2^X \to 2^{\Box X}$ is a homotopy equivalence.*

Proof. Let $y = \{y^1, y^2, \ldots, y^k\} \in 2^{\Box X}$. Then by Lemmas 3 and 4, $(\pi_X)_*^{-1}(y) = \prod_{i=1}^{k} 2^{\pi_X^{-1}(y^i)}$ is a compact metric AR-space. It is easy to see that for any different $y, y' \in 2^{\Box X}$, $(\pi_X)_*^{-1}(y)$ and $(\pi_X)_*^{-1}(y')$ are disjoint and $2^{\Box X}$ is finite. Thus $(\pi_X)_*$ is a homotopy equivalence.

Proof of Theorem 1. We shall prove the first part of Theorem 1 (the proof of the second part is similar, only simpler). Let $\underline{X} = \{X_\alpha, \pi_\alpha^\beta, \Omega\}$ be an ANR-system associated with a compact space X. Then it is easy to prove that

$$\Box X = \varprojlim \{\Box X_\alpha, \Box \pi_\alpha^\beta, \Omega\}.$$

By Lemma 2 we have $2^X = \varprojlim \{2^{X_\alpha}, (\pi_\alpha^\beta)_*, \Omega\}$ and $2^{\Box X} = \varprojlim \{2^{\Box X}, (\Box \pi_\alpha^\beta)_*, \Omega\}$. For any $\alpha, \beta \in \Omega$, $\alpha \leqslant \beta$, the following diagram commutes

$$\begin{array}{ccc} 2^{X_\alpha} & \xleftarrow{(\pi_\alpha^\beta)_*} & 2^{X_\beta} \\ \downarrow{\scriptstyle (\pi_{X_\alpha})_*} & & \downarrow{\scriptstyle (\pi_{X_\beta})_*} \\ 2^{\Box X_\alpha} & \xleftarrow[(\Box \pi_\alpha^\beta)_*]{} & 2^{\Box X_\beta} \end{array}$$

because $\pi_{X_\alpha} \pi_\alpha^\beta = (\Box \pi_\alpha^\beta) \pi_{X_\beta}$. Thus the system of maps $\{(\pi_{X_\alpha})\}_{\alpha \in \Omega}$ is the map from the system $\{2^{X^\alpha}, (\pi_\alpha^\beta)_*, \Omega\}$ to the system $\{2^{\Box X_\alpha}, (\Box \pi_\alpha^\beta)_*, \Omega\}$, which is a homotopy equivalence in the sense of Mardešić (because every $(\pi_{X_\alpha})_*$ is a homotopy equivalence). It is easy to see that the map $(\pi_X)_*: 2^X \to 2^{\Box X}$ is the inverse limit of the map (of systems) $\{(\pi_{X_\alpha})_*\}_{\alpha \in \Omega}$. Thus $(\pi_X)_*$ is the shape equivalence.

Next we shall give alternative proof of Theorem 1. We start with the following lemma.

LEMMA 7. *Let X be a paracompact space. Suppose that there is a closed map f from X onto a space Y with $\dim Y = 0$ such that for each $y \in Y$ $f^{-1}(y)$ is of trivial shape. Let H be an* ANR-*space.*

(3.1) *If $g: X \to H$, then there is a map $g': Y \to H$ such that $g'f \simeq g$ and the homotopy class of g' is determined uniquely by the homotopy class of g.*

(3.2) *Let $g, h: X \to H$. Then $g \simeq h$ if and only if $\pi_H g = \pi_H h: X \to \square H$, where $\pi_H: H \to \square H$ is the decomposition map.*

Proof. Let $g: X \to H$. Take any point $y \in Y$. Since $f^{-1}(y)$ is connected, $g(f^{-1}(y))$ is connected. Let H_y be the component of H containing $g(f^{-1}(y))$. Then H_y is an ANR-space. Since $f^{-1}(y)$ is of trivial shape and X is paracompact, it is easy to show that there is an open neighborhood U_y of $f^{-1}(y)$ in X and a homotopy $h_y: U_y \times I \to H_y$ such that

(3.3) $h_y(x, 0) = g(x)$ and $h_y(x, 1) = p_y (=$ *a* point of $H_y)$ for each $x \in U_y$.

