Axiomatizability of second order arithmetic with ω-rule

by

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Abstract. We prove that the system of Analysis with ω-rule and without the axiom of choice is not finitely axiomatizable. In proof, we use an interpretation of constructible reals in the system under consideration.

Section 0. Let A denote the system of second order arithmetic formulated in two-sorted predicate logic with number and function quantifiers. System A contains the induction axiom and the comprehension schema but no axiom of choice. A_ω is A with the rule ω added. We prove below that A_ω is not finitely axiomatizable, which anwers a question of Schwabhäuser [4].

Section 1. For the proof, we construct in A an internal well-ordered model, namely the constructible reals. To do this, some form of the axiom of choice is required. Thus, we prove

LEMMA 1.1. *The Σ_2^1-choice is provable in A, that is,*

$$A \vdash (n)(\mathrm{E}x)F(n, x) \to (\mathrm{E}z)(n)F(n, z^{(n)}) \tag{C}$$

for arbitrary Σ_2^1-formula F of A.

Proof. The theorem of Kondô says that each complementary analytic set of irrationals admits a complementary analytic uniformization. Representing CA sets in A by Π_1^1-formulas (with parameters), we obtain the following schema:

For every Π_1^1-formula $H(x, y)$, there is a Π_1^1-formula $H^*(x, y)$ such that

$$A \vdash (x, y)[H^*(x, y) \to H(x, y)],$$

$$A \vdash (x)[(\mathrm{E}y)H(x, y) \to (\mathrm{E}!y)H^*(x, y)] \tag{K}$$

(H may contain number or function parameters. H^* contains the same parameters as H.)

Schema (K) immediately implies the choice schema (C) for Π_1^1-formulas F. Indeed, let $F(n, x)$ be Π_1^1 and assume the antecedent of (C). Let F^* be a uniformization of F, which exists in view of (K). Thus:

$$A \vdash F^*(n, x) \to F(n, x),$$
$$A \vdash (n)(\mathrm{E}x)F(n, x) \to (n)(\mathrm{E}!x)F^*(n, x).$$

By comprehension, there is a z such that

$$z(J(n, m)) = k \equiv (\mathrm{E}x)[F^*(n, x) \,\&\, x(m) = k]$$

(J is a standard pairing function). Thus

$$A \vdash (n)(\mathrm{E}x)F(n, x) \to (n)F^*(n, z^{(n)})$$

and hence

$$A \vdash (n)(\mathrm{E}x)F(n, x) \to (\mathrm{E}z)(n)F(n, z^{(n)}).$$

Finally, the Π^1_1-form of choice implies the Σ^1_2-form, for if $F(n, x)$ is Σ^1_2, then F is of the form $(\mathrm{E}y)F_1(n, x, y)$ with F_1, Π^1_1. Now, if $(n)(\mathrm{E}x)F(n, x)$, then $(n)(\mathrm{E}z)F_2(n, z)$, where $F_2(n, z)$ is

$$(x, y)[x = (z)_0 \,\&\, y = (z)_1 \to F_1(n, x, y)].$$

Applying the Π^1_1-choice, we get a z such that $(n)F_2(n, z^{(n)})$; thus $(n)F(n, (z^{(n)})_0, (z^{(n)})_1)$ and hence $(n)F(n, u^{(n)})$ follows by putting $u^{(n)} = (z^{(n)})_0$. It remains to prove schema (K), which will be done in the next section.

Section 2. In order to prove schema (K), it is sufficient to repeat the standard Lusin–Novikoff construction as given, say, in Schoenfield [5] with obvious modifications, such as e.g. replacing ordinal numbers by well-orderings, and to make sure that the corresponding arithmetical statements thus obtained are provable in A. This is done below.

We recall some notions of the theory of sieves. We may assume that the positive integers enumerate in a recursive way finite sequences of integers, or, which is the same, the binary fractions from the interval 0, 1. In this way we obtain a recursively definable ordering $<^*$ of positive integers of type η, isomorphic to the ordering $>$ of binary fractions. An arbitrary 0-1 function f such that $f(0) = 1$ is then a code for the ordering $<^*$ restricted to the set $\{n: f(n) = 0\}$. The Π^1_1-formula $\mathrm{Bord}(f)$ means that f is a well-ordering. For an arbitrary ordering f we define also orderings $f|n$ by:

$$f|0 = f \quad \text{and} \quad (f|n)(m) = 0 \equiv f(m) = 0 \,\&\, m <^* n$$

for n such that $f(n) = 0$. If $f(n) = 1$ and $n \geqslant 1$ then $f|n$ is constantly 1. (Thus $f|n$ is a code for the initial segment of f corresponding to n.) Let $H(x, y)$ be a Π^1_1-formula (we omit the parameters occurring in H, since if makes no difference for the proof). Thus, H can be written in an equivalent form:

$$H(x, y)\colon\ (f)(\mathrm{E}n)p(\overline{f(n)}, \overline{x(n)}, \overline{y(n)}),$$

with p recursive. We put

$$E_{x,y} = \{n>0\colon (j)[j<lh(n) \to \neg p(n \restriction j, \overline{x(j)}, \overline{y(j)})]\}.$$

Now define the ordering g in such a way that

(1) $$g(n) = 0 \equiv n \in E_{x,y}.$$

Since the above notions are definable, we infer that there is a formula $\Phi(x, y, g)$ (Φ is in fact recursive) such that:

$$A \vdash (x, y)(\mathrm{E}!g)\Phi(x, y, g),$$
(2) $$A \vdash H(x, y) \equiv (\mathrm{E}g)[\Phi(x, y, g) \,\&\, \mathrm{Bord}(g)].$$

We shall also use the relation $\preccurlyeq$ ($\prec$), which holds for orderings f, g if and only if f is embeddable into (a proper initial segment of) g. Both $\preccurlyeq$ and $\prec$ are Σ^1_1. Also $\simeq$ denotes the isomorphism of f, g.

We now construct a uniformizing formula H^*. For brevity, let $\Gamma_\Phi(x, y)$ denote the unique g such that $\Phi(x, y, g)$ (comp. (2)). The formula

$$\neg H(x, y) \vee \{H(x, y) \,\&\, (\mathrm{E}z, n)_{\geqslant 1}[(j)_n(z(j) = y(j) \,\&\, \Gamma_\Phi(x, z)|j \simeq \Gamma_\Phi(x, y)|j)$$
$$\&\, (z(n) < y(n)) \vee (z(n) = y(n) \,\&\, \Gamma_\Phi(x, z)|n \prec \Gamma_\Phi(x, y)|n)]\}$$

is equivalent to the Σ^1_1-formula

$$\neg H(x, y) \vee (\mathrm{E}z, n)_{\geqslant 1}\{(j)_n[z(j) = y(j) \,\&\, \Gamma_\Phi(x, z)|j \simeq \Gamma_\Phi(x, y)|j] \,\&$$
$$\&\, [(z(n) < y(n)) \vee (z(n) = y(n) \,\&\, \Gamma_\Phi(x, z)|n \prec \Gamma_\Phi(x, y)|n)]\},$$

which we denote by $\neg H^*$. Hence $H^*(x, y)$ is Π^1_1 and is equivalent to

$$H(x, y) \,\&\, (z, n)_{\geqslant 1}\{(j)_n[z(j) = y(j) \,\&\, \Gamma_\Phi(x, z)|j \simeq \Gamma_\Phi(x, y)|j]$$
$$\to [(z(n) \geqslant y(n)) \,\&\, (z(n) = y(n) \to \Gamma_\Phi(x, y)|n \preccurlyeq \Gamma_\Phi(x, z)|n)]\}.$$

(Note that the statement $\neg(f \prec g) \to (g \preccurlyeq f)$ is provable in A for well orderings f, g.) Thus, we have

$$A \vdash (x, y)[H^*(x, y) \to H(x, y)].$$

It remains to show that

$$A \vdash (x)[(\mathrm{E}y)H(x, y) \to (\mathrm{E}!y)H^*(x, y)].$$

To see this let $F(x, y, n)$ be

$$H(x, y) \,\&\, (z)\{(j)_n[z(j) = y(j) \,\&\, \Gamma_\Phi(x, z)|j \simeq \Gamma_\Phi(x, y)|j]$$
$$\to [(z(n) \geqslant y(n)) \,\&\, (z(n) = y(n) \to \Gamma_\Phi(x, y)|n \preccurlyeq \Gamma_\Phi(x, z)|n)]\}.$$