This is done by using a bridge map theorem (see for example [4, Theorem 5]). Put $V_y = Y - f(X - U_y)$, $y \in Y$. By the closedness of f Y is paracompact and $\{V_y: y \in Y\}$ forms an open cover of Y. Since $\dim Y = 0$, there is a locally finite open cover $\mathscr{W} = \{W_\alpha: \alpha \in \Omega\}$ such that order of $\mathscr{W} = 1$ and $\mathscr{W}$ refines $\{V_y: y \in Y\}$. For each $\alpha \in \Omega$, choose a point y_α of Y such that $W_\alpha \subset V_{y_\alpha}$. Define $g': Y \to H$ by $g'(y) = p_{y_\alpha}$ for $y \in W_\alpha$, $\alpha \in \Omega$ (cf. (3.3)). Since order of $\mathscr{W} = 1$, g' is continuous. Since $\{f^{-1}(W_\alpha); \alpha \in \Omega\}$ forms a locally finite open cover of X whose order $= 1$, and $g'f | f^{-1}(W_\alpha) \simeq g | f^{-1}(W_\alpha)$ for each $\alpha \in \Omega$ by (3.3) and the definition of g', we know $g'f \simeq g$. This completes the proof of the first part of (3.1). Next, let us prove (3.2). Since $\square H$ is a discrete space by the local connectedness of H, it follows that $g \simeq h$ implies $\pi_H g = \pi_H h$. Suppose that $\pi_H g = \pi_H h$. Let g' and h' be maps of Y into H constructed for g and h in the proof of (3.1) respectively. Let $\mathscr{W}_g$ and $\mathscr{W}_h$ be locally finite open covers of Y used for the constructions of g' and h'. Take a locally finite open refinement $\mathscr{W}$ of $\mathscr{W}_g \wedge \mathscr{W}_h$ such that order of $\mathscr{W} = 1$. From $\pi_H g = \pi_H h$ and the definition of g' and h', we know for each $W \in \mathscr{W}$ two points $g'(W)$ and $h'(W)$ belong to the same component of H. Since each component of H is arcwise connected, $g' \simeq h'$ and hence $g \simeq fg' \simeq fh' \simeq h$. This completes the proof of (3.2). The second half of (3.1) is a consequence of (3.2).

By Lemma 7 we obtain the following theorem. In case X and Y are metrizable and X is finite dimensional, it is a consequence of [6, Theorem 1]. Note that we do not assume the finite dimensionality of X in the theorem.

THEOREM 2. *Assume that X, Y and f satisfy the same hypothesis as in Lemma 7. Then the shaping $\tilde{f}: X \to Y$ induced by f* (cf. [11]) *is a shape equivalence. In particular* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

Proof. We have to construct a shaping $\varphi: Y \to X$ such that $\varphi\tilde{f} = \tilde{1}_X$ and $\tilde{f}\varphi = \tilde{1}_Y$, where $\tilde{1}_X$ and $\tilde{1}_Y$ are the shapings induced by the identities $1_X: X \to X$ and $1_Y: Y \to Y$. For a map g of X into an ANR-space K, define $\varphi(g) = g'$, where g' is a map of Y into K constructed for g in Lemma 7 (3.1). To show φ is a shaping, let L be an

ANR-space and let $\xi: K \to L$ and $h: X \to L$ be maps such that $\xi g \simeq h$. Since $\pi_L \xi\varphi(g) f = \pi_L \xi g = \pi_L h = \pi_L \varphi(h) f$, we have $\xi\varphi(g) f \simeq \varphi(h) f$ by (3.2). Hence, by the uniqueness of g' in (3.1), we know $\xi\varphi(g) \simeq \varphi(h)$. This implies that φ is a shaping. It is easy to prove by (3.1) and the definition of $\tilde{f}$ that $\varphi\tilde{f} = \tilde{1}_X$ and $\tilde{f}\varphi = \tilde{1}_Y$. This completes the proof.

EXAMPLE 1. Consider the following sets in the plane R^2: $A_0 = \{(0, 0)\}$,

$$A_i = \{(x, y): x \geqslant 0,\ (x-1)^2 + y^2 = (1 + 1/i)^2\},$$

$i = 1, 2, \ldots$, $X = \bigcup_{i=0}^{\infty} A_i$; $Y = \{(0, 0)\} \cup \{(0, 1/i);\ i = 1, 2, \ldots\}$. We define $f: X \to Y$ by $f(A_0) = (0, 0)$ and $f(A_i) = (0, 1/i)$, $i = 1, 2, \ldots$ Next, let

$$A_0' = \{(x, y): x \neq 2,\ (x-1)^2 + y^2 = 1\}$$

and put $X' = A_0' \cup \bigcup_{i=1}^{\infty} A_i$. Define $g: X' \to Y$ by $g(A_i) = (0, 1/i)$, $i > 0$ and $g(A_0') = (0, 0)$. Then f and g are continuous and open maps and for each $y \in Y$ $f^{-1}(y)$ and $g^{-1}(y)$ are a point or an open interval or a closed interval. However, since $\check{H}^1(X)$ and $\check{H}^1(X')$ are both infinite groups and $\check{H}^1(Y) = 0$, each of $\tilde{f}$ and $\tilde{g}$ is not a shape equivalence, where $\check{H}^*$ is the integral Čech cohomology. We know that X' is locally compact. These examples are shown that we can not replace the closedness of a map f in Theorem 2 by the condition (i) or (ii):

(i) f is open and for each $y \in Y$ $f^{-1}(y)$ is compact;

(ii) X is locally compact and f is open.

(Note that if two conditions (i) and (ii) are satisfied then f becomes a closed map.)

Alternative proof of Theorem 1. Since X is compact, $\square X$ is a compact space and $\dim \square X = 0$ by Ponomarev [16]. Hence $2^{\square X}$ is compact and $\dim 2^{\square X} = 0$. Therefore by Theorem 2, it is enough to prove that for each point y of $2^{\square X}$ or $C(\square X) = \square X$, $(\pi_X)_*^{-1}(y)$ is of trivial shape. However it is easily proved by Lemmas 2 and 6.

There are several corollaries of Theorem 1. The first concerns an absolute shape retract (ASR) and an absolute neighborhood shape retract (ANSR) (see [10] for the definitions.)

COROLLARY 1. *Let X be compact. Then*:

(3.4) 2^X *and* $C(X)$ *are* ASR (*equivalently of trivial shape* [10, Theorem 4]) *if and only if X is connected.*

(3.5) 2^X *and* $C(X)$ *are* ANSR *if and only if X has a finite number of components.*

Proof. The if part is an immediate consequence of Theorem 1 because $\square X$ is a singleton or a finite set. Next, let us prove the only if part of (3.5). Then by Mardešić [10, Corollary 2] there exists an ANR (compact) Y such that $\mathrm{Sh}(2^X) \leqslant \mathrm{Sh}(Y)$ or

$\mathrm{Sh}(C(X))\leqslant\mathrm{Sh}(Y)$. Since for compact spaces A and B $\mathrm{Sh}(A)\leqslant\mathrm{Sh}(B)$ implies $\mathrm{Sh}(\square A)\leqslant\mathrm{Sh}(\square B)$, we have $\mathrm{Sh}(\square 2^X)\leqslant\mathrm{Sh}(\square Y)$ or $\mathrm{Sh}(\square C(X))\leqslant\mathrm{Sh}(\square Y)$. Also it is easy to know that $\mathrm{Sh}(\square 2^X)=\mathrm{Sh}(2^{\square X})$ and $\mathrm{Sh}(\square C(X))=\mathrm{Sh}(C(\square X))$. By Theorem 1 we have $\mathrm{Sh}(2^{\square X})\leqslant\mathrm{Sh}(\square Y)$ or $\mathrm{Sh}(\square X)\leqslant\mathrm{Sh}(\square Y)$. Since Y is an ANR (compact), $\square Y$ is a finite set. Since $\dim 2^{\square X}=\dim\square X=0$, by Mardešić and Segal [8, Theorem 20] we can conclude $\square X$ is a finite set. The proof of (3.4) is similar.