Thus, we have

$$A \vdash n \leqslant m \to [F(x, y, m) \to F(x, y, n)],$$
(3) $$A \vdash F(x, y, n) \,\&\, F(x, y', n) \to (j)_n[y(j) = y'(j) \,\&\, \Gamma_\Phi(x, y)|j \simeq \Gamma_\Phi(x, y')|j].$$

Note that if $W(f)$ is a formula such that

$$A \vdash (\mathrm{E}f)W(f),$$
$$A \vdash W(f) \rightarrow \mathrm{Bord}(f),$$

then also

$$A \vdash (\mathrm{E}g)\{W(g) \,\&\, (f)[W(f) \rightarrow g \preccurlyeq f]\},$$

i.e. it is provable in A that any nonempty class of well-orderings contains a shortest element. Using this, we infer at once

$$A \vdash (\mathrm{E}y)F(x, y, n) \rightarrow (\mathrm{E}y)F(x, y, n+1),$$

because, going from n to $n+1$, we first select the y's which take the smallest value at the point n and then from among such y's we choose those for which the well-ordering $\Gamma_\Phi(x, y)|n$ is the shortest possible. Since also $F(x, y, 0)$ is equivalent to $H(x, y)$, induction gives

$$A \vdash (\mathrm{E}y)H(x, y) \rightarrow (n)(\mathrm{E}y)F(x, y, n).$$

Now, by using (3) and comprehension

$$A \vdash (\mathrm{E}y)H(x, y) \rightarrow (\mathrm{E}z)(n, m)[z(n) = m \equiv (\mathrm{E}y)(F(x, y, n+1) \,\&\, y(n) = m)].$$

We show that

(4) $$A \vdash (\mathrm{E}y)H(x, y) \rightarrow (n)F(x, z, n).$$

First, we show that

(5) $$A \vdash \mathrm{Bord}(\Gamma_\Phi(x, z)).$$

Let $W(f, n)$ be

(6) $$(\mathrm{E}y)H(x, y) \rightarrow (\mathrm{E}y)[F(x, y, n+1) \,\&\, f \simeq \Gamma_\Phi(x, y)|n].$$

Thus, by (3):

$$A \vdash W(f, n) \,\&\, W(f', n) \rightarrow f \simeq f',$$

(7) $$A \vdash W(f, n) \rightarrow \mathrm{Bord}(f).$$

Now, suppose that $n <^* m$ and

$$\Gamma_\Phi(x, z)(n) = \Gamma_\Phi(x, z)(m) = 0.$$

If $k = \max\{n, m\}$ and $y(j) = z(j)$ for $j \leqslant k$, then by (1)

$$\Gamma_\Phi(x, y)(n) = \Gamma_\Phi(x, y)(m) = 0.$$

Thus, if y is such that $F(x, y, k)$ holds, then n, m are in the domain of the well-ordering $\Gamma_\Phi(x, y)$ and

$$\Gamma_\Phi(x, y)|n \prec \Gamma_\Phi(x, y)|m$$

follows. Consequently, using (3) and (6), we infer

(8) $$A \vdash n <^* m \,\&\, \Gamma_\Phi(x, z)(n) = \Gamma_\Phi(x, z)(m) = 0 \rightarrow R(n, m),$$

where $R(n, m)$ is defined thus:

$$(\mathrm{E}f, g)[W(f, n) \,\&\, W(g, m) \,\&\, f \preccurlyeq g].$$

R is a well-ordering by (7). (8) implies that $\Gamma_\Phi(x, z)$ can be embedded into a well-ordering, and thus $A \vdash \mathrm{Bord}(\Gamma_\Phi(x, z))$ follows. The same argument also shows that

$$A \vdash (\mathrm{E}y)H(x, y) \rightarrow F(x, y, n+1) \rightarrow \Gamma_\Phi(x, z)|n \preccurlyeq \Gamma_\Phi(x, y)|n.$$

From this and (5) we get (4), and hence

$$A \vdash (\mathrm{E}y)H(x, y) \rightarrow (\mathrm{E}z)H^*(x, z).$$

Since the statement

$$A \vdash (n)F(x, z, n) \,\&\, F(x, z', n) \rightarrow z = z'$$

is obvious, the proof of schema (K) is completed.

Section 3. Let S be a partial system of set theory consisting of the following axioms:

extensionality, foundation, pairing, union, infinity, subsets and Σ_1-collection.