COROLLARY 2. *For every compact space X, 2^X and $C(X)$ are movable.*

Proof. By Theorem 1 we know that each of 2^X and $C(X)$ has the same shape type as a 0-dimentional compact space. The corollary follows from Mardešić and Segal [7, Example 2].

COROLLARY 3. *Let X and Y be compact spaces. If $\mathrm{Sh}(X)\geqslant\mathrm{Sh}(Y)$ (resp. $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$), then $\mathrm{Sh}(2^X)\geqslant\mathrm{Sh}(2^Y)$ and $\mathrm{Sh}(C(X))\geqslant\mathrm{Sh}(C(Y))$ (resp. $\mathrm{Sh}(2^X)=\mathrm{Sh}(2^Y)$ and $\mathrm{Sh}(C(X))=\mathrm{Sh}(C(Y))$).*

Proof. By the proof of Lemma 6 and Mardešić and Segal [8, Theorem 20], we know that any shaping $\varphi\colon X\to Y$ determines uniquely a continuous map $f_\varphi\colon\square X\to\square Y$ such that $\tilde f_\varphi\tilde\pi_X=\tilde\pi_Y\varphi$, where $\tilde g$ denotes the shaping determined by a map g. If $\psi\colon Y\to X$ is a shaping such that $\psi\varphi=\tilde 1_X$, where 1_X is the identity of X, then the map $f_{\psi\varphi}=f_\psi f_\varphi\colon\square X\to\square X$ is the identity. Thus $f_{\psi\varphi *}=f_{\psi *}f_{\varphi *}\colon 2^{\square X}\to 2^{\square X}$ is the identity so that $\mathrm{Sh}(2^{\square X})\leqslant\mathrm{Sh}(2^{\square Y})$. The corollary follows from Theorem 1.

COROLLARY 4. *If $|\square X|=|\square Y|=\aleph_0$, then $\mathrm{Sh}(2^X)=\mathrm{Sh}(2^Y)$, where $|Z|$ denotes the cardinality of Z.*

Proof. Since $\square X$ and $\square Y$ are compact, it follows from Arhangel'skiĭ [1] that $\square X$ and $\square Y$ are metrizable. Since both $\square X$ and $\square Y$ have dense sets of isolated points, $2^{\square X}$ and $2^{\square Y}$ are homeomorhic by Pełczyński [15]. Thus $\mathrm{Sh}(2^X)=\mathrm{Sh}(2^{\square X})=\mathrm{Sh}(2^{\square Y})=\mathrm{Sh}(2^Y)$.

Denote by $\mathcal{M}$ the class of all compact spaces X such that $\square X$ is metrizable. We note that the hypothesis of Corollary 4 can be replaced by the following: $X, Y\in\mathcal{M}$ and both $\square X$ and $\square Y$ have countable infinite dense sets of isolated points (cf. Pełczyński [15]).

COROLLARY 5. *If $X, Y\in\mathcal{M}$, then $\mathrm{Sh}(2^X)\geqslant\mathrm{Sh}(2^Y)$ or $\mathrm{Sh}(2^X)\leqslant\mathrm{Sh}(2^Y)$ and also $\mathrm{Sh}(C(X))\geqslant\mathrm{Sh}(C(Y))$ or $\mathrm{Sh}(C(X))\leqslant\mathrm{Sh}(C(Y))$. Moreover, if both $\square X$ and $\square Y$ are infinite, then $\mathrm{Sh}(2^X)\equiv\mathrm{Sh}(2^Y)$, that is, $\mathrm{Sh}(2^X)\geqslant\mathrm{Sh}(2^Y)$ and $\mathrm{Sh}(2^X)\leqslant\mathrm{Sh}(2^Y)$.*

Proof. Since $X, Y\in\mathcal{M}$, $\square X$ and $\square Y$ are 0-dimensional compact metric spaces. If $|\square X|\geqslant\aleph_1$, then $\square X$ contains a Cantor discontinuum. Hence $\square Y$ is embedded into $\square X$ so that $\square Y$ is a retract of $\square X$ (see for example [5, Theorem 4]) and $2^{\square Y}$ is a retract of $2^{\square X}$. Thus $\mathrm{Sh}(\square X)\geqslant\mathrm{Sh}(\square Y)$ and $\mathrm{Sh}(2^{\square X})\geqslant\mathrm{Sh}(2^{\square Y})$. Therefore $\mathrm{Sh}(C(X))\geqslant\mathrm{Sh}(C(Y))$ and $\mathrm{Sh}(2^X)\geqslant\mathrm{Sh}(2^Y)$ by Theorem 1. If $|\square X|\leqslant\aleph_0$ and $|\square Y|\leqslant\aleph_0$, then $\square X$ and $\square Y$ are homeomorhic to ordered compacta by Mazurkiewicz and Sierpiński [12, Théorème, p. 21] and hence it holds that there is an embedding: $\square X\to\square Y$ or $\square Y\to\square X$. This completes the proof of the first part of the corollary. Next, let $\square X$ and $\square Y$ be infinite sets. By Pełczyński [15] both 2^X and 2^Y contain Cantor discontinua. Hence there are embeddings: $2^{\square X}\to 2^{\square Y}$ and $2^{\square Y}\to 2^{\square X}$ so that both the relations $\mathrm{Sh}(2^{\square X})\geqslant\mathrm{Sh}(2^{\square Y})$ and $\mathrm{Sh}(2^{\square X})\leqslant\mathrm{Sh}(2^{\square Y})$ hold. The corollary is a consequence of Theorem 1.

EXAMPLE 2. Let X be a Cantor discontinuum and let Y be a countably infinite compact set. Then $\mathrm{Sh}(2^X)\equiv\mathrm{Sh}(2^Y)$ by Corollary 5. However $\mathrm{Sh}(2^X)\neq\mathrm{Sh}(2^Y)$ because 2^X has no isolated points by Michael [13, 4.13.4] and on the other hand 2^Y has isolated points (every isolated point of Y is isolated in 2^Y). This example shows that we can not replace the relation $\mathrm{Sh}(2^X)\equiv\mathrm{Sh}(2^Y)$ in Corollary 5 by $\mathrm{Sh}(2^X)=\mathrm{Sh}(2^Y)$.

The following example shows that the hypothesis $X, Y\in\mathcal{M}$ in Corollary 5 is essential.

EXAMPLE 3. Let X' be a discrete space whose cardinality $|X'|=\aleph_1$ and let X be a one point compactification of X'. Next, let D be a set consisting of exactly two points and let $Y=\prod_{\alpha\in\Omega}D_\alpha$, where $|\Omega|=\aleph_1$ and each D_α is a copy of D. Then both X and Y are 0-dimensional compact spaces with an infinite number of points and hence $C(X)=X$ and $C(Y)=Y$. Since Y has no isolated points, there is no embedding of Y into X so that $\mathrm{Sh}(X)\ngeqslant\mathrm{Sh}(Y)$. Suppose that $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$. Then there is an embedding $i\colon X\to Y$ and a retraction $r\colon Y\to i(X)$ by Mardešić and Segal [8, Theorem 20]. Since Y has Souslin property (= the countable chain condition) by Šanin [18], $i(X)$ must have Souslin property. This contradiction means $\mathrm{Sh}(X)\nleqslant\mathrm{Sh}(Y)$. Finally, suppose $\mathrm{Sh}(2^X)\leqslant\mathrm{Sh}(2^Y)$. Then there is an embedding $i\colon 2^X\to 2^Y$ and a retraction $r\colon 2^Y\to i(2^X)$. Note that, by the definition of the finite topology, if Z is separable then 2^Z is separable. Since Y is separable by Ross and Stone [17], 2^Y is separable so that 2^X must be separable. However it is easy to see that each point of X' is isolated in 2^X and hence 2^X is not separable. This contradiction shows that $\mathrm{Sh}(2^X)\not\equiv\mathrm{Sh}(2^Y)$.