Σ_1-collection and subsets imply Σ_1-substitution. S is strong enough to define the constructible sets and prove their fundamental properties. Thus, we can prove

LEMMA 3.1. *There is a formula $L(x)$ defining an internal transitive well-ordered model of S plus the axiom of choice.*

Proof. In reconstructing the constructible sets within S we follow C. Karp [1] with some necessery cautions, since S lacks both the power set and full substitution. Thus, for instance, defining the $(\xi+1)$-th level of L, we assign to each sequence $\langle F, a_1 \ldots a_n\rangle$, where F is a restricted formula and $a_1 \ldots a_n$ is a suitable sequence of parameters from L_ξ, the set $\{x \in L_\xi\colon F(x, a_1 \ldots a_n)\}$ and note that this assignment is Σ_1 (even if F is arbitrary). Hence, $L_{\xi+1}$ exists by Σ_1-substitution. Similarly, we extend the hierarchy at limit ordinals. This process yields a formula $\mathscr{L}$ (which is Δ_1) such that

$$S \vdash (\xi)(\mathrm{E}a)\mathscr{L}(\xi, a),$$

i.e. the constructible sets are a proper class. Putting $L(x) = (\mathrm{E}\xi)\mathscr{L}(\xi, x)$, we obtain a Σ_1-formula such that

$S \vdash F^L$ for each axiom F of S plus choice, where F^L is the relativization of F to L.

We first prove the schema of reflection principle, which gives the axiom of subset. This implies in turn Σ_1-substitution, notifying again that the suitable assignment is Σ_1. Next, we prove the absoluteness lemma and construct a Δ_1-formula which well-orders L. This implies the axiom of choice and Σ_1-collection.

Remark 1. Actually, it can be proved that full substitution as well as the schema of choice holds in L. Hence L is a model of ZFC(−) (see Marek [2]).

System S can be interpreted in A by using well-founded graphs (or trees). We refer the reader to Zbierski [6], where this interpretation is developed. Let us only note that the Σ_1-formulas of S correspond to the Σ_2^1-formulas of A under this interpretation and hence the Σ_1-collection is provable by using the Σ_2^1-axiom of choice, which is valid in A in view of the lemma of Section 1 (cf. Marek [2]). There is also a formula of A which selects those graphs which are codes of reals (see Zbierski [6]). Since system A is obviously interpretable (in a natural way) in S, we get an interpretation of A in A by means of graphs.

Finally, let L_A be the restriction of the above interpretation to constructible graphs. Thus we obtain

COROLLARY. *$A \vdash F^{L_A}$, for each theorem F of A.*

In fact, in view of the remark following Lemma 1 the full schema of the axiom of choice (C) restricted to L_A is provable in A. In other words, one can define in A an internal model of A with choice (C).

Remark 2. In (Zbierski [6]) it is also shown that the set

$$\{S(f):\ M \vdash L_A[f]\},$$

where M is the principal model of A, consists precisely of constructible reals ($S(f)$ is the real encoded by the graph f).

LEMMA 3.2. *$A \vdash F^{L_A} \rightarrow (Ef)\,\mathrm{Mod}(f, \ulcorner F \urcorner)$ for an arbitrary formula F of A.*

($\mathrm{Mod}(f, \ulcorner F \urcorner)$ is a hyperarithmetical formula meaning "the family $\{f^{(n)}: n \in \omega\}$ is a model of F". $\ulcorner F \urcorner$ is the Gödel number of F, see Mostowski [3]).

Proof. It is easier to prove the corresponding form of Lemma 2 in S. Using the axiom of choice, apply within the constructible sets a Skolem–Löwenheim argument to obtain a countable (and hence enumerated by a single real) model of F. Now, we may prove our main

THEOREM. *Neither A nor A_ω is finitely axiomatizable.*

Proof. Suppose, for contradiction, that there is a formula F of A whose set of ω-consequences $\mathrm{Cn}_\omega(F)$ is equal to A_ω. Since $A \subseteq A_\omega$ and since the constructible reals is an ω-model, we get

$$A_\omega \vdash F^{L_A} \quad \text{and} \quad A_\omega \vdash (Ef)\,\mathrm{Mod}(f, \ulcorner F \urcorner)$$

in view of Lemma 3.2. Now, if $\mathrm{Con}(F)$ is a suitable formula of A expressing the consistency of the set of ω-consequences of F, then in view of our assumption we obtain $F \vdash_\omega \mathrm{Con}(F)$, which gives the desired contradiction.

References

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