THEOREM 3. *Let n be a positive integer. If X and Y are compact, then $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$ (resp. $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$) implies $\mathrm{Sh}(X(n))\leqslant\mathrm{Sh}(Y(n))$ (resp. $\mathrm{Sh}(X(n))=\mathrm{Sh}(Y(n))$).*

Proof. Let $\underline{X}=\{X_\alpha,\pi_\alpha^\beta,\Omega\}$ and $\underline{Y}=\{Y_\gamma,\mu_\gamma^\delta,\Gamma\}$ be ANR-systems consisting of finite dimensional compact ANR's X_α and Y_γ associated with X and Y respectively. Suppose that $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$. There are maps $\underline{f}\colon\underline{X}\to\underline{Y}$ and $\underline{g}\colon\underline{Y}\to\underline{X}$ such that $\underline{g}\underline{f}\simeq 1_{\underline{X}}$. (See Mardešić and Segal [8] for notations.) Let $\underline{f}=\{f_\gamma,\Gamma\}$ and $\underline{g}=\{g_\alpha,\Omega\}$. For each $\alpha\in\Omega$ there is an index $\alpha'\in\Omega$ such that $\alpha'>fg(\alpha)$, α and

$$g_\alpha f_{g(\alpha)}\pi_{fg(\alpha)}^{\alpha'}\simeq\pi_\alpha^{\alpha'}\colon X_{\alpha'}\to X_\alpha. \tag{3.6}$$

Consider the systems $\underline{X}(n)=\{X_\alpha(n),\tilde\pi_\alpha^\beta,\Omega\}$ and $\underline{Y}(n)=\{Y_\gamma(n),\tilde\mu_\gamma^\delta,\Gamma\}$, where $\tilde\pi_\alpha^\beta=\pi_{\alpha *}^\beta|X_\beta(n)$ and $\tilde\mu_\gamma^\delta=\mu_{\gamma *}^\delta|Y_\delta(n)$. By Lemmas 5 and 2, $\underline{X}(n)$ and $\underline{Y}(n)$ are ANR-systems associated with $X(n)$ and $Y(n)$ respectively. For each $\gamma\in\Gamma$, define

$f_\gamma(n)\colon X_{f(\alpha)}(n)\to Y_\gamma(n)$ by $f_\gamma(n) = f_{\gamma*}|X_{f(\gamma)}(n)$ and similarly define $g_\alpha(n)\colon Y_{g(\alpha)}(n) \to X_\alpha(n)$, $\alpha \in \Omega$, by $g_\alpha(n) = g_{\alpha*}|Y_{g(\alpha)}(n)$. By (3.6) and Lemma 1 it holds that

$$g_\alpha(n) f_{g(\alpha)}(n) \tilde{\pi}^{\alpha'}_{fg(\alpha)} \simeq \tilde{\pi}^{\alpha'}_\alpha\colon X_{\alpha'}(n)\to X_\alpha(n). \tag{3.7}$$

Also Lemma 1 shows that $\underline{f}(n) = \{f_\gamma(n), \Gamma\}$ and $\underline{g}(n) = \{g_\alpha(n), \Omega\}$ are maps of $\underline{X}(n)$ into $\underline{Y}(n)$ and of $\underline{Y}(n)$ into $\underline{X}(n)$ respectively. Since (3.7) implies $\underline{g}(n)\underline{f}(n)\simeq \underline{1}_{\underline{X}}$, $\mathrm{Sh}(X(n))\leqslant \mathrm{Sh}(Y(n))$. That $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ implies $\mathrm{Sh}(X(n)) = \mathrm{Sh}(Y(n))$ is proved similarly.

COROLLARY 6. *For a positive integer n and a compact space X, the followings hold.*

(i) *If X is an* ASR, *then X(n) is an* ASR.

(ii) *If X is an* ANSR, *then X(n) is an* ANSR.

(iii) *If X is movable* [7], *then X(n) is movable.*

(iv) *If X is uniform movable* [14], *then X(n) is uniform movable.*

Proof. Suppose that X is an ANSR. By Mardešić [10, Theorem 6] there is a finite dimensional compact ANR-space Y such that $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$. From Theorem 3 it follows that $\mathrm{Sh}(X(n))\leqslant\mathrm{Sh}(Y(n))$. Since by Lemma 5 $Y(n)$ is a compact ANR-space, by applying Mardešić [10, Theorem 6] again we know $X(n)$ is an ANSR. The proof of (i) is similar. The assertions (iii) and (iv) are proved by the same argument as in the proof of Theorem 3.

References

[1] A. V. Arhangel'skiĭ, *An addition theorem for weight of sets lying in bicompacta*, Dokl. Akad. Nauk SSSR 126 (1959), pp. 239–241 (Russian).
[2] K. Borsuk, *Theory of shape*, Lecture Note Series 28, Math. Inst. Aarhus Univ. 1971.
[3] T. Ganea, *Symmetrische Potenz topologisher Räume*, Math. Nachr. 11 (1954), pp. 305–316.
[4] Y. Kodama, *Mappings of a fully normal space into an absolute neighborhood retract*, Sci. Rep. Tokyo Koiku Daigaku 5 (1955), pp. 37–47.
[5] — *On* LC^n *metric spaces*, Proc. Japan Acad. 33 (1957), pp. 79–83.
[6] — *Decomposition spaces and shape of Fox*, Fund. Math. 97 (1977), pp. 199–208.
[7] S. Mardešić and J. Segal, *Movable compacta and* ANR-*systems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), pp. 649–654.
[8] — — *Shapes of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.
[9] — — *Equivalence of Borsuk and the* ANR-*system approach to shapes*, Fund. Math. 72 (1971), pp. 61–68.
[10] — *Retracts in shape theory*, Glasnik Math. 6 (1971), pp. 153–163.
[11] — *Shapes for topological spaces*, Gen. Topology and Appl. 3 (1973), pp. 265–282.
[12] S. Mazurkiewicz and W. Sierpiński, *Contribution à la topologie des ensembles dénombrables*, Fund. Math. 1 (1920), pp. 17–27.
[13] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1951), pp. 152–181.
[14] M. Moszyńska, *Uniformly movable compact spaces and their algebraic properties*, Fund. Math. 77 (1972), pp. 125–144.
[15] A. Pełczyński, *A remark on spaces* 2^X *for zerodimensional* X, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965), pp. 85–89.

[16] V. I. Ponomarev, *On continuous decompositions of bicompacta*, Uspehi Math. Nauk (N. S.) 12 (1957), 4 (76), pp. 335–340; Amer. Math. Soc. Translation, Ser. 2, 30 (1963), pp. 235–240.
[17] K. A. Ross and A. H. Stone, *Products of separable spaces*, Amer. Math. Monthly 71 (1964), pp. 398–403.
[18] N. H. Šanin, *On intersection of open subsets in the product of topological spaces*, C. R. (Doklady) Acad. Sci. URSS (N. S.) 53 (1946), pp. 499–501 (Russian).
[19] J. Segal, *Hyperspaces of the inverse limit spaces*, Proc. Amer. Math. Soc. 10 (1959), pp. 706–709.
[20] M. Wojdysławski, *Rétractes absolus et hyperespaces des continus*, Fund. Math. 32 (1939), pp. 184–192.

